Determination of Alpha-Alpha Scattering Phase Shifts

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A program is described which makes it possible to carry out a phase shift analysis of alpha-alpha scattering on the ILLIAC (University of Illinois Graduate College High Speed Electronic Digital Computer). In approximately 5–10 min the program can carry out the analysis in terms of up to 6 phase shifts (δ_0, δ_2, $\cdots$, δ_{10}) with a precision comparable to an extremely careful human analysis. The number of scattering angles and the energy of the scattering are limited by 32 and 1–150 Mev, respectively.

THE method usually used to determine a set of phase shifts for alpha-alpha scattering from an experimental measurement of the differential cross section $\sigma^{\exp}(\theta)$, is the graphical one originally used by Wheeler.[1] This method is beset by several limitations. (a) Only the three lowest phase shifts δ_0, δ_2, and δ_4 can be obtained directly because of the limitation of two-dimensional graph paper. Higher phase shifts must be obtained by trial and error involving laborious computation, curve fitting, and adjustment. (b) Only part of the experimental data is directly used, since the cross section at three angles is sufficient for the determination of the three phase shifts. The data at other angles are used only in adjusting the higher phase shifts and in making a choice between the several sets of alternate solutions that arise, both of which again involve fitting a theoretical curve to the experimental data. (c) No consistent account is taken of the different experimental errors at the different angles. (d) Although implied above, explicit mention should be made of the lengthy and tedious hand computation involved. A careful fit at 20 experimental angles using four phase shifts can take two to four weeks of steady, conscientious work. Hence, it is very laborious to investigate extensively the energy variation of the phase shifts since many such analyses at different energies must be made.

To facilitate such an investigation,[2] the Digital Computer, Graduate College, University of Illinois (ILLIAC) has been programmed to carry out the phase shift analysis. The program finds the phase shifts δ_l which minimize the function:

$$f(\delta_0,\delta_2,\cdots,\delta_{l_{\max}})\equiv\sum_{i=1}^{n}\frac{[\sigma^{\exp}(\theta_i)-\sigma^{\text{th}}(\theta_i;\delta_0,\delta_2,\cdots,\delta_{l_{\max}})]^2}{\Delta_i^2},\qquad(1)$$

where $\sigma^{\exp}(\theta_i)$ and Δ_i are the observed cross section and root-mean-square error at θ_i in the laboratory system and

$$\sigma^{\text{th}}(\theta;\delta_0,\delta_2,\cdots,\delta_{l_{\max}})=\left(\frac{z^2e^2}{mv^2}\right)^2\bigg|\csc^2\theta\exp(-i\alpha\log\sin^2\theta)+\sec^2\theta\exp(-i\alpha\log\cos^2\theta)+\frac{2i}{\alpha}\sum_{l=0\ (\text{even})}^{l_{\max}}e^{2i(\xi_l-\xi_0)}\times(e^{2i\delta_l}-1)(2l+1)P_l(\cos 2\theta)\bigg|^2,\qquad(2)$$

with

$$\xi_l-\xi_0\equiv\sum_{s=1}^{l}\tan^{-1}(\alpha/s),\qquad(3)$$

$$\alpha=4e^2/\hbar v,\qquad(4)$$

where v is the speed of the incident alpha particle. The capacity of ILLIAC and the scaling provided by the program allow wide limits to be placed on the energy of the incident alpha particles, the number n and position θ_i of the experimental observations, and the number of phase shifts used in analyzing the data. These quantities are all adjustable within the ranges:

$$\begin{aligned}10.5^\circ\leq\theta_i\ (\text{lab})&\leq 45^\circ\\ 0\leq\ n\ &\leq 32\\ 0\leq\ l_{\max}\ &\leq 10\\ 0.1\leq\ \alpha\ &\leq 1.4\end{aligned}\qquad(5)$$

i.e.,

$$1\text{ Mev}\leq E\ (\text{lab})\leq 150\text{ Mev}.$$

The numerical minimization program used by the machine[3] first computes the gradient of the function f in δ space, and then alters the δ's so as to approach the

TABLE I. A comparison between the first four phase shifts for α-α scattering at 22.9 Mev as computed by Briggs and as computed by ILLIAC.

	Solution I		Solution II	
	Briggs[a]	ILLIAC	Briggs[a]	ILLIAC
δ_0	$-10.5°$	$-10.7°$	86.5°	86.4°
δ_2	94.1°	94.0°	$-61.5°$	$-61.8°$
δ_4	59.1°	59.2°	$-52.0°$	$-51.3°$
δ_6	0.95°	1.09°	1.10°	1.62°
ϵ	$\cdots$	~1.5	$\cdots$	~2.1

[a] See reference 5.

[1] J. A. Wheeler, Phys. Rev. 59, 16 (1941).
[2] Nilson, Briggs, Jentschke, Kerman, and Snyder (to be published).
[3] James N. Snyder, ILLIAC Program Library, Programs H3–H6 (unpublished).

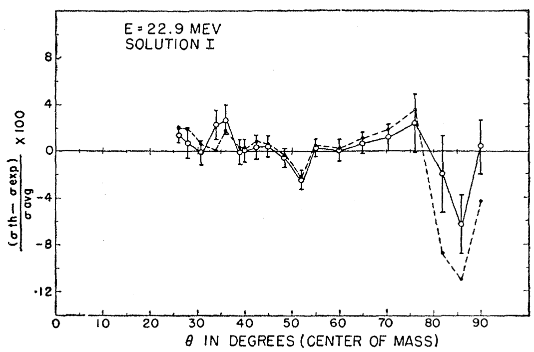

FIG. 1. Comparison of the phase shift analysis of Briggs (see reference 5) (solid lines) and ILLIAC (broken lines). Best solution.

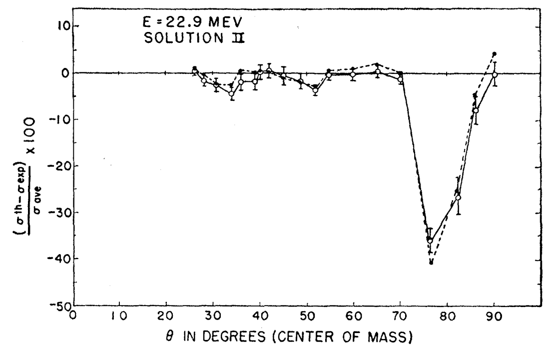

FIG. 2. Comparison of the phase shift analysis of Briggs (see reference 5) (solid lines) and ILLIAC (broken lines). Alternate solution.

minimum of the function in the most efficient manner. This procedure is repeated with ever decreasing mesh size until some predetermined mesh size is reached. It is known from tests on various functions that such a method of steepest descent can be deceived by functions of complicated structure. To eliminate this possibility in the case at hand the program, at the conclusion of the minimization process outlined above, makes another attempt to minimize the function by treating it as a function of only one of the δ_i at a time, proceeding through all the δ_i cyclically again and again with ever decreasing mesh size. In all of the cases run to date, this precaution has proved superfluous. To use the program the experimenter must spend about one-half hour preparing a teletype tape of input data consisting of several parameters which instruct the program how to proceed, the list of experimental angles, the list of cross sections and a list of weighting factors w_i such that:

$$\sum_{i=1}^{n} w_i = 1, \quad w_i = k/\Delta_i^2, \tag{6}$$

where k is a normalizing constant. ILLIAC then carries out the phase shift analysis in the order of 5 min and prints the best set of phase shifts, the numerical value of the minimum of the function, and a table of the cross section at the experimental angles computed using this set of phase shifts.

It is necessary to provide the machine with a set of initial phase shifts $\delta_i^{(0)}$ from which it proceeds to the nearest minimum. These initial values may be chosen in several ways, e.g., educated, intuitive guesses or rough results from a preliminary analysis. Usually the initial $\delta_i^{(0)}$ for the analysis at a given energy were chosen to be the final δ_i from an already completed analysis at a nearby energy. The function being minimized has many local minima in δ space representing possible alternate solutions to the problem. Since the machine carries out an analysis so rapidly, it is possible to start at many different initial points which form a rough net in the δ space and find thereby the various alternate solutions. The most suitable of these can then be chosen by examining which set best fits the energy dependence of the phase shifts and by examining which set yields the best visual fit of the experimental curves. When the alpha-alpha scattering data taken at Illinois were processed by this program, it was invariably found that the best set of phase shifts on these two bases also yielded the absolute minimum value of the function f out of the various relative minima found.

According to the Theory of Least Squares,[4] the root-mean-square error to be feared in a measurement of unit weight is

$$\epsilon \equiv (f_{\min}/n-m)^{\frac{1}{2}}, \tag{8}$$

where n is the number of observations and m is the number of parameters used in an attempt to fit them. Since the rms error associated with the numerical fit σ^{th} is given by $\epsilon\Delta$, a fit commensurate with the accuracy of the experimental data demands that ϵ be of order 1. The cases run to date have been characterized by ϵ in the range 1–3. Hence, it is possible for ILLIAC to quickly carry out a phase shift analysis which is comparable in accuracy to that obtainable by a most careful human.

As an illustration, the 22.9-Mev data taken by Briggs[5] and analyzed by him with extreme care were analyzed by ILLIAC. The comparative results of Briggs and ILLIAC for the best solution (I) and an alternate solution (II) are shown in Table I and Figs. 1 and 2.

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[4] See, for example, E. T. Whittaker and G. Robinson, *The Calculus of Observations* (Blackie, London, 1944), fourth edition, Chap. IX.

[5] Briggs, Singer, and Jentschke, Phys. Rev. **91**, 438 (1953).