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Reflection of Waves by an Inhomogeneous Medium*

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A new approximate solution is given for the general linear second-order differential equation which is especially appropriate in treating the reflection of waves by an inhomogeneous medium. The well-known approximations to a fundamental pair of solutions made by Liouville, Rayleigh, and Jeffreys, which suffer from singularities at the zeros of a particular function, are replaced by another pair of simple approximations u_1, u_2, which in general agree well with the first pair but remain finite at the zeros. Then a corresponding approximation ρ_1 is obtained for ρ, the coefficient of reflection of plane waves by a specified inhomogeneous medium. Also iterative processes are given which from ρ_1 (or any other approximation) derive a sequence of approximations ρ_2, ρ_3, $\cdots$, which rapidly converge on ρ. Lastly it is shown that the approximation u_1 for a particular equation leads to a good, simple approximation to the Hankel function $H_n^{(2)}(nz)$ which agrees well with the approximations of Hankel, Debye, and Carlini but has a wider range of validity.

1

THE determination of the reflection of plane waves by a medium with a refractive index μ which is an assigned function $\mu(x)$ of but one coordinate x is a problem of great practical and theoretical importance in physics and engineering. The theoretical problem is essentially linked with that of finding a solution to the general linear differential equation of the second order which satisfies two conditions at a given point. The same mathematical problem is fundamental in quantum mechanics.

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Its solution can always be reduced to that of the normal form

$$d^2u/dx^2+p^2u=0, \tag{1}$$

where $p\equiv p(x)$ is a given function of x (which may be complex). The quantity $p(x)$ may be equal to $2\pi fc^{-1}\mu(x)$ where $\mu(x)$ is the complex refractive index for a radio wave of frequency f which has reached the height x in the ionosphere[1]; or $p^2(x)$ may be proportional to the kinetic energy of a particle in quantum mechanics.

In many important problems, $p^2(x)$ is such that solutions in terms of well-known functions cannot be found. In such problems it is necessary to use approximate solutions such as those discovered by Liouville,[2] Rayleigh,[3] Jeffreys,[4] Langer,[5] and others.[5]

Langer's approximations remain finite near only one zero of $p^2(x)$ and are not easy to use when $p^2(x)$ is complex. Liouville's and Rayleigh's approximations are

$$u_{LR}\sim p^{-\frac{1}{2}}\exp\left(\pm i\int^x pdx\right), \tag{2}$$

and are special cases of Jeffreys' approximations,

$$u_J\sim p_0^{-\frac{1}{2}}\exp\left[\pm i\int^x\left(hp_0+\frac{p_1}{2p_0}\right)dx\right], \tag{3}$$

which relate to $p^2(x)$ in the form

$$p^2(x)=h^2p_0^2(x)+hp_1(x),$$

where h is a parameter.

Both u_{LR} and u_J have singularities, at the zeros of $p^2(x)$ and $p_0^2(x)$, respectively, which make it necessary

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[1] For oblique propagation, with the initial angle of incidence θ, we have $p(x)=2\pi fc^{-1}[\mu(x)^2-\sin^2\theta]^{\frac{1}{2}}$.

[2] J. Liouville, J. de Math. 2, 16 and 418 (1837).

[3] Lord Rayleigh, Proc. Roy. Soc. (London) A86, 207 (1912).

[4] H. Jeffreys and B. Jeffreys, *Methods of Mathematical Physics*, Sec. 17.122 (1950).

[5] See E. C. Kemble, *Fundamental Principles of Quantum Mechanics* (1937). It is not clear why so many writers still refer to the approximations of Liouville, Rayleigh, and Jeffreys as the "B.W.K." or "W.B.K." approximations.

to "connect" or "match" them with approximate solutions in terms of Airy integrals or Bessel functions. This procedure has had successes in quantum theory but has not been found very convenient for the study of the reflection of very long waves by the ionosphere.

A pair of simple approximations which are free from these difficulties are the following:

$$u_1 = \exp\left(\int^x q dx\right), \tag{4}$$

$$u_2 = u_1 \int^x u_1^{-2} dx, \tag{5}$$

where

$$q = -\frac{p'}{2p}\left[1 - \left\{1 - \frac{4p^4}{p'^2}\right\}^{\frac{1}{2}}\right], \tag{6}$$

and $p' \equiv dp/dx$.

For u_1 and u_2 are continuous at any zero of p^2, they agree with Jeffreys' approximations u_J, within quantities of order $(1/h)$, at points distant from any zero, and lastly, when $(p^2 = -x^m)$, u_1 and u_2 approximate to the known solutions of (1) in terms of cylinder functions of order $n = 1/(m+2)$ both when x is very small and when x is very large. [This is shown for u_1 by (31) and (32) below.]

The approximation given by (4) and (6) is suggested by the following discussion. If we set

$$u = \exp\left(\int y dx\right), \tag{7}$$

we obtain the Riccati equation,

$$y' + y^2 + p^2(x) = 0. \tag{8}$$

The substitutions

$$y = pv, \qquad z = \int p dx, \tag{9}$$

transform (8) into

$$dv/dz + v^2 + 2rv + 1 = 0, \tag{10}$$

where

$$r = \frac{1}{2p}\frac{dp}{dz} = \frac{1}{2p^2}\frac{dp}{dx}. \tag{11}$$

When r is constant, solutions of (10) are

$$v_1, v_2 = -r \pm (r^2 - 1)^{\frac{1}{2}}, \tag{12}$$

and the corresponding solutions of (8) are

$$y_1, y_2 = -\frac{p'}{2p}\left[1 \mp \left\{1 - \frac{4p^4}{p'^2}\right\}^{\frac{1}{2}}\right]. \tag{13}$$

When also $|r^2| \ll 1$, we obtain the approximations

$$y_1^{LR},\ y_2^{LR} = -\frac{p'}{2p} \pm ip, \tag{14}$$

which lead to the well-known approximations due to Liouville, Rayleigh, etc., which are given in (2).

Clearly (13) must in general lead to a better approximation than (14). Moreover, by choosing the negative sign before the radical in (13), the corresponding wave function u_1 *is finite at all zeros* and simple poles. This choice of sign is that used in the formula (6).

3

The physical problem is to determine the coefficient of reflection ρ of any kind of plane wave (of wavelength $2\pi k$ in free space) which starts in a uniform medium A of refractive index kp_A, enters at $x = a$ the nonuniform medium B of index $kp(x)$, and emerges at $x = c$ into the uniform medium C of index kp_C.

To satisfy the physical conditions we must take the function

$$y(x) = u'/u \tag{15}$$

as continuous everywhere, and in C take

$$u = u_C = e^{ip_C x}. \tag{16}$$

From (1), (15), and (16) we see that y is that solution of (8) for which

$$y(c) = ip_C. \tag{17}$$

Since in A we have

$$u_A = A_1 e^{ip_A(x-a)} + \rho A_1 e^{-ip_A(x-a)},$$

therefore

$$y(a) = y_A(a) = ip_A\left(\frac{1-\rho}{1+\rho}\right),$$

and so

$$\rho = \frac{ip_A - y(a)}{ip_A + y(a)}, \tag{18}$$

where $y(x)$ satisfies (8) and (17).

If now

$$y_0 = u_0'/u_0$$

is a *particular* solution of (8) ,then the general solution of (1) is

$$u = u_0 + K u_0 \int^x u_0^{-2} dx,$$

where K is any constant, and so by (15) the general solution of (8) is

$$y(x) = y_0 + u_0^{-2} \Big/ \int_\alpha^x u_0^{-2} dx, \tag{19}$$

where α is an arbitrary constant.

From (19) and (17) we obtain

$$ip_C = y_0(c) + u_0(c)^{-2} \Big/ \int_\alpha^c u_0^{-2} dx. \tag{20}$$

On eliminating α between (19) and (20), setting $x = a$ in the resulting expression for $y(x)$, and substituting for

$y(a)$ in (18), we obtain

$$\rho=\frac{\{ip_A-y_0(a)\}f-1}{\{ip_A+y_0(a)\}f+1}, \tag{21}$$

where

$$f=\int_c^a \exp\left(-2\int_a^x y_0 dx\right)dx + \frac{\exp\left(-2\int_a^c y_0 dx\right)}{ip_C-y_0(c)}. \tag{22}$$

If now we adopt for y_0 the approximation q defined in (6), we obtain the following approximation to ρ:

$$\rho_1=\frac{\{ip_A-q(a)\}f_1-1}{\{ip_A+q(a)\}f_1+1}, \tag{23}$$

where

$$f_1=\int_c^a \exp\left(-2\int_a^x q dx\right)dx+\frac{\exp\left(-2\int_a^c q dx\right)}{ip_C-q(c)}. \tag{24}$$

In a particular case we obtain:

$$\rho_1=[ip_A-q(a)]/[ip_A+q(a)], \tag{25}$$

when p and p' are continuous at $x=c$.

It follows that when p and p' are continuous at both the points $x=c$ and $x=a$, then $\rho_1=0$. This may mean nothing more than that under these conditions $|\rho|$ is small. It is easy to show by means of (18) that $\rho=0$ when p^2 has the complex form

$$p^2=-(x-a)(x-c)i\varphi(x) + \left[\int_a^x (x-a)(x-c)\varphi(x)dx\right]^2, \tag{26}$$

where $\varphi(x)$ is real and finite in the interval $a\leqslant x\leqslant c$, and

$$p_A=\int_\alpha^a (x-a)(x-c)\varphi(x)dx,$$

$$p_C=\int_\alpha^c (x-a)(x-c)\varphi(x)dx.$$

In this example only p is continuous at $x=a$ and c. In order to make also p' continuous we may here set $\varphi(x)=(x-a)(x-c)\psi(x)$, where $\psi(x)$ is finite in the same interval. Then both ρ and ρ_1 are zero.

4

When it is required to obtain a more accurate value of ρ than ρ_1 (or any other approximation), we may use the following iterative process when $p(x)$ is continuous at $x=a$ and $x=c$:

$$\rho_{n+1}(a)=\int_c^a \frac{p'(x)}{2p(x)}\{1+\rho_n(x)^2\} \cdot\exp\left[\int_a^x \left(2ip(s)+\frac{p'(s)}{p(s)}\rho_n(s)\right)ds\right]dx, \tag{27}$$

where $\rho_1(a)$ is given by (23). For it can be shown, first that when $p(x)$ is continuous at $x=a$ and $x=c$ the exact reflection coefficient $\rho(a)$, given by (18), is that solution of the equation[6]

$$\rho'(a)+2ip(a)\rho(a)+\frac{p'(a)}{2p(a)}\{\rho^2(a)-1\}=0, \tag{28}$$

for which $\rho(c)=0$, and then that the sequence of numbers $\rho_n(a)$ given by (27), with a good first approximation $\rho_1(a)$, will converge on this solution with convergence of the second order, i.e., with

$$|\rho-\rho_{n+1}|=O|\rho-\rho_n|^2.$$

When $p(x)$ is discontinuous at $x=a$, (27) and (28) can still be used if we take $p'(x)$ to be very large over a very small range of x and then proceed to the limit at which $p'\to\infty$ and the range$\to 0$.

The relations (27) or (28) may be used also to test the accuracy of any given approximation.

It is interesting to note that when in (27) we set $\rho_1(a)=0$, then we obtain for $\rho_2(a)$ the approximation made by Rayleigh[3] and by Schelkunoff[7] for weak reflection.

Similarly a second-order iteration can be constructed which yields successive approximations u_2, u_3, $\cdots$, to a solution of Eq. (1) starting with any approximation u_1 such as those given in (4) and (5) or those given by Langer[5] and by Miller and Good.[8]

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If we apply the foregoing results to Rayleigh's example in which

$$p^2=n^2x^{-2},$$
$$a=n/p_A, \quad c=\infty,$$

we obtain from (6) the expression $q=\beta p$, where

$$\beta=\frac{1}{2n}-\left(\frac{1}{4n^2}-1\right)^{\frac{1}{2}},$$

and so, by (23) or (25),

$$\rho=\frac{i-\beta}{i+\beta}=\frac{i}{2n+(4n^2-1)^{\frac{1}{2}}},$$

[6] A similar equation for ρ has been published by L. R. Walker and N. Wax in J. Appl. Phys. **17**, 1043 (1946).

[7] S. Schelkunoff, Report No. EM-37, New York University, November, 1951, (unpublished), p. 117, Eq. (34).

[8] S. C. Miller, Jr., and R. H. Good, Jr., Phys. Rev. **91**, 174 (1953).

which is Rayleigh's exact result (if we note that he used $-i$ where we use i). This expression for ρ is also found to satisfy (28) exactly.

As another application of the approximation u_1, given in (4), we shall use it to determine a particular approximation to the second Hankel function which appears to have a wider range of validity than any one of the well-known approximations. On taking

$$p^2 = -x^m, \tag{29}$$

we obtain

$$q = -\frac{m}{4x}[1-(1+16m^{-2}x^{m+2})^{\frac{1}{2}}], \tag{30}$$

and on setting

$$m = n^{-1}-2, \quad x^{m+2} = -\tfrac{1}{4}z^2,$$

and choosing a suitable constant of integration, we obtain, for $m \neq -2$,

$$\begin{gathered} u_1 = 2^{\frac{1}{2}}[(1-\alpha^2z^2)^{\frac{1}{2}}+1]^{n-\frac{1}{2}} \exp\{(\tfrac{1}{2}-n)[(1-\alpha^2z^2)^{\frac{1}{2}}-1]\}, \\ z = 2ix^{1/2n}, \quad \alpha = 2n/(2n-1). \end{gathered} \tag{31}$$

On studying the form of u_1 when $\alpha^2z^2 \ll 1$ or $\gg 1$, we arrive at the following results for n not an integer:

$$H_n^{(2)}(nz) \doteqdot i\pi^{-1}\Gamma(n)n^{-n}z^{-n}u_1, \tag{32}$$

when

(1) $\alpha^2z^2 \ll 1$ and n positive but not in the range 17/18 to 15/14;
(2) $\alpha^2z^2 \gg 1$ and n large;
(3) $n = \frac{1}{2}$, and then (32) is exact for all z.

Also, when x is not large or small but is outside the range 17/18 to 15/14, the approximation (32) is still good if, when $\alpha^2z^2 \gg 1$, the right-hand side is multiplied by a certain function of n alone.

Equation (32) agrees with the asymptotic approximations of Hankel, Debye, and Carlini. Moreover, (32) remains finite when $z^2 = 1$ whereas the last two mentioned approximations then become infinite.

In general we may expect by means of (4) and (5) to be able to obtain similar kinds of approximations to certain functions which arise from a second-order, linear differential equation.

In conclusion I wish to acknowledge the valuable help received from discussions of the general problem with Professors J. Gibbons and N. Davids.

Additional Note. In a paper read by the author on May 5th, at a meeting of the American Committee of the International Union for Scientific Radio, the following second-order iteration was also presented:

$$\begin{aligned} v_{n+1}(a) = \int_c^a \{p_C^2 - p^2(x) + v_n^2(x)\} \\ \times \exp\left\{2ip_C(x-a) + \int_a^x 2v_n(s)ds\right\} \cdot dx, \end{aligned} \tag{33}$$

combined with

$$\rho_n(a) = \frac{p_A - p_C + iv_n(a)}{p_A + p_C - iv_n(a)}. \tag{34}$$

This is not subject to the restriction that $p(x)$ is continuous at $x=a$ and $x=c$. Also it is particularly convenient for computation when $p^2(x)$ is given by curves representing its real and imaginary parts as in the current work on radio-wave reflection by the D region of the ionosphere. These iterations are at present being used for such work by Mr. J. Wolfe and the results will be reported in due course.

A more detailed account of these approximations and iterations is given by the author in Scientific Report No. 67 of the Ionosphere Research Laboratory.