differ by approximately a factor of 2, the ionization method yielding the smaller value. The energy flux calculated on the basis of the latitude effect depends critically on the form of the primary proton spectrum assumed for the region above 14 Bev—where accurate experimental data are lacking—and depends also upon the assumption that the counter telescopes measure directly the actual flux of primary particles. While the present data give no better information than has been available heretofore on the form of the particle spectra, there is strong evidence that the equatorial proton flux is approximately one-half that which has been assumed in the energy flux calculations based on the latitude effect. This observation suggests that the discrepancy noted above may arise from errors in particle flux measurements rather than from any serious oversight in the interpretation of the ionization *vs* depth data.

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Production Spectra of Cosmic-Ray Mesons in the Atmosphere*

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Recent experimental data on the differential vertical intensities of μ mesons in the atmosphere make it possible not only to improve the original Sands' production spectrum of μ mesons but also to study its dependence on the geomagnetic latitude. It is shown that, for the residual ranges, R', between 100 g cm^{-2} and 6000 g cm^{-2}, the production spectrum may be approximated by an empirical formula of the type: $C(a+R')^{-\gamma}$ where C and γ are numerical constants practically independent of the geomagnetic latitude. The latitude dependence of the production spectrum is thus expressed through the parameter a which displays a monotonic decrease with increasing latitude. With numerical values of a, C, and γ, compatible with experimental data, the production spectrum of μ mesons is then used as a basis for the derivation of the differential and integral energy spectra of charged π mesons. The latitude dependence of the π-meson spectrum is linked to the geomagnetic cut-off of the primary cosmic radiation, which leads to some crude conclusions on the average multiplicity of π mesons produced in proton collisions with air nuclei. In particular, the dependence of the multiplicity on the primary energies between 2 and 13 Bev is studied in detail and compared with Fermi's statistical theory of π-meson production.

I. INTRODUCTION

THE original idea of M. Sands to introduce an empirical μ-meson spectrum at production[1] has been proven to be very useful in dealing with many problems concerning the mesonic components of cosmic rays in the atmosphere. The knowledge of this empirical production spectrum makes possible not only the computation of the vertical intensities of μ mesons at any desired altitude, but it also offers a basis for testing various theories of π-meson production in nucleonic collisions. The latter aspect is demonstrated, for example, by P. Budini and G. Molière in their review article in Heisenberg's *Kosmische Strahlung*[2] where the authors test Heisenberg's theory of multiple meson production.

Sands derived his production spectrum from early experimental data (partly his own, partly those of others) taken in the proximity of 45° geomagnetic latitude. Unfortunately, these data were then rather imcomplete and not quite suitable for Sands' task. Moreover, the values of the physical constants (like the rest mass or the lifetime of the μ meson) employed in Sands' calculations were, at the time, burdened by large experimental errors.

The purpose of this paper is, therefore, twofold: (a) to bring Sands' spectrum more up-to-date (Sec. II), (b) to extend its applicability to various geomagnetic latitudes (Sec. III). In addition, in Sec. IV we shall utilize the obtained latitude dependence of the μ-meson spectrum at production to study the relationship between the geomagnetic cut-off of the primary radiation and the integral energy spectrum of π mesons at production.

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[1] M. Sands, Technical Report No. 28, Laboratory for Nuclear Science, Massachusetts Institute of Technology, 1949 (unpublished); Phys. Rev. **77**, 180 (1950).

[2] *Kosmische Strahlung*, edited by W. Heisenberg (Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953), second edition, p. 391 ff.

The latter discussion will lead us to some crude conclusions concerning the problem of multiple meson production in air nuclei for energies ranging from about 2 to 13 Bev.

II. SANDS' SPECTRUM AT 50°N

(a) Unidimensional Equation of the Vertical μ-Meson Intensity

Consider the number of μ mesons produced per cm^2-sec-sterad in an infinitesimal horizontal layer dx at the atmospheric depth x, which have residual ranges between R' and $R'+dR'$ and move toward the earth along the vertical. This number will then, in general, depend on three parameters: the atmospheric depth x, the residual range R', and the geomagnetic latitude λ, so that it may be denoted by $H(x,R';\lambda)dxdR'$. Following Sands, let us assume that $H(x,R';\lambda)$ may be represented by the following product:

$$H(x,R';\lambda)=G(R';\lambda)\exp(-x/L), \tag{1}$$

where L is a constant. Equation (1) is equivalent to the assumption that the meson-producing component is absorbed exponentially throughout the atmosphere with an effective attenuation mean free path L. This assumption is justified experimentally[3] for all atmospheric depths larger than about 250 g cm^{-2}. At higher altitudes it is subject to some uncertainty. However, one should expect a sizeable deviation from the exponential law only at atmospheric depths smaller than 100 g cm^{-2}, a region which contributes relatively little to the μ-meson intensities in the lower parts of the atmosphere (provided that the μ-meson energies are not exceedingly high). In what follows, we shall use for L Tinlot's numerical value of 120 g cm^{-2}.[3]

It is worth noting that the assumption on an exponential attenuation of the N-component does not necessarily imply that the theoretical absorption mean free path $L_a(x)$ remains constant throughout the atmosphere. For instance, the altitude dependence of $L_a(x)$ describable by a formula: $L_a(x)=L_0x/(b+x)$, where L_0 and b are free parameters, still leads to an exponential absorption of the N-component, since $\exp[-x/L_a(x)]=\exp(-b/L_0)\times\exp(-x/L_0)$. Note that, even if the parameter b should vary with the geomagnetic latitude, Eq. (1) still retains its original form.

Following Sands we shall refer to the quantity $G(R';\lambda)$ defined by Eq. (1) as the "production spectrum of μ mesons." However, we shall bear in mind that its physical significance is not so direct as the term might suggest. It is sometimes well to remember that it is the quantity $H(x,R';\lambda)$, rather than the quantity $G(R';\lambda)$, which has an unambiguous physical meaning.

Having accepted Eq. (1) as a sufficient approximation to reality, one can express the differential vertical intensity of μ mesons, observed at the atmospheric depth s, by the following equation:

$$i_v(R,s;\lambda)=\int_0^s G(R+s-x;\lambda)\times\exp(-x/L)w(x,s,R)dx, \tag{2}$$

where $R=R'+x-s$ is the residual range of the μ meson at the depth s, and $w(x,s,R)$ is the survival probability of the μ meson produced at x and arriving at s with the residual range R.[4] It is evident from the form of Eq. (2) that it contains the following simplifying assumptions: (a) the air layer traversed by π mesons before their decay into μ mesons is negligible; (b) the production and propagation of π and μ mesons is restricted to a narrow cone about the vertical (unidimensional model). It is readily seen that the above assumptions place an upper and lower limit on the validity of Eq. (2). However, we shall show in the Appendix that Eq. (2) represents a satisfactory approximation to reality, when one restricts oneself to the interval of residual ranges between $R=100$ g cm^{-2} and $R=6{,}000$ g cm^{-2}. It is fortunate that the preponderant part of the mesonic component is contained within this interval.

(b) Production Spectrum for Large Residual Ranges of μ Mesons

Consider the differential vertical intensity of μ mesons at sea level ($s=x_0$) for relatively large residual ranges, say 2000 g cm$^{-2}<R<6000$ g cm^{-2}. In this case one can solve Eq. (2) analytically with respect to the production spectrum $G(R';\lambda)$, i.e., one can express $G(R';\lambda)$ explicitly in terms of i_v and w. This is possible because, at large R, the argument of $G(R';\lambda)$, $R'=R+x_0-x$, does not change its relative value by a large amount within the interval of integration $0\leqslant x\leqslant x_0$. Consequently, $G(R';\lambda)$ may be considered in this interval as a slowly varying function of x, and one may apply the mean-value theorem of integral calculus to the right-hand side of Eq. (2), *viz.*,

$$i_v(R,x_0;\lambda)=G(R+x_0-x_m;\lambda)\times\int_0^{x_0}\exp(-x/L)w(x,x_0,R)dx. \tag{3}$$

One can estimate the value of the (unknown) quantity x_m ($0<x_m<x_0$) by expanding $G(R+x_0-x;\lambda)$ in Eq. (2) into the Taylor's series at $x=x_m$; neglecting higher derivatives of G, one then finds:

$$x_m=\int_0^{x_0}x\exp(-x/L)w(x,x_0,R)dx\Big/\int_0^{x_0}\exp(-x/L)w(x,x_0,R)dx. \tag{4}$$

[3] See, e.g., J. H. Tinlot, Phys. Rev. **73**, 1476 (1948); **74**, 1197 (1948).

[4] For a detailed discussion on the survival probability the reader is referred to S. Olbert, Phys. Rev. **92**, 454 (1953).

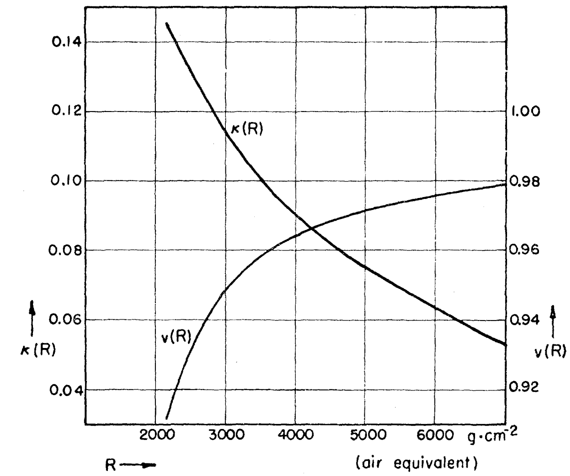

FIG. 1. The empirical functions $v(R)$ and $\kappa(R)$ describing the survival probability, w, of μ mesons produced at x and arriving at sea level with residual ranges $R>2{,}000$ g cm^{-2}, *viz.*: $w=v(R)\times(x/x_0)^{\kappa(R)}$. The functions $v(R)$ and $\kappa(R)$ correspond to the annual mean of the vertical temperature distribution at 40° geographic latitude (see reference 4).

The above estimate of x_m is quite accurate due to the additional fact that the integrand $\exp(-x/L)w(x,x_0,R)$ displays a single, sharp maximum in the vicinity of $x=L$. For the evaluation of the integrals occurring on the right-hand side of Eqs. (3) and (4), it is convenient to make use of the following empirical representation of the survival probability:

$$w(x,x_0,R)=v(R)(x/x_0)^{\kappa(R)}, \tag{5}$$

where the quantities $v(R)$ and $\kappa(R)$ depend only on the residual ranges at sea level and can be computed numerically for any given geographic location (see Fig. 1). Equation (5) reproduces the actual values of the survival probability accurately for all production levels $x<500$ g cm^{-2}. At the production levels closer to the ground [contributing very little to the values of $i_v(R,x_0;\lambda)$], the empirical formula (5) introduces errors amounting only to a few percent. Substituting Eq. (5) into Eqs. (3) and (4), one finds:

$$G(R+x_0-x_m;\lambda)=\frac{i_v(R,x_0;\lambda)}{Lw(L,x_0,R)\Gamma(1+\kappa)} \tag{6}$$

with:

$$x_m=(1+\kappa)L. \tag{7}$$

Here we made use of the fact that $x_0/L\gg1$, i.e., we extended the integrals appearing in Eqs. (3) and (4) up to infinity. Equation (6) represents the required solution of Eq. (2) with respect to the production spectrum of μ mesons for large residual ranges.

In order to study the explicit behavior of G with respect to R' we have chosen the experimental data of Caro, Parry, and Rathgeber[5] on the distribution-in-momentum of μ mesons at sea level (geomagnetic latitude $\lambda\approx50°$S) as the most reliable source of information on $i_v(R,x_0;\lambda)$. The results of the numerical evaluation of the right-hand side expression of Eq. (6) show then that the production spectrum $G(R';50°)$, may be represented by an empirical formula of the type:

$$G(R';50°)=C/(a+R')^{\gamma}, \tag{8}$$

where C, a, and γ are positive constants. Unfortunately, the range interval for which Eq. (6) is valid (see below) is not sufficient to yield an unambiguous set of unique numerical values of these constants. Explicit calculations show that one can reproduce $G(R';50°)$ for 2000 g cm$^{-2}<R'<6000$ g cm^{-2} with *equal* accuracy by assuming for the parameter a any arbitrary value between 200 and 700 g cm^{-2}, provided one adjusts correspondingly the two remaining constants, C and γ. In order to eliminate this ambiguity, one must turn to experimental data which are more sensitive to the assumed values of the parameter a. This will be done in the next section where we shall make use of the observations on the altitude dependence of the differential intensity of μ mesons with small residual ranges.

The degree of accuracy that one may claim for the solution (6) can be estimated from the relative error made in our derivation of $G(R';50°)$. This error is given by:

$$\tfrac{1}{2}(1+\kappa)L^2\left[\frac{1}{G}\frac{\partial^2G}{\partial R'^2}\right]_{R'=R+x_0-x_m}=\tfrac{1}{2}\gamma(1+\gamma)(1+\kappa)\left[\frac{L}{a+R+x_0-x_m}\right]^2.$$

With proper choice of γ and a, one finds that, for $R>2000$ g cm^{-2}, this error is smaller than 1.2 percent. Hence, for large residual ranges, the production spectrum of μ mesons is known almost as well as the intensity at sea level itself.

(c) Production Spectrum for Small Residual Ranges of μ Mesons

One of the experiments which can yield a sufficiently accurate information on the behavior of the production spectrum at small residual ranges is that dealing with the altitude dependence of differential intensities of relatively slow μ mesons. We have chosen the measurements of M. Conversi[6] as most suitable for the purpose of our analysis. Combining the techniques of delayed coincidences and anticoincidences Conversi was able to determine the absolute number of μ mesons (per g-sec-sterad), stopped in 10 cm of graphite after traversing 15.2 cm of lead, as a function of the atmospheric depth, s. The data were taken at 50°N geomagnetic latitude. The vertical allowed cone was relatively narrow and the statistical accuracy of the data exceeded that of earlier

[5] Caro, Parry, and Rathgeber, Australian J. Sci. Research **A4**, 16 (1950).

[6] M. Conversi, Phys. Rev. **79**, 750 (1950).

measurements of this kind[7] considerably. Thus, according to Eq. (2), one may express Conversi's "altitude curve" by the following equation:

$$i_v(R_0,s;\lambda)=\int_0^s G(R_0+s-x;\lambda)\times\exp(-x/L)w(x,s,R_0)dx, \quad (9)$$

where $R_0=100$ g cm^{-2} (air equivalent) and $\lambda=50°$N are now fixed parameters so that the intensity, i_v, is a function of the atmospheric depth s, only.

Equation (9) represents a linear integral equation for the unknown function $G(R_0+s-x;\lambda)$. Due to the complicated character of the kernel function $e^{-x/L}w(x,s,R_0)$ and the limited knowledge of $i_v(R_0,s;\lambda)$, it is practically impossible to apply successfully the ordinary methods of solving an equation of this type. We have attempted to solve Eq. (9) by assuming: (a) that the dependence of G on $R'=R_0+s-x$ may be expressed by a formula of the type of Eq. (8); (b) that there exists a unique set of values for the parameters C, a, and γ, which not only solves Eq. (9) but also reproduces solution (6) at large R'.

Using thus as a guide the solution at large residual ranges, we have chosen several tentative sets for the numerical values of these three parameters and then, for each of these sets, we have computed the right-hand side of Eq. (9).[8] By this procedure we could find a final set of constants, C, a, and γ, which yielded an $i_v(R_0,s;\lambda)$-curve compatible with the experimental data of Conversi, and which was consistent with Eq. (6). This set is the following:

$$\begin{aligned}&C=7.3\times10^4[\mathrm{g}^{-2+\gamma}\,\mathrm{cm}^{2-2\gamma}\,\mathrm{sec}^{-1}\,\mathrm{sterad}^{-1}],\\&a=520\ \mathrm{g\ cm}^{-2}\ \text{(air equivalent)},\\&\gamma=3.58.\end{aligned} \quad (10)$$

Conversi's experimental points covered the atmospheric depths from sea level up to about $s=230$ g cm^{-2}. Unfortunately, the data show a gap at atmospheric depths in the proximity of 700 g cm^{-2}. To complete Conversi's curve in this region we made use of the measurements of Kraushaar,[9] which give the differential vertical intensity of μ mesons as a function of residual ranges at Echo Lake, Colorado (see Fig. 2). The experimental points and the curve computed on the basis of Eqs. (9) and (10) are shown in Fig. 2. One sees that the agreement is practically complete except for one point at the depth of 230 g cm^{-2} where Conversi's point falls above the calculated curve. We believe, however, that this point is to be corrected for the μ mesons stemming from the π mesons locally produced in the lead absorber by the nucleons (and π mesons) which are still abundant at this high altitude.[10] Corrections of this kind may be expected to be of the order of 20 percent, so that we do not consider this single discrepancy as crucial.

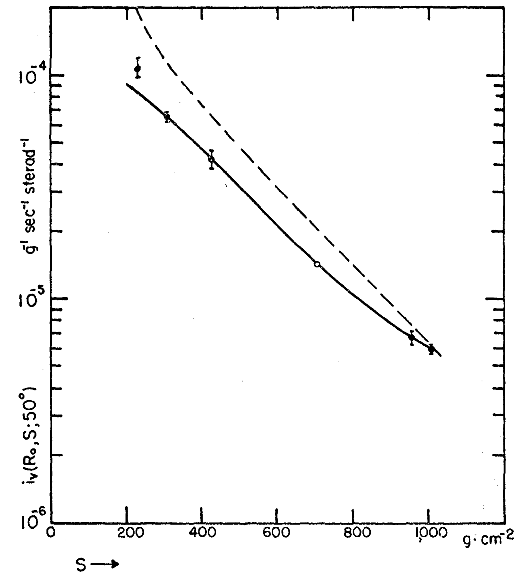

FIG. 2. Differential vertical intensity of μ mesons with residual ranges $R_0=100$ g cm^{-2} (air equivalent) as a function of the atmospheric depth, s, at 50°N geomagnetic latitude. The full circles, with corresponding errors, indicate the measurements of Conversi (see reference 6); the open circle represents one of Kraushaar's measurements (see reference 9) at Echo Lake, Colorado. The dashed line represents Sands' measurements (see reference 1) on μ mesons with residual ranges between 5 and 83 g cm^{-2}. The solid line was computed by means of Eqs. (9) and (11).

To summarize, we conclude that, at the geomagnetic latitude of 50°N, the production spectrum of μ mesons can be expressed by the following formula:

$$G(R';50°)=\frac{7.3\times10^4}{(520+R')^{3.58}}[\mathrm{g}^{-2}\,\mathrm{cm}^2\,\mathrm{sec}^{-1}\,\mathrm{sterad}^{-1}]. \quad (11)$$

According to Eq. (9) and the above requirement (b), Eq. (11) is valid for the residual ranges between $R'=100$ g cm^{-2} and $R'=6000$ g cm^{-2} (air equivalent). In order to check the validity of Eq. (11) in this interval, we have computed the vertical differention intensity of μ mesons at sea level for various residual ranges from $R=100$ g cm^{-2} up to $R=5000$ g cm^{-2}. The results obtained are indicated in Fig. 3 by open circles. The solid line represents the experimental curve deduced from the data of Caro *et al.*;[5] the dashed line represents Rossi's curve.[11] Note a significant discrepancy between these two curves at $R>2000$ g cm^{-2}.

Figure 4 shows a comparison between the original Sands' spectrum (dashed line) and the production spectrum given by Eq. (11)—(solid line). One notices that the two spectra are similar except for the Sands'

[7] Rossi, Sands, and Sard, Phys. Rev. **72**, 120 (1947); see also reference 1.
[8] For detailed curves of the survival probability used in these calculations, the reader is referred to S. Olbert, Technical Report No. 61, Laboratory for Nuclear Science, Massachusetts Institute of Technology, 1954 (unpublished).
[9] W. L. Kraushaar, Phys. Rev. **76**, 1045 (1949).
[10] For remarks on this problem, see also reference 6.
[11] B. Rossi, Revs. Modern Phys. **20**, 537 (1948).

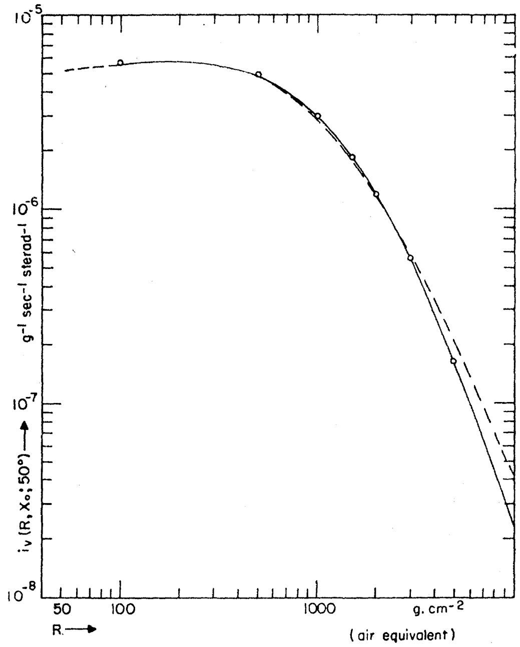

FIG. 3. Differential vertical intensity of μ mesons at sea level, x_0, as a function of the residual range R ($\lambda \approx 50°S$). The solid line corresponds to the measurements of Caro *et al.*;[5] the dashed line represents Rossi's curve.[11] The open circles represent the intensity computed by means of Eqs. (2) and (11).

"dip" at $R'=100$ g cm^{-2}. We cannot make any definite statement about the behavior of our spectrum at this point since our solution is valid only for $R'>100$ g cm^{-2}. At any rate, according to the investigations of Ascoli,[12] the dip obtained by Sands is too pronounced to correspond to a real phenomenon.

III. THE LATITUDE DEPENDENCE OF THE PRODUCTION SPECTRUM OF μ MESONS

Thus far we have considered the production spectrum of μ mesons only in the proximity of 50° geomagnetic latitude. We now turn to the discussion concerning the latitude dependence of $G(R';\lambda)$.

In contrast to the abundance of the experimental material concerning the latitude dependence of the *total* cosmic radiation, the information on the latitude dependence of the differential μ-meson intensity itself is incomplete. This is especially true for the high-altitude measurements. To our knowledge, there exists, at present, only one experiment of this type, namely, that of Conversi.[6] Conversi's data were taken at a constant altitude of 30 000 feet for several latitudes ranging from the geomagnetic equator up to 70°N. The measurements were made with the apparatus and technique discussed in the foregoing section, so that the data can be interpreted as a direct measure of the latitude dependence of $i_v(R_0,s_0;\lambda)$ for $R_0=100$ g cm^{-2} and $s_0=307$ g cm^{-2}. The statistical accuracy achieved in this series of measurements was, unfortunately, smaller than that at 50°N; thus, one cannot give too much weight to the "latitude curve" chosen by Conversi as the best fit (see reference 6).

One is offered primarily two advantages by studying the latitude dependence of differential intensities of slow mesons at *high* altitudes: (1) the effect is large compared with that at low altitudes, (2) the atmospheric latitude effect is practically negligible.

The latter effect stems from the fact that, in general, the survival probability of μ mesons varies with the (geographic) latitude due to the different air-mass distribution at different latitudes. According to meteorological observations, one finds an increase in the annual mean of the atmospheric temperature when one moves southward along any given isobar *below* the tropopause and, inversely, a decrease when one moves southward *above* the tropopause. This implies that the existing temperature gradients can reduce or enhance the μ-meson intensity depending on whether one makes the measurements below or above a certain atmospheric depth. One can show[8] that this atmospheric latitude effect practically disappears for a layer between 250 and 300 millibars.

It follows from the above remarks that the observed latitude variation of $i_v(R_0,s_0;\lambda)$ originates almost exclusively from the geomagnetic effect on the production spectrum of μ mesons. We may, therefore, use Conversi's latitude curve as a direct index of the latitude dependence of this quantity. For the purpose of the derivation of this dependence, let us assume that the production spectrum $G(R';\lambda)$ can be expressed at all latitudes, λ, by an empirical formula of the type given by Eq. (8). If this is so, then, among the three parameters C, γ, and a, only the latter may vary with the geomagnetic latitude. The following argument shows that the

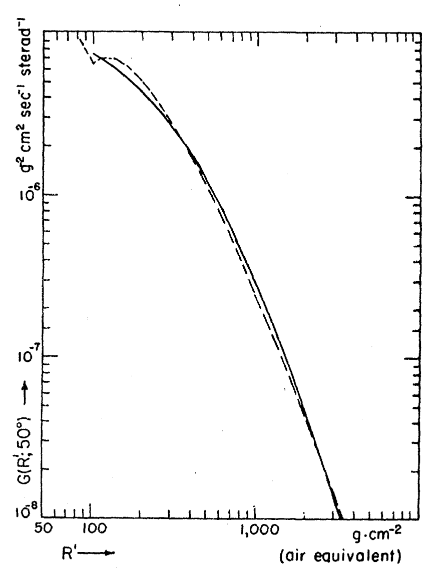

FIG. 4. Comparison of Sands' production spectrum (dashed line) with that computed by means of Eq. (11) (solid line).

[12] G. Ascoli, Phys. Rev. 79, 812 (1950).

parameters C and γ ought to be practically independent of λ:

A μ meson with a residual range larger than, say, 3000 g cm^{-2} has for its parent a π meson of kinetic energy of not less than 8 Bev; a π meson of that energy can be produced by the primary particle only if the kinetic energy of the latter exceeds 12 Bev.[13] Thus, most of the μ mesons with residual ranges above 3000 g cm^{-2} are originated by the primaries which are sufficiently energetic to overcome the magnetic field of the earth at all latitudes. This implies that the production spectrum of μ mesons with $R'>3000$ g cm^{-2} should be practically the same for all geomagnetic latitudes. Since a noticeable latitude variation of the characteristic constants C and γ would violate this conclusion, we see that the latitude dependence of $G(R';\lambda)$ must be expressed through the parameter a, the only parameter with respect to which $G(R';\lambda)$ is insensitive at large residual ranges R'. Using the numerical values for C and γ determined at $\lambda=50°$, we may, thus, write for the production spectrum of μ mesons at any given latitude:

$$G(R';\lambda)=\frac{7.3\times10^4}{[a(\lambda)+R']^{3.58}}[\mathrm{g^{-2}\,cm^2\,sec^{-1}\,sterad^{-1}}]. \quad (12)$$

Inserting Eq. (12) into Eq. (2) with constant $R=R_0=100$ g cm^{-2} and constant $s=s_0=300$ g cm^{-2}, one obtains a unique functional relation between $a(\lambda)$ and $i_v(R_0,s_0;\lambda)$ (the other quantities occurring on the right-hand side of Eq. (2) are, in the present case, practically independent of λ). Figure 5 shows the parameter $a(\lambda)$ as a function of $i_v(R_0,s_0;\lambda)$. On the abscissa of Fig. 5 we have also indicated the geomagnetic latitudes corresponding to the values of $i_v(R_0,s_0;\lambda)$ observed by Conversi.[6] One sees that $a(\lambda)$ decreases monotonically from 646 g cm^{-2} at the geomagnetic equator to 513 g cm^{-2} at $\lambda=60°$N (see also Table II). Of course, among these values of $a(\lambda)$, the value of 520 g cm^{-2} at 50° geomagnetic latitude is to be considered as most reliable. This value has been derived in the preceding section where we had at our disposal a more accurate and rather complete set of experimental data.

IV. PRODUCTION SPECTRUM AND THE AVERAGE MULTIPLICITY OF CHARGED π MESONS

The above empirical production spectrum of μ mesons may be used as a basis for the derivation of the differential energy spectrum $S(E_\pi;\lambda)$ of charged π mesons at production. The functional relationship between $S(E_\pi;\lambda)$ and $G(R';\lambda)$ is well defined, if one assumes that *all* μ mesons stem from the decay of charged π mesons. In view of the recent discovery of K mesons decaying into μ mesons,[14] this assumption may now be considered only as an approximation to reality. However, the relative abundance of K mesons is probably too small to affect significantly our results.

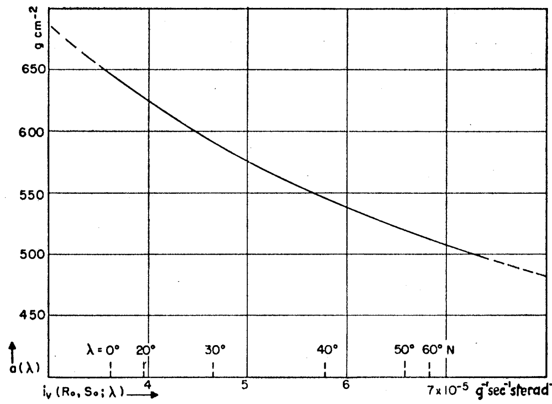

FIG. 5. Latitude dependence of the parameter $a(\lambda)$ defined by Eq. (12). The parameter $a(\lambda)$ is plotted *versus* the intensity of μ mesons with the residual range $R_0=100$ g cm^{-2} at the atmospheric depth $s_0=300$ g cm^{-2}. The geomagnetic latitudes indicated on the abscissa correspond to the μ-meson intensities given by Conversi's latitude curve.[6]

The numerical evaluation of $S(E_\pi;\lambda)$ is simplified further by the fact that, in the energy interval of interest, the energy spectrum of π mesons at decay is practically identical to that at production (see Appendix). Following Ascoli,[12] we find that $G(R';\lambda)$ is related to $S(E_\pi;\lambda)$ by the equation:

$$G(R';\lambda)dR'=\frac{m_\pi^2}{m_\pi^2-m_\mu^2}dE_\mu \times\int_{E_{\min}}^{E_{\max}}\frac{S(E_\pi;\lambda)}{(E_\pi^2+2m_\pi c^2E_\pi)^{\frac{1}{2}}}dE_\pi, \quad (13)$$

where m_μ, E_μ and m_π, E_π are the rest mass and the kinetic energy of μ and charged π mesons respectively, and $E_{\min}$ and $E_{\max}$ are the lower and upper limits of the energies of π mesons that can give rise to a μ meson of energy E_μ.

The solution of Eq. (13) with respect to $S(E_\pi;\lambda)$ can be obtained by means of the method developed by Ascoli.[12] Table I shows the results for various energies, E_π, and various geomagnetic latitudes, λ. Again, among these energy spectra, $S(E_\pi;\lambda)$ at $\lambda=50°$N is to be looked upon as most accurate. It is interesting to note that this energy spectrum behaves in a manner similar to that observed at high altitudes in photographic emulsions.[15] The numerical values of $S(E_\pi;\lambda)$ at lower energies, occurring in parenthesis of Table I, were obtained by an arbitrary extrapolation of Eq. (12) beyond $R'=100$ g cm^{-2}.

With the aid of Table I we can estimate the number (per cm^2-sec-sterad) of charged π mesons produced

[13] One obtains the value of 12 Bev as a lower limit for the primary energy by assuming that the inelastic nucleon-nucleon collision gives rise to only one π meson of 8 Bev produced in foreward cone.

[14] M. G. K. Menon and C. O'Ceallaigh, Proc. Roy. Soc. (London) **221**, 292 (1954).

[15] Camerini, Fowler, Lock, and Muirhead, Phil. Mag. **41**, 413 (1950).

TABLE I. The differential energy spectra, $S(E_\pi;\lambda)$, of charged π mesons at decay (or approximately at production) for various kinetic energies, E_π, and for various geomagnetic latitudes, λ. The parenthesized values of $S(E_\pi;\lambda)$ were obtained by an arbitrary extrapolation of Eq. (12) beyond $R'=100$ g cm^{-2}. In order to obtain $S(E_\pi;\lambda)$ in Mev^{-1} g^{-1} sec^{-1} sterad^{-1}, multiply the tabulated values by 10^{-6}.

E_π Bev \ λ	0°	20°	30°	40°	50°	60°
0.030					(2.61)	
0.065					(4.16)	
0.108	(2.09)	(2.32)	(2.84)	(3.73)	(4.41)	(4.60)
0.186					4.16	
0.266	1.76	1.93	2.32	2.96	3.44	3.58
0.409					2.43	
0.547	1.01	1.09	1.28	1.56	1.78	1.84
0.840					1.07	
1.06	0.44	0.47	0.53	0.62	0.69	0.71
1.51					0.36	
1.96	0.14	0.15	0.17	0.19	0.20	0.20
3.53	0.039	0.040	0.043	0.046	0.049	0.049
6.32	0.009	0.009	0.009	0.009	0.010	0.010
11.3					0.002	

throughout the atmosphere and moving *downward* with kinetic energies greater than E_π. Denoting this number by $\Pi(E_\pi;\lambda)$, we have, according to the definition of $S(E_\pi;\lambda)$ and Eq. (1),

$$\Pi(E_\pi;\lambda)=L\int_{E_\pi}^{\infty}S(E_\pi{}';\lambda)dE_\pi{}'. \tag{14}$$

Since we do not know the behavior of $S(E_\pi;\lambda)$ below a certain energy, $E_\pi<E_0$, we cannot determine the total number of π mesons of all energies, i.e., the quantity $\Pi(0;\lambda)$. In what follows, we shall limit ourselves to $\Pi(E_0;\lambda)$ with $E_0\approx 260$ Mev. For this fixed value of the kinetic energy of π mesons, the quantity $\Pi(E_0;\lambda)$ is a function of the geomagnetic latitude only. Qualitatively, it is evident that the latitude dependence of Π is a direct consequence of the geomagnetic cut-off imposed upon the flux of the primary radiation. Assuming that the primary differential energy spectrum displays a sharp cut-off at E_p (E_p being well-defined by λ) we can consider $\Pi(E_0;\lambda)$ also as a function of the cut-off energy, E_p (see Table II).

TABLE II. Geomagnetic data pertinent to the primary radiation and the μ- and π-meson components. The symbols below have the following meaning: λ is the geomagnetic latitude; E_p is the kinetic cut-off energy of vertical primary protons [from M. S. Vallarta, Phys. Rev. 74, 1837 (1948)]; J is the integral intensity of the primary radiation;[16] the parameter a is defined by Eq. (12); $\Pi(E_0;\lambda)$ represents the number of charged π mesons with energies greater than $E_0=260$ Mev produced throughout the atmosphere in the forward cone.

λ	E_p (Bev)	J cm^{-2} sec^{-1} sterad^{-1}	a g cm^{-2}	Π cm^{-2} sec^{-1} sterad^{-1}
0°	14.2	~0.026	646	0.135
10°	13.7	~0.027	642	0.137
20°	12.0	~0.031	627	0.145
30°	8.4	~0.046	591	0.165
40°	4.2	~0.080	546	0.197
50°	1.6	~0.18	520	0.217
60°	0.43	~0.29	513	0.225

The functional relationship between $\Pi(E_0;\lambda)$ and E_p enables one to draw some crude conclusions with respect to the effective multiplicity of charged π mesons produced by primary particles with energies between 2 and 13 Bev. For this purpose consider the following quantity

$$\bar{n}(E_0;E_p)=-\frac{d\Pi}{dE_p}\bigg/\frac{dJ}{dE_p}, \tag{15}$$

where $J(E_p)$ is the integral energy spectrum of the N-component of cosmic rays averaged over the whole atmospheric depty. According to the definition of $\Pi(E_0;\lambda)$, $\bar{n}(E_0,E_p)$ represents approximately the average number of charged π mesons with energies greater than E_0, produced in the foreward cone by a particle with energy E_p.

One can get a rough idea about the dependence of $\bar{n}$ on E_p by substituting the energy spectrum of the *primary*

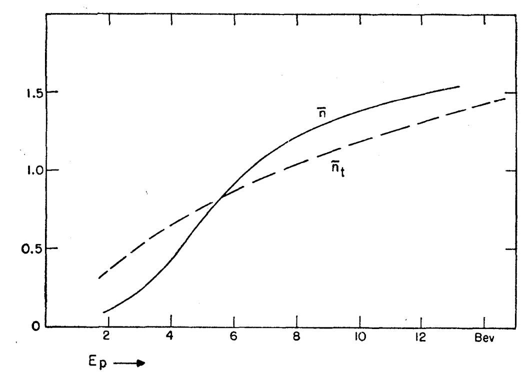

FIG. 6. Multiplicities of charged π mesons produced in the foreward cone by the primary particle with the kinetic energy E_p. $\bar{n}$ is the empirical multiplicity estimated by means of Eq. (15); $\bar{n}_t$ is the theoretical multiplicity predicted by Fermi's statistical model of nucleon-nucleon collisions.

cosmic rays[16] for the quantity $J(E_p)$ in Eq. (15). This procedure, although very crude, should yield the values of $\bar{n}$ which are correct within a factor of two. It is probably not worthwhile to use a more elaborate expression for $J(E_p)$ because the present data on the absolute values of primary intensities are rather uncertain and may be in error by about the same factor (*albido* effect, etc.). The results of our estimate are indicated by the solid line in Fig. 6. One sees that $\bar{n}$ increases with increasing E_p much more rapidly at lower than at higher energies. The flattening-off begins at about 6 Bev.

It is interesting to compare the above results with those predicted by Fermi's statistical theory of meson production in nucleon-nucleon collisions.[17] According to this theory, the probability of observing n π mesons

[16] We made use of the data compiled by G. Puppi and N. Dallaporta in *Progress in Cosmic Ray Physics* (North-Holland Publishing Company, Amsterdam, 1952), p. 320.
[17] E. Fermi, Progr. Theoret. Phys. (Japan) 5, 570 (1950).

(both charged and neutral) produced in a nucleon-nucleon collision is approximately given by:

$$f_n(w)=\frac{A}{(3n+\frac{1}{2})!}\left[\frac{250(w-2)^3}{w}\right]^n, \tag{16}$$

where $wMc^2=(4M^2c^4+2Mc^2E_p)^{\frac{1}{2}}$ is the total energy carried by both nucleons before the collision in the center-of-mass system, and A is normalization constant, given by:

$$A^{-1}=\sum_{n=0}^{\infty}\frac{1}{(3n+\frac{1}{2})!}\left[\frac{250(w-2)^3}{w}\right]^n. \tag{17}$$

By means of Eqs. (16) and (17), one then can compute the expected mean number of charged π mesons produced in the foreward cone in a nucleon-nucleon collision, *viz.*,

$$\bar{n}_t\simeq\tfrac{1}{3}\sum_{n=1}^{\infty} nf_n(w). \tag{18}$$

The factor $\frac{1}{3}$ in Eq. (18) arises from the fact that, on the average, among the produced mesons only $\frac{2}{3}$ are charged, and among them only $\frac{1}{2}$ are emitted in the foreward cone The energy dependence of $\bar{n}_t$ is indicated in Fig. 6 by a dashed line. Note that the discrepancy between $\bar{n}$ and $\bar{n}_t$ is more pronounced at lower than at higher energies. The theoretical multiplicity, $\bar{n}_t$, decreases less rapidly with decreasing E_p than the empirical multiplicity, $\bar{n}$.

In view of the crudeness with which both multiplicities, $\bar{n}$ and $\bar{n}_t$, are derived, it is difficult to draw any definite conclusion with regard to the found discrepancy at lower energies. This difficulty is enhanced by the lack of a theory which would enable us to properly correct $\bar{n}_t$ for the plural processes inside the air nucleus. The simple models of the air nucleus considered recently by Haber-Schaim[18] and Amaldi *et al.*[19] are not applicable to the energies of several Bev.

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APPENDIX

The assumption (*a*) in Sec. II(a) represents a good approximation if the momenta of π mesons are not exceedingly high. The mean free path, L_d, of the π meson before decay is given by:

$$L_d=\tau_\pi\frac{p_\pi}{m_\pi}\rho(x)\simeq 10^{-3}\frac{p_\pi}{m_\pi c}x \;[\text{g cm}^{-2}], \tag{A-1}$$

where τ_π, p_π, m_π are the mean life, the momentum, and the mass of the π meson, respectively, and $\rho(x)$ is the air density in the (isothermal) atmosphere. L_d is only of the order of a few grams per cm^2 for the π mesons produced above the 100-millibar level if $p_\pi<14$ Bev/c. For momenta larger than this value, L_d becomes comparable with the mean free path for nuclear collisions, and the assumption that almost *all* π mesons decay into μ mesons before interacting with air nuclei is no longer valid. By taking 14 Bev/c as the largest possible value of p_π still justifying the assumption (*a*), we thus restrict Eq. (2) to μ mesons with residual ranges at production smaller than about 6000 g cm^{-2}.

The assumption (*b*) in Sec. II(a) places a lower limit on Eq. (2) The unidimensional treatment is justifiable only if the production and the decay of π mesons as well as the propagation of μ mesons are collimated along the vertical.

The experimental observations in photographic emulsions at high altitudes[20] indicate that most of the π mesons with momenta larger than about 300 Mev/c (minimum ionizing secondary tracks) are omitted in the laboratory system at angles smaller than 15° with respect to the direction of the primary particle.

Secondly, the maximum angle of emission of a μ meson in the decay process of a π meson is given by:

$$\sin\vartheta_{\max}=\tfrac{1}{2}\left(\frac{m_\pi}{m_\mu}-\frac{m_\mu}{m_\pi}\right)\frac{m_\pi c}{p_\pi}, \tag{A-2}$$

i.e., $\vartheta_{\max}<8°$ if $p_\pi>300$ Mev/c.

Finally, the mean-square angle of multiple scattering of μ mesons in the atmosphere is given by:

$$\langle\theta^2\rangle_{\text{Av}}=\frac{E_s^2}{X_0}\int_{\pi_\mu(x_2)}^{\pi_\mu(x_1)}\frac{d\pi_\mu}{\pi_\mu^2[-d\pi_\mu/dx]}, \tag{A-3}$$

where $E_s\approx 21$ Mev, $X_0=38$ g cm^{-2} (the radiation length of air), π_μ is the product of the velocity times the momentum of the μ meson, and x_1, x_2 are the levels of production and observation, respectively. The ionization loss of π_μ, $-d\pi_\mu/dx$, is practically constant for all π_μ's larger than 200 Mev (its numerical value for air is 2.0 Mev g^{-1} cm^2) so that one has

$$\langle\theta^2\rangle_{\text{Av}}=\frac{E_s^2}{[-d\pi_\mu/dx]X_0}\left[\frac{1}{\pi_\mu(x_2)}-\frac{1}{\pi_\mu(x_1)}\right]. \tag{A-4}$$

Equation (A-4) implies, for example, that a μ meson produced at the 100-millibar level and arriving at sea level with a momentum of about 300 Mev/c (corresponding residual range about 100 g cm^{-2}) will have a rms angle of scattering of about 7°.

Combining the above three angular spreads, we see that, if we exclude μ mesons with residual ranges smaller than 100 g cm^{-2}, we may expect the local μ-meson intensity, produced by the vertical N-component, to be contained within a cone of about 30° zenith angle. The unidimensional treatment of the μ-meson component can be considered valid within these limits.

[18] U. Haber-Schaim, Phys. Rev. **84**, 1199 (1951).
[19] Amaldi, Mezzetti, and Stoppini, Nuovo cimento **10**, 803 (1953).
[20] Brown, Camerini, Fowler, Heitler, King, and Powell, Phil. Mag. **40**, 862 (1949).