Thus we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{near } 0^\circ} \simeq \frac{e^4}{M^2}+\frac{2q^2e^4}{3M^2}\frac{1}{m_\pi{}^2}\left\{-1+\frac{3m_\pi{}^2\beta_A{}^4}{8M^2}\right\}+\cdots, \tag{3.26}$$

where β_A is the anaomalous magnetic moment in nuclear Bohr magnetons. Since $3m_\pi{}^2\beta_A{}^4/8M^2\simeq 0.08$, we see that near the forward direction the scattering by the magnetic moment is greatly overcompensated by the interference of Thomson and Rayleigh scattering. We cannot say with certainty, of course, that this situation persists at most angles, but it seems likely.

So far we have treated the scatterer as a "particle," but it seems reasonable that Eq. (3.2) should be valid for any system of spin $\frac{1}{2}$, whether elementary or complex, and that the expansion in frequency should be possible when the photon energy is small compared to the energy of the first excited state of the system. Thus complex nuclei of spin $\frac{1}{2}$ may be treated, as well as the proton.

The generalization of the result to other spins is an interesting problem. For spin zero, the term in the scattering amplitude linear in frequency must *vanish* in order that the symmetry condition (3.5) be satisfied. For spins higher than $\frac{1}{2}$, we conjecture that it is still possible to express the linear term in terms of static properties of the system.

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Calculation of Potentials from the New Tamm-Dancoff Equation

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This note is about the construction of energy-independent potentials from the Tamm-Dancoff equations, by a systematic method of Klein's. It is shown that, when the method is applied to the new Tamm-Dancoff equation of Dyson, the difficulty of the spurious singularities can be overcome. It is proved further that the old and new Tamm-Dancoff methods give the same potential in the adiabatic limit.

INTRODUCTION

IN this note, we are concerned with the use of the Tamm-Dancoff methods[1] to derive, for the two-nucleon system, a single integral equation like that used, for instance, by Lévy.[2] The old Tamm-Dancoff method (O.T.D.), which has been used in the past, suffers from infinite vacuum-to-vacuum terms. These do not occur in the new Tamm-Dancoff method (N.T.D.) of Dyson.[3] N.T.D. has also the advantage of being generally closer than O.T.D. to the covariant, renormalizable theories.

The kernel of the N.T.D. integral equation contains "spurious" poles,[4] which are not at the threshold for any real physical process. We thus have the problem of imposing boundary conditions on the amplitude which satisfies the integral equation, so that it should have no singularities arising from the spurious poles. This problem, which amounts to the definition of the vacuum state in the Dyson amplitudes, has not yet been solved.

The purpose of this note is to examine what form the problem of the spurious singularities takes when the kernel of the N.T.D. integral equation is considered as a generalized potential. The kernel is dependent upon the total energy, and so does not give a straightforward eigenvalue equation for bound states. We adopt a systematic method of eliminating the energy dependence, which has been proposed by Klein.[5] Klein's actual suggestion was for the adiabatic limit; but in Sec. 1 we give a natural generalization of his method, which, while it would probably be useful only near the adiabatic limit, enables one to obtain some general properties of the energy-independent potential.

In Sec. 1, we also give a method of generating energy-independent potentials, using the four-dimensional formalism of N.T.D. of Mathews and Salam.[6] This method, which agrees with Klein's except for a few special terms, is the basis of our general results.

In Sec. 2, we show that, for the energy-independent potential, the problem of the spurious singularities reduces to the simpler question of how to define the poles in a certain integral. It is no longer a boundary condition problem. The essential step in this demonstration is a generalization of a result already given by Klein[7] for the fourth order. This states that, if the total energy is replaced by the kinetic energy, the kernel is free of spurious poles on a suitable rearrangement of terms.

[1] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[2] M. M. Lévy, Phys. Rev. **88**, 72 (1952).
[3] F. J. Dyson, Phys. Rev. **90**, 994 (1953).
[4] F. J. Dyson, Phys. Rev. **91**, 1543 (1953).
[5] A. Klein, Phys. Rev. **94**, 195 (1954).
[6] Quoted by the present author in reference 9.
[7] A. Klein, Phys. Rev. **95**, 1653 (1954).

In Sec. 3, we prove that, in the adiabatic limit, O.T.D. and N.T.D. are identical. This result has been found by Kurşunoğlu[8] for the fourth order.

1. THE POTENTIAL

In three-dimensional momentum-space, the two-nucleon integral equation derived from O.T.D. or N.T.D. has the general form

$$\{W-H(p)-H(q)\}\chi(p,q) = \sum_n \int V_n(W;\, p,q,k)\chi(p+k,\, q-k)dk, \quad (1)$$

where W is the total energy, $H(p)=\alpha\cdot p+\beta M$, M is the nucleon mass, and V_n is of the nth order in the coupling constant. In O.T.D., or in N.T.D., if the negative energy components have been eliminated, $H(p)$ should be replaced by $E(p)=(p^2+M^2)^{\frac{1}{2}}$, etc. But we can leave this open by supposing χ to contain positive-energy projection operators if necessary.

We write (1) shortly as

$$(W-T)\chi=\sum V_n(W)\chi. \quad (2)$$

Then our generalization of Klein's[5] equation with the energy-independent potential U is given by the successive approximations:

$$\begin{aligned}(W-T)\chi&=V_1(T)\chi=U_1\chi,\\ (W-T)\chi&=\{V_1(T)+V_2(T)+V_1'(T)(W-T)\}\chi \quad (3)\\ &\simeq\{V_1(T)+V_2(T)+V_1'(T)V_1(T)\}\chi\\ &=(U_1+U_2)\chi, \quad \text{etc.}\end{aligned}$$

In the right hand side of Eq. (3), the operator T takes the value $T(k)=H(p+k)+H(p-k)$. (In the adiabatic limit actually considered by Klein, $T=2M$ everywhere.)

V contains poles for values of the energy and momenta at the thresholds for the creation of new particles. Klein's method is only applicable for values of W well away from these poles, i.e., for bound states and low energy scattering. In N.T.D., spurious poles occur in V for all values of W.

We now translate the above procedure for generating U into the four-dimensional form of N.T.D. of Matthews and Salam.[9] The starting point is Eq. **A**(7). When the approximation which gives an integral equation is made, **A**(7) becomes, with the use of **A**(15),

$$\exp(-iWt)\chi(p,q)=(i)^n\int_{-\infty}^{t}dt_1\cdots\int_{-\infty}^{t_{n-1}}dt_n\int dk \times F(t;\, t_1,\cdots t_n;\, p,q,k) \times\exp\{-it_n[W-T(k)]\}\chi(p+k,\, q-k). \quad (4)$$

In Eq. (4) we have transformed to momentum-space in the spatial components, but preserved the form of the time integrations. F, which is independent of W, is an integral over the internal momenta of a sum of exponentials, each of which, when $t_1=t_2=\cdots=t_n$, is proportional to

$$\exp[-it_1\{T(k)-T(0)\}-itT(0)]. \quad (5)$$

Integrating by parts in (4) successively over the t_n, t_{n-1}, $\cdots t_2$-integrations,[10] we get

$$\begin{aligned}&\exp(-iWt)\chi(p,q)\\ &=(i)^n\int_{-\infty}^{t}dt_1\exp[-it_1\{W-T(k)\}]\\ &\quad\times\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n\int dk\\ &\quad\times F(t;\, t_1,\cdots t_n;\, p,q,k)\chi(p+k,\, q-k)-R, \quad (6)\end{aligned}$$

where the remainder term R has $(n+1)$ time integrations, and contains the factor $\{W-T(k)\}\chi(p+k,\, q+k)$. It follows from (5) that, when the time integrations are done in (6), the resulting equation has the form

$$(W-T)\chi=V_n(T)\chi+V_n^{\dagger}(W)(W-T)\chi, \quad (7)$$

where $V_n^{\dagger}$ comes from R. Repeating the integration by parts in R, $R_n^{\dagger}(W)$ gives $V_n'(T)$ together with higher order terms. Thus the general equivalence of the two methods is clear on comparing (7) with (3).

However, there are certain exceptional terms for which the four-dimensional formalism gives different results from Klein's procedure. These are terms arising from the elimination of an amplitude which contains the destruction operators for the initial nucleons in their initial state, and also both the creation and destruction operators for the same particle with the same momentum. A typical energy denominator corresponding to such an intermediate state is

$$\{W-T(k)-E(p+k+k')+E(p+k+k')\}^{-1}. \quad (8)$$

For the one-nucleon equation, an example of such a term is Eq. (14) of reference 7.

We cannot immediately put $W=T$ in expression (8), but terms of this type always appear in pairs whose sum is nonsingular at $W=T$. Graphically, the relation of such a pair is shown by Figs. 1(c) and (d) of reference 7. In Klein's procedure, then, these pairs must be combined before we put $W=T$.

In the four-dimensional formalism, such terms show themselves up after the integration by parts as the time integral of a constant. This would be infinite if it were not for the factor

$$\exp\{\epsilon(t_1+t_2+\cdots t_n)\},$$

[8] B. Kurşunoğlu, Phys. Rev. (to be published).

[9] J. C. Taylor, Phys. Rev. **95**, 1313 (1954). Hereafter referred to as **A**.

[10] This device is due to M. Cini and S. Fubini, Nuovo cimento **10**, 1695 (1953).

which must always be inserted in (4) to insure convergence (ϵ is a small positive number). Again, such terms appear in pairs which, after combination and integration, are finite as $\epsilon \to 0$.

For these terms, Klein's procedure gives just twice the result of the four-dimensional formalism. This discrepancy arises because the limits $\epsilon \to 0$ and $W \to T$ are taken in different orders in the two cases. We shall adopt the four-dimensional method, because it enables us to deduce general results. However, the whole matter is somewhat academic, because the terms concerned are all infinite nucleon self-energy terms with which it is not known how to deal at present.

Finally we remark that in the Matthews-Salam form of the Tamm-Dancoff equations all the terms in the kernel are of the same order in the coupling constant; so that some are reducible and may be regarded as iterations of a lower-order kernel. In the usual, and more useful, form of the equation these do appear as lower-order kernels. It is always possible formally to pass from one equation to the other by iteration or the reverse process.

2. SPURIOUS POLES

We begin this section by proving that $V(T)$ is free from spurious poles. From the relation established in the last section between (3) and (6), we have, using (5),

$$V_n(T)=\{T(k)-T(0)\}\exp\{itT(k)\}(i)^n \times\int_{-\infty}^{t} dt_1\cdots\int_{-\infty}^{t_{n-1}} dt_n F(t;\, t_1\cdots t_n;\, p,q,k). \quad (9)$$

The integral on the right hand side of (9) is the same as the integral in (4), except that the exponential in the latter is replaced by unity. From the derivation of (4) from **A**(7), one sees that the dependence of the intergal in **A**(7) upon the state vectors $|t_n\rangle$ and ${}_0\langle t_n|$ of **A** has dropped out in (9). It follows that $V_n(T)$ is proportional to the momentum-space form of the right hand side of **A**(21).

Now consider Eq. **A**(22). The four-dimensional momentum-space form of this contains only Feynman functions S_F and Δ_F. The integrations over the fourth-components of the internal energy-momenta may be performed, so as to obtain a three-dimensional form. Because of the definition of the Feynman functions, the only combinations of internal energies that can result are sums, for differences would have ambiguous poles. Thus, in the three-dimensional momentum-space form of **A**(22) there are no spurious poles.

A(21) is obtained from **A**(22) by a Dyson[11] Heisenberg-operator analytical continuation process. This changes only the way the integration is done round the poles and leaves their position unchanged. Therefore, **A**(21), and so also $V(T)$, are free of spurious poles.

This is the result obtained by Klein[7] in the fourth order. He finds additional terms which happen also, in this case, to be without spurious poles. As explained in Sec. 1, the extra terms are due to a slightly different way of defining $V(T)$.

Thus the spurious poles are connected with the W dependence of V. We will now show that Klein's method for expressing the W dependence allows us to apply the boundary conditions immediately, and obtain a potential free from spurious singularities.

To see this, note that all terms in the potential, U, defined by Eqs. (3), are of the form

$$U(p,q;\, k)=\int dk' Y(p,q;\, k')V(p,q;\, k,k'), \quad (10)$$

where V is free of spurious poles. Thus the k' integration may be done, giving a potential U free of spurious poles. It is still necessary to define the path of the integration round the spurious poles in Y. Following Dyson,[4] we suggest that the vacuum state is correctly defined in this case if the spurious poles are taken as principal values.

3. ADIABATIC LIMIT

The results of the previous sections hold, in particular, for the adiabatic limit. We now prove more strongly that N.T.D. yields the same potential, $V(W)$, as O.T.D., in the adiabatic limit; so that $V(W)$ itself is free from spurious poles.

Suppose that we know that the wave function is small except for values of the momenta of the order of magnitude of μ, the meson mass. Then, by the adiabatic limit, we mean here that we work to the zeroth order in μ/M.

V, defined by Eq. (2), contains energy denominators of two kinds, typified by

$$\{W-E(p)-E(q-k)-\omega(k)\}^{-1} \quad (11)$$

and

$$\{W-E(p)-E(q)-E(q-k)-E(q-k-k')-\omega(k')\}^{-1}, \quad (12)$$

where $\omega(k)=(k^2+\mu^2)^{\frac{1}{2}}$. In the adiabatic limit, these become, respectively,

$$\{\mathcal{E}-\omega(k)\}^{-1} \quad (13)$$

and

$$(-2M)^{-1}, \quad (14)$$

where $\mathcal{E}=W-2M$ is the binding energy. Whereas the $\mathcal{E}$ dependence of (13) must be retained, the neglect of $\mathcal{E}$ in (14) means the neglect of a term of relative order μ/M, and is consistent with the adiabatic approximation.

Our proof is based upon the relation between the Tamm-Dancoff equations and a three-dimensional approximation (L.K.), given by Lévy[2] and Klein,[12] to the

[11] F. J. Dyson, Phys. Rev. **92**, 428 (1951).

[12] A. Klein, Phys. Rev. **90**, 1101 (1953).

Bethe-Salpeter equation. This relation has been investigated in **A**. There, use was made of the four-dimensional formalism, which, as explained in Sec. 1, gives in a few cases results different from the three-dimensional formalism. This discrepancy is unimportant here, since the relevant terms are all neglected in the adiabatic limit.

Consider first O.T.D. It differs from L.K. by changes in the energy denominators described by a rule of Klein's.[12] According to this rule, denominators of the type of (11) are unchanged, whereas (12) becomes

$$\{-E(q-k)-E(q-k-k')-\omega(k')\}^{-1}. \quad (15)$$

(15) has the same adiabatic limit, (14), as (12); and it is true generally that O.T.D. and L.K. agree in the adiabatic limit.

For N.T.D. there is a corresponding L.K. equation, given in **A**, to be called N.L.K. An argument similar to the above shows that N.T.D. and N.L.K. agree in the adiabatic limit. Thus the problem is now reduced to the relation between L.K. and N.L.K.

According to Klein,[12] the L.K. kernel is given by

$$\int dp_0[\{\tfrac{1}{2}W+p_0-E(p+k)+i\epsilon\}^{-1} +\{\tfrac{1}{2}W-p_0-E(q-k)+i\epsilon\}^{-1}]K(p,q,k;\,p_0), \quad (16)$$

where K is the corresponding kernel of the Bethe-Salpeter equation, and p_0 is a relative energy. Correspondingly, the N.L.K. kernel is given by (16) with K replaced by K'; where K' is an analytic continuation of K of the sort introduced by Dyson[11] for Heisenberg-operators. We are assuming that the negative energy components in N.T.D. have been eliminated.

In general, K is given as an integral over a number of internal energy-momenta. With the aid of the Feynman prescription for the poles, we may integrate over the internal energy-components. In the resultant expression, p_0 occurs in denominators of two kinds, typified by

$$\{\tfrac{1}{2}W-p_0-E(p+k')-\omega(k')+i\epsilon\}^{-1} \quad (17)$$

and

$$\{\tfrac{1}{2}W-p_0+E(q)+\omega(k)+\omega(k')-i\epsilon\}^{-1}; \quad (18)$$

the point being that (17) contains only positive energies and (18) only negative energies. No term can contain both because of the Feynman prescription. K' is identical with K, except that $-i\epsilon$ is replaced by $i\epsilon$ in terms of the type (18).

When the p_0-integration is done in (16), terms arise from each pair of coincident poles chosen from among the denominators like (17) and (18) and those shown explicitly in (16). The only terms that can come from a pair containing a term of type (18) are of type (12).

The pairs of coincident poles that are affected by the change from K to K' are the pairs containing just one factor of type (18). If for K the other factor is one of the terms shown explicitly in (16), say the second, then for K' it is replaced by the first. However, in the adiabatic limit, the contribution to (16) is in either case the same, $(2M)^{-1}$.

There remains the case in which the pair of poles contains one of type (17) and one of type (18). For any term from such a pair of poles, there is always a term arising from the pole of type (17) and one of the poles shown in (16). The contribution from the former pair is of order μ/M times the contribution from the latter pair, and so is consistently neglected in the adiabatic limit.

This completes the proof that the contribution to (16) is the same when K is replaced by K'. It follows that L.K. and N.L.K., and therefore O.T.D. and N.T.D., agree in the adiabatic limit.

Finally, the remark at the end of Sec. 1 is relevant here. In particular, it is because we are using the "iterated" form of the Tamm-Dancoff equations, that we need consider only the positive energy components of Dyson's amplitude in the N.T.D. equation; in the "iterated" form, terms appear automatically that otherwise would have to be found by eliminating negative-energy matrix elements of the lower order parts of the kernel.

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