turbation theory. Here again the states most greatly affected are the 1s states and for these the error is less than 3 percent for $Z=12(\pi)$, and less than 2 percent for $Z=12(\mu)$.

It should be noted that our results for the μ states, 2p and 2s in carbon, are in agreement with the values published by Pomeranchuk.[1] However, since no account is taken of the effect of the Coulomb field on the motion of the virtual electron-positron pairs, the values given here includes a fractional error of order $(Z\alpha)^2$.

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Quantization of Multiple Frequency Linear Systems

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The commutation relations and constants of motion for a wave equation whose frequencies have a common degeneracy are discussed. Although the quantum relations separate into those of simple subsystems, the corresponding decomposition holds only for a restricted class of constants of motion, excluding the energy.

A DETAILED investigation of the quantum mechanics of linear systems[1] has brought to light some points of interest relating to the multiple frequency systems first considered by Pais and Uhlenbeck.[2]

The quantization formalism employed by the writer[3] for E.B. systems is briefly as follows: Suppose the linear system to be given by $\Sigma M_{\alpha\beta}\psi_\beta(x)=0$, or $M\psi=0$, where $M=(M_{\alpha\beta})$ is a self-adjoint matric differential operator; x stands for all n independent variables one of which, x_0, is timelike in the sense that there is a complete set of solutions ψ which vanish strongly at ∞ of any surface $x_0=$ constant, and M is of finite degree d in $\partial_0=\partial/\partial x_0$. If we now introduce the expression

$$\mathcal{T}(\omega,\psi)=i/\hbar\int\omega^T[M,\,Z(x_0-x_0')]\psi dx^n, \tag{1}$$

where $Z(y)$ is the step function $[1+\mathrm{sgn}(y)]/2$ and ω^T is the transpose of ω, the commutation relations may be written in the form,

$$\mathcal{T}(\omega,\psi)\cdot F\psi(x')=F\omega(x'), \tag{2}$$

for any vector $\omega=(\omega_\alpha(x))$ and any linear operator F of degree $<d$ in ∂_0; here $A\cdot B$ stands for $[A,B]$. (Actually, it suffices to consider only ω for which $M\omega=0$; in such a case, $\mathcal{T}(\omega,\psi)$ is independent of x_0 and F may be taken as arbitrary.) Furthermore, if the system is invariant under the operator T ($T\psi$ is a solution whenever ψ is), then if $\mathcal{T}(\psi,T\psi)+\mathcal{T}(T\psi,\psi)=0$ as well, the constant of motion corresponding to T is given by

$$Q_T=\tfrac{1}{2}i\hbar\mathcal{T}(\psi,T\psi), \tag{3}$$

which transforms correctly and also satisfies

$$[F\psi,Q_T]=i\hbar FT\psi. \tag{4}$$

Now consider the multiple frequency linear system $\mathrm{N}^r\psi=0$, where N is self-adjoint, of degree s in ∂_0; since an operator F of degree $<d=rs$ in ∂_0 can generally be written as $F=\sum_{b=0}^{r}\mathrm{H}_b\mathrm{N}^b$, where H_b is of ∂_0-degree $<s$, (1) and (2) are equivalent to

$$(i/\hbar)\int\omega^T[\mathrm{N}^r,\,Z(x_0-x_0')]\psi dx^n\cdot\mathrm{HN}^b\psi(x')=\mathrm{HN}^b\omega(x'),$$

or to

$$(i/\hbar)\sum_{k=0}^{r-1}\int(\mathrm{N}^k\omega)^T[\mathrm{N},\,Z(x_0-x_0')]\mathrm{N}^{r-k-1}\psi dx^n\cdot\mathrm{HN}^b\psi(x')=\mathrm{HN}^b\omega(x'), \tag{5}$$

where H is of ∂_0-degree $<s$. The kth integral in (5) depends only on $\mathrm{N}^k\omega$, $\partial_0\mathrm{N}^k\omega$, $\cdots$, $\partial_0^{s-1}\mathrm{N}^k\omega$, evaluated at $x_0=x_0'$; since these functions may be chosen independently, (5) decomposes into

$$(i/\hbar)\int(\mathrm{N}^a\omega)^T[\mathrm{N},\,Z(x_0-x_0')]\mathrm{N}^{r-a-1}\psi dx^n\cdot\mathrm{HN}^b\psi(x')=\delta_{a,\,b}\mathrm{HN}^b\omega(x'). \tag{6}$$

According to (6), only $\mathrm{KN}^a\psi\cdot\mathrm{HN}^{r-1-a}\psi\neq0$ for K, H of degree $<s$ in ∂_0; thus the commutation relations decompose in a pairwise manner, with an additional single set if r is odd. Moreover, we obtain from (6), on redefining ω, the relations,

$$\begin{aligned}(/\hbar)\int\omega^T[\mathrm{N},\,Z(x_0-x_0')]\mathrm{N}^{r-1-a}\psi dx^n\cdot\mathrm{HN}^a\psi(x')&=\mathrm{H}\omega(x'),\\ (i/\hbar)\int(\mathrm{N}\omega)^T[\mathrm{N},\,Z(x_0-x_0')]\mathrm{N}^a\psi dx^n\cdot\mathrm{HN}^{r-1-a}\psi(x')&=\mathrm{HN}\omega(x'),\end{aligned} \tag{7}$$

[1] J. K. Percus, Columbia University Dissertation, 1954 (unpublished).
[2] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).
[3] The formalism used here may be shown to be equivalent to those of other authors when they are applicable to a given linear system; a more detailed discussion of this point will appear in a subsequent paper.

between $KN^a\psi$ and $HN^{r-1-a}\psi$, precisely those which would result between $K\varphi$ and $HN\varphi$ for $N^2\varphi=0$ (unless $a=r-1/2$, in which case the first of (7) corresponds to the relations between $K\varphi$ and $H\varphi$ for $N\varphi=0$).

The Eqs. (6) may also be written in the form

$$\mathcal{T}^{(a)}(\omega,\psi)\cdot H\begin{Bmatrix} N^b \\ N^{r-1-b}\end{Bmatrix}\psi(x')=\delta_{a,b}H\begin{Bmatrix} N^b \\ N^{r-1-b}\end{Bmatrix}\omega(x'),$$

where

$$\mathcal{T}^{(a)}(\omega,\psi)$$
$$=(i/\hbar)\int\{(N^a\omega)^T[N,Z(x_0-x_0')]N^{r-1-a}$$
$$+(N^{r-1-a}\omega)^T[N,Z(x_0-x_0')]N^a\}\psi dx^n, \quad (8)$$
$$\mathcal{T}^{(r-1/2)}(\omega,\psi)$$
$$=(i/\hbar)\int(N^{(r-1)/2}\omega)^T[N,Z(x_0-x_0')]N^{(r-1)/2}\psi dx^n$$

respectively for $a<r-1/2$ and for $a=r-1/2$ if r is odd. Now let us write $Q_T=\Sigma Q_T^{(a)}$, where $Q_T^{(a)}=\frac{1}{2}i\hbar\mathcal{T}^{(a)}(\psi,T\psi)$, no longer a constant; it readily follows from (8) that if the ∂_0-degree of T is 0, then $[Q_T^{(a)},Q_T^{(b)}]=0$, and the commutation relations between $Q_T^{(a)}$ and the $KN^a\psi$, $HN^{r-1-a}\psi$ are identical with those between Q_T, $K\varphi$, $HN\varphi$ for $N^2\varphi=0$, so that e.g. eigenvalues are also the same. Such a decomposition may not however obtain if T is of ∂_0-degree>0: Thus, if Q_{∂_0} (=energy) is a constant of the motion, we would require, for $a,b\leq r-1/2$, $[Q_{\partial_0}^{(a)},Q_{\partial_0}^{(b)}]=0$ as well as $[HN^b\psi,Q_{\partial_0}^{(a)}]=i\hbar HN^b\partial_0\psi\delta_{a,b}$; now there certainly exists an H of ∂_0-degree$<s$ such that $H\partial_0=N+f$, where f is of ∂_0-degree 0. We then have $[H\psi,Q_{\partial_0}^{(0)}]=i\hbar H\partial_0\psi=i\hbar N\psi+i\hbar f\psi$, whence $[[H\psi,Q_{\partial_0}^{(0)}],Q_{\partial_0}^{(1)}]=-\hbar^2N\partial_0\psi$; on the other hand, since $[Q_{\partial_0}^{(0)},Q_{\partial_0}^{(1)}]=0$, this should equal $[[H\psi,Q_{\partial_0}^{(1)}],Q_{\partial_0}^{(0)}]=[0,Q_{\partial_0}^{(0)}]=0$, which of course cannot be.

This last result contradicts one of Pais and Uhlenbeck ([2]Appendix) for the case $r=3$; the discrepancy is due to the fact that in their "contact transformation," $[P_0,P_2]\neq 0$. While a pairwise decomposition of the Hamiltonian, equivalent to expressing a solution of $N^r\psi=0$ as a sum of solutions of $N^2\psi=0$, is in principle impossible, we have at least verified their conjecture, insofar as commutation relations and ∂_0-independent constants of motion are concerned, as to the decomposition of the system into twofold subsystems.