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Solution of Many-Body Scattering Problems by Fredholm's Equation

P. SWAN
Physics Department, University of Melbourne, Melbourne, Australia

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A reduction is made of the second-order linear integro-differential equation describing many-body elastic collisions to a Fredholm equation, for which an analytic solution can be constructed. The Fredholm equation is easier to solve numerically, an example being solved by iteration as an illustration of its use.

An important type of equation which occurs in the treatment of many-body collisions is the second order linear integro-differential equation,

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-U(r)\right]f_l(r)=\lambda\int_0^\infty K_l(r,r')f_l(r')dr', \quad (1)$$

where $K_l(r,r')$ is a symmetric kernel such that $\int_0^\infty K_l(r,r')dr'$ is finite for all r and approaches 0 asymptotically for $r\to\infty$, and λ is an arbitrary parameter. $U(r)$ is a short range potential function such that $|U(r)|<A/r$ for $r\to\infty$, and $U(r)$ is less singular than $1/r^2$ at $r=0$.

Equations of the form (1) appear in the treatment of the scattering of electrons by atoms[1] and in the collisions of neutrons and protons with deuterons,[2] tritons,[3,4] and other nuclei.

A function $G(R,r)=(R-r)$ exists which satisfies the usual Green's function conditions except that

$$\frac{d^2}{dr^2}G(R,r)=0.$$

Its use on (1) leads to

$$f_l(R)=f_l(0)+f_l'(0)R+\int_0^\infty\left[M_l(R,r)+\lambda N_l(R,r)\right]f_l(r)dr, \quad (2)$$

where

$$\begin{aligned} M_l(R,r)&=(R-r)[U(r)+l(l+1)/r^2-k^2], \quad (r<R)\\ &=0 \qquad\qquad\qquad\qquad\qquad\quad , \quad (r>R)\\ N_l(R,r)&=\int_0^R(R-r')K_l(r,r')dr'. \end{aligned} \quad (3)$$

As $f_l(0)=0$ and $f_l'(0)$ can be regarded as a fixed parameter for S-wave interactions ($l=0$), Eq. (2) has the form of a Fredholm equation or integral equation of the second kind, the solution of which is well-known.[5] If $l>0$, then $f_l'(0)=0$ and this solution breaks down; but Whittaker and Watson[5] have constructed one valid for this case. However, these solutions are impractical for actual numerical calculations so that iterative or difference-equation methods must be used.

The iterative process used on (3) has advantages as compared to (1) provided the kernel $N_l(R,r)$ can be evaluated analytically. This may be done for elementary cases of electron-atom collisions in terms of exponential and $[\mathrm{Ei}(x)]$ functions, and for nuclear collisions such as n-H^3 scattering in terms of Gaussian and error functions, provided appropriate ground state functions of Gaussian form are used.[3,4]

Whereas iteration of (1) involves the repetition of the two steps of integrating with a trial function over the kernel product and solving a second order differential equation, (3) does not require the latter step. This simplifies calculation if the number of steps needed to solve the differential equation is large.

It has been shown[6] that neutron-deuteron scattering is very approximately described by the equation

$$\left(\frac{d^2}{dr^2}+k^2\right)f(r)=\lambda\int_0^\infty rr'e^{-\alpha(r+r')}f(r')dr'. \quad (4)$$

The scattering phase-shift η involved in the asymptotic solution for large r, $\sin(kr+\eta)$, is given by

$$\cot\eta=-(\alpha^2-k^2)\{8\alpha^4\lambda+(\alpha^2+k^2)^2[4\alpha^3(\alpha^2+k^2)^2-(3\alpha^2+k^2)\lambda]\}/16\alpha^5\lambda k. \quad (5)$$

For $\lambda=-1$, $\alpha=1$, $k=0.1$, one obtains $\eta=2.912$. Eq. (4) reduces to the Fredholm Eq. (2) with

$$\begin{aligned} M(R,r)&=-k^2(R-r), \quad (r<R),\\ N(R,r)&=r(r-2)e^{-\alpha r}[1-(1+R)e^{-R}]+re^{-\alpha r}R^2e^{-R}. \end{aligned} \quad (6)$$

During the iterative process, one may either maintain a fixed initial gradient $f'(0)$, or adjust $f'(0)$ at each iteration so as to obtain unit asymptotic amplitude. Adopting the former method and using an initial trial function $f_0(r)=\sin kr$ in (2), one obtains successively $\eta=0$, 3.11, 2.88, 2.92, so that the convergence of the iteration is reasonable and comparable to its use on (1).

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