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Limiting Processes in the Formal Theory of Scattering

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The equivalence of different formal limiting processes employed in the theory of scattering is discussed by comparing the integral equations which hold for finite values of the parameter.

IN the theory of scattering, one always introduces or implies formal limiting processes such as the adiabatic switching on of the interaction. Recent work[1–4] has emphasized the importance of these limiting processes. The relations between alternative limiting processes, however, are not as clear as one might desire. It is the purpose of this note to clarify these relations by comparing the integral equations which hold for finite values of the parameter.

Let $H=K+V$ be the Hamiltonian of the system. V is the interaction and $V(t)$ is the Hamiltonian in the interaction representation,

$$V(t)=e^{iKt}Ve^{-iKt}.$$

We consider the unitary[5] matrix $U(t,t_0)$ which satisfies the Schrödinger equation,

$$i\partial U(t,t_0)/\partial t=V(t)U(t,t_0) \tag{1}$$

($\hbar=1$), and the initial condition $U(t,t_0)=1$. It satisfies the integral equation[6]

$$U(t,t_0)=1-i\int_{t_0}^{t}dt'V(t')U(t',t_0), \tag{2}$$

and

$$U(t_0,t)=1+i\int_{t_0}^{t}dt'U(t_0,t')V(t'). \tag{3}$$

An explicit solution is given by[7]

$$U(t,t_0)=e^{iKt}e^{-i(K+V)(t-t_0)}e^{-iKt}. \tag{4}$$

Our concern is with the definitions of $U(t,-\infty)$ and $U(\infty,t)$. These operators may be defined either by adiabatic switching on of the interaction or by a certain average over initial or final times proposed by Gell-Mann and Goldberger.[1] In either case, we obtain an operator family $U_\epsilon(t,-\infty)$ and $U_\epsilon(\infty,t)$ depending on a parameter ϵ ($\epsilon>0$) and we define the operators $U(t,-\infty)$ and $U(\infty,t)$ by

$$U(t,-\infty)=\lim_{\epsilon\to 0}U_\epsilon(t,-\infty), \tag{5}$$

and

$$U(\infty,t)=\lim_{\epsilon\to 0}U_\epsilon(\infty,t). \tag{6}$$

Such a procedure is necessary since the limit $\lim_{t_0\to-\infty} U(t,t_0)$ does not exist in general. Our operator family should be defined, however, in such a way that

$$\lim_{\epsilon\to 0}U_\epsilon(t,-\infty)=\lim_{t_0\to-\infty}U(t,t_0),$$

if the latter limit exists. It is easy to verify that this is true for the definition (8) below.[8]

The operator $U_\epsilon(t,-\infty)$ may, for instance, be defined as the solution of the integral equation,

$$U_\epsilon(t,-\infty)=1-i\int_{\infty}^{t}dt'e^{\epsilon(t'-t)}V(t')U_\epsilon(t',-\infty), \tag{7}$$

which is obtained from (2) by switching on the interaction adiabatically and letting t_0 go to $-\infty$. The qualitative idea of switching on the interaction does not determine the factor $e^{\epsilon(t'-t)}$ uniquely. If one starts from the differential Schrödinger equation, one is led to substitute for instance, $V(t)\to e^{-\epsilon|t|}V(t)$.[2,3] It is easy to see that for finite t a factor $e^{\epsilon|t'|}$ in (7) instead of $e^{\epsilon(t'-t)}$ would give the same $U(t,-\infty)$ in the limit $\epsilon\to 0$, if this limit exists. Our choice (7) will prove to be particularly useful.

An alternative definition has been proposed by Gell-Mann and Goldberger:[9]

$$U_\epsilon(t,-\infty)=\epsilon\int_{-\infty}^{t}dt_0e^{\epsilon(t_0-t)}U(t,t_0). \tag{8}$$

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[1] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[2] F. J. Belinfante and C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, No. 6 (1954).

[3] S. Fubini, Atti accad. naz. Lincei **12**, 298 (1952).

[4] The formal theory of scattering is also discussed in detail in a forthcoming book by J. M. Jauch and F. Rohrlich, on *Quantum Electrodynamics* (Addison-Wesley Press, Cambridge, 1955), Chap. 7.

[5] For a proof of the unitarity of U see, for instance, reference 2, p. 28.

[6] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

[7] S. T. Ma, Phys. Rev. **87**, 652 (1952), Eq. (42).

[8] It is important to realize that

$$\lim_{\epsilon\to 0}\epsilon\int_{t_1}^{t_2}dte^{\epsilon t}f(t)=0$$

for any function f for which the integral exists provided $\epsilon t_1\to 0$ and $\epsilon t_2\to 0$.

[9] Reference 1, Eq. (3:19). This equation gives

$$U_\epsilon(t,-\infty)=\epsilon\int_{-\infty}^{0}dt_0e^{\epsilon t_0}U(t,t_0)$$

which is different from (8). However, the limit $\epsilon\to 0$ is the same if it exists. See reference 8.

We may substitute the right-hand side of (4) for $U(t,t_0)$ in (8) and obtain a closed expression for $U_\epsilon(t,-\infty)$:

$$U_\epsilon(t,-\infty)=e^{iKt}U_\epsilon(0,-\infty)e^{-iKt}, \tag{9}$$

where

$$U_\epsilon(0,-\infty)=\epsilon\int_{-\infty}^{0} d\tau e^{\epsilon\tau}e^{i(K+V)\tau}e^{-iK\tau}. \tag{10}$$

We shall now show that the two definitions of the operator $U_\epsilon(t,-\infty)$, [Eqs. (7) and (8)] are in fact identical. This can be shown by proving that the operator defined by (8) satisfies the integral equation (7): Multiply (2) by $\epsilon e^{\epsilon(t_0-t)}$ and integrate over t_0 from $-\infty$ to t. The first term on the right-hand side is then equal to 1, the second term is identical with the second term in (7) after the order of the two integrations has been interchanged. Similarly we can show that the two definitions of the operator $U_\epsilon(\infty,t)$,

$$U_\epsilon(\infty,t)=1-i\int_t^{\infty} dt' U_\epsilon(\infty,t')V(t')e^{-\epsilon(t'-t)} \tag{11}$$

and

$$U_\epsilon(\infty,t)=\epsilon\int_t^{\infty} dt_0 e^{-\epsilon(t_0-t)}U(t_0-t), \tag{12}$$

are identical. We have thus shown that the averaging process (8) or (12) is equivalent to an adiabatic switching on or switching off of the interaction.

The connection with the time-independent scattering theory is easily established as follows: Inserting (9) in (7), we find,

$$U_\epsilon(0,-\infty)=1-i\int_{-\infty}^{0} dt e^{\epsilon t}e^{iKt}VU_\epsilon(0,-\infty)e^{-iKt}, \tag{13}$$

which proves that $U(0,-\infty)$ is identical with Møller's[10] wave matrix $\Omega^{(+)}$. Indeed $\Omega^{(+)}$ is *defined* by the integral equation

$$(q|\Omega_\epsilon^{(+)}|q')=(q|1|q')+(E_{q'}-E_q+i\epsilon)^{-1}\times(q|U\Omega_\epsilon^{(+)}|q'), \tag{14}$$

and $\Omega^{(+)}=\lim_{\epsilon\to 0}\Omega_\epsilon^{(+)}$. Equation (14) is written in a representation in which K is diagonal. If we evaluate the integration in (13) in this representation we see that $U_\epsilon(0,-\infty)=\Omega_\epsilon^{(+)}$.

In analogy to (9), we have from (4) and (12)

$$U_\epsilon(\infty,t)=e^{iKt}U_\epsilon(\infty,0)e^{-iKt}, \tag{15}$$

and one verifies easily that $U(\infty,0)^*=\Omega^{(-)}$, where $\Omega^{(-)}$ is defined by

$$(q|\Omega_\epsilon^{(-)}|q')=(q|1|q')+(E_{q'}-E_q-i\epsilon)^{-1}\times(q|V\Omega_\epsilon^{(-)}|q'), \tag{16}$$

and $\Omega^{(-)}=\lim_{\epsilon\to 0}\Omega_\epsilon^{(-)}$. We have thus shown that the insertion of a small imaginary part in the energy denominator of the time-independent scattering theory is equivalent to an adiabatic switching of the interaction.

The S matrix in the time-dependent scattering theory is usually defined by[11]

$$S=U(\infty,-\infty). \tag{17}$$

It appears that the definition (17) may be ambiguous without detailed prescriptions about the limits involved. We shall show that there is no ambiguity if the limits are taken in any one of the several ways discussed below. We may define $U(\infty,-\infty)$, for instance, by

$$\begin{aligned}U(\infty,-\infty)&=\lim(\epsilon\to 0,\ \epsilon'\to 0)U_{\epsilon'}(\infty,t)U_\epsilon(t,-\infty)\\&=\lim(\epsilon\to 0,\ \epsilon'\to 0)\epsilon\epsilon'\int_t^{\infty} dt_0'\int_{-\infty}^{t} dt_0\\&\quad\times e^{\epsilon t_0-\epsilon' t_0'+t(\epsilon'-\epsilon)}U(t_0',t_0).\end{aligned} \tag{18}$$

We see that the right-hand side of (18) is independent of t and the ratio ϵ'/ϵ for finite t in the limit $\epsilon\to 0$, $\epsilon'\to 0$. We see, also, that it is equal to

$$\lim(t_0'\to+\infty,\ t_0\to-\infty)U(t_0',t_0),$$

if these limits exist.[8]

We can further show that

$$\begin{aligned}U(\infty,-\infty)&=\lim(t\to\infty,\ \epsilon t\to 0)U_\epsilon(t,-\infty)\\&=\lim(t\to-\infty,\ \epsilon t\to 0)U_\epsilon(\infty,t).\end{aligned} \tag{19}$$

Inserting (9) and (15) in (7) and (11), respectively, we see that the double limit $t\to\pm\infty$, $\epsilon t\to 0$ exists provided the energy dependence of $VU_\epsilon(0,-\infty)$ and $U_\epsilon(\infty,0)V$ is sufficiently regular. Since these limits exist, (19) follows from (18).[8]

The equivalence of the S matrix defined by (17) and the S matrix of the time-independent scattering theory follows from the relations $U(0,-\infty)=\Omega^{(+)}$ and $U(\infty,0)=\Omega^{*(-)}$.[12]

We are indebted to Dr. J. M. Jauch for stimulating discussions and critical comments.

[10] C. Møller, Kgl. Danske Videnskab. Selskab, Mat-fys. Medd. **23**, No. 1 (1945). Møller denotes the wave matrix by Ψ; we follow the notation of reference 1.

[11] This definition was first given by E. C. G. Stueckelberg, Helv. Phys. Acta **17**, 3 (1944). See also J. Schwinger, reference 5, and F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

[12] This equivalence was first demonstrated by E. C. G. Stueckelberg, Helv. Phys. Acta **18**, 195 (1945). See also S. Fubini, reference 3.