Transverse Hall and Magnetoresistance Effects in p-Type Germanium

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Modification of the two-band model to include existence of a small number of high-mobility holes permits calculations of the magnitude, temperature dependence, and magnetic-field dependence of Hall and magnetoresistance effects in high-purity p-type germanium, which are in excellent quantitative agreement with experimental data. For a specimen containing 4.3×10^{13} acceptors/cm^3, the data indicate at 300°K the presence of 1.1×10^{12} positive carriers/cm^3 with a mobility of 15 000 cm^2/volt-sec. At 205°K these high-mobility holes increase the magnetoresistance at weak fields by a factor of 25 and the Hall coefficient by 1.6. Results indicate that Hall mobilities computed by taking the product $R\sigma$ must be interpreted with caution, or unwarranted inferences may be drawn. For the specimens studied, mobilities computed from the Hall data, by using equations based on spherical energy surfaces but taking into account the fast holes and the values of the magnetic fields, agree within experimental error with the values obtained by other methods. Discussion of the fast holes in connection with valence band degeneracy, cyclotron resonance observations, and interband optical transitions is given. The desirability of using data taken at large and at very small magnetic fields in analyzing galvanomagnetic phenomena is emphasized.

INTRODUCTION

In the customary development of Hall coefficient equations for calculations of charge-carrier densities, a weak magnetic field approximation is used, since the integration over the distribution of electron velocities is comparatively simple for this limiting case.[1] In order for this approximation to be valid, the Hall coefficient must be independent of magnetic field. Ordinarily, it is assumed that fields of from one to six kilogauss satisfy these restrictions. However, measurements of Hall coefficients as a function of magnetic field in p-type germanium have shown significantly large variations in this range of field strengths.[2,3] More recently, data at 77°K on p-type high-purity germanium specimens show increases of over 50 percent for the Hall coefficient as the magnetic field is decreased from 1000 to 100 gauss.[4] It is thus apparent that the use of the weak-field expressions for the Hall effect is not justified in this case. An obvious alternative is the employment of the exact relationships valid for arbitrary values of magnetic field.[5] It has been established, however, that such a treatment does not predict the behavior in p-type germanium, although it does explain the Hall and magnetoresistance effects in indium antimonide.[6] The disagreement appears quite fundamental and cannot be accounted for by such considerations as impurity, or other types of scattering mechanisms having a power-law dependency of mean-free path on energy.

MAJOR FACTORS TO BE CONSIDERED

In obtaining an explanation of the behavior of Hall and magnetoresistance effects in p-type germanium, the following previously established considerations are significant, namely:

(1) For ordinary magnetic fields the temperature at which the Hall coefficient becomes zero decreases with increasing magnetic field, and in this temperature region, the effect of an increased magnetic field is to make the Hall coefficient less positive (or more negative).

(2) In the extrinsic region there occur large decreases in Hall coefficient with increasing magnetic field. At lower temperatures the rate becomes more rapid, so that a limiting value is approached for moderate fields.

(3) Measurements at a fixed, weak magnetic field in the extrinsic region give a Hall coefficient which decreases as the temperature is lowered.

As a result of this behavior, the use of weak-field Hall measurements to determine extrinsic carrier concentrations, using the conventional equations, gives results of dubious validity. For example, the decrease of Hall coefficient in the extrinsic region as the temperature is lowered would be interpreted as an increase in carrier concentration. Furthermore, calculations of hole mobilities yield results which are in serious disagreement with those determined by drift techniques.[7] It is significant to note, however, that if extrinsic carrier concentrations are determined using the limiting values of Hall coefficients for strong magnetic fields, neither of these inconsistencies is present.[4]

The magnetic-field dependence of the Hall coefficient in the extrinsic region is indicative of unusually high mobility carriers. On the other hand, drift-mobility measurements and calculations from conductivity data indicate that most of the p-type conductivity results from carriers with room-temperature mobility of approximately 1900 cm^2/volt-sec.[7,8]

It is possible to reconcile these apparently conflicting observations by assuming that there exist a relatively

[1] See, for example, F. Seitz, *The Modern Theory of Solids* (McGraw-Hill Book Company, New York, 1940), p. 192.
[2] W. C. Dunlap, Phys. Rev. **71**, 471 (1947); **79**, 286 (1950); **82**, 329 (1951).
[3] Purdue University Theses; W. M. Scanlon, 1948 (unpublished); J. W. Cleland, 1949 (unpublished).
[4] Harman, Willardson, and Beer, Phys. Rev. **94**, 1065 (1954).
[5] See J. W. Harding, Proc. Roy. Soc. (London) **A140**, 205 (1933).
[6] Harman, Willardson, and Beer, Phys. Rev. **95**, 699 (1954).
[7] M. B. Prince, Phys. Rev. **92**, 681 (1953).
[8] L. P. Hunter, Phys. Rev. **91**, 579 (1953).

small number of holes with a mobility considerably higher than those either of the majority holes or of the electrons. As will be shown subsequently, such a model can explain quite effectively the behavior enumerated above. Such a high-mobility carrier would be expected to possess an effective mass significantly less even than that of the electrons in germanium. That this state of affairs might exist has recently been given support by a number of investigations on germanium. For example, calculations by Herman and Callaway[9] indicate a triple degeneracy in the valence band. Such a structure was also discussed by Briggs and Fletcher[10] in connection with their data on the infrared absorption by free holes. A requirement was stated that holes from one set of these states have a rather small effective mass, perhaps $\frac{1}{10}$ that of the free electron mass. This scheme has also been discussed by Burstein[11] in connection with observations on zinc- and indium-doped germanium.

The most substantial support, however, comes from recent experiments on cyclotron resonance.[12] The data indicate the existence of two discrete effective mass values ($m^*=0.04m$ and $m^*=0.30m$) in p-type germanium.

DEVELOPMENT OF EQUATIONS

The well-known expressions for the current densities j_x and j_y with an electric field E_x and E_y, and magnetic field H_z may, in the absence of temperature gradients, be written in the form:

$$j_x=|e|\sum_k A_kE_x+\sum_k e_kB_kE_y, \tag{1}$$

$$j_y=-\sum_k e_kB_kE_x+|e|\sum_k A_kE_y, \tag{2}$$

where the summations are over each charge carrier having a specific mobility, and where the electronic charge e_k is negative for electrons and positive for holes.

For a mean-free path independent of energy, classical statistics, and spherical energy surfaces, the coefficients A_k and B_k can be expressed in terms of carrier densities n_k and zero magnetic-field mobilities μ_k^0 as follows:[13]

$$A_k=n_k\mu_k^0\int_0^\infty x^2e^{-x}(x+\gamma_k)^{-1}dx\equiv n_k\mu_k^0K_k, \tag{3}$$

$$B_k=n_k\mu_k^0\gamma_k^{\frac{1}{2}}\int_0^\infty x^{3/2}e^{-x}(x+\gamma_k)^{-1}dx\equiv \tfrac{1}{2}\pi^{\frac{1}{2}}n_k\mu_k^0\gamma_k^{\frac{1}{2}}L_k, \tag{4}$$

where

$$\gamma_k\equiv(9\pi/16)(\mu_k^0H_z)^2. \tag{5}$$

The magnetic field parameter γ is dimensionless, and if H is given in gauss, then μ^0 must also be expressed in electromagnetic units.

[9] F. Herman and J. Callaway, Phys. Rev. **89**, 518 (1953).
[10] H. B. Briggs and R. C. Fletcher, Phys. Rev. **91**, 1342 (1953).
[11] Burstein, Henvis, and Sclar, Phys. Rev. **94**, 750 (1954).
[12] Dresselhaus, Kip, and Kittel, Phys. Rev. **92**, 827 (1953).
[13] V. A. Johnson and W. J. Whitesell, Phys. Rev. **89**, 941 (1953).

The dimensionless integrals K and L, which arise from integrations over the charge-carrier energies, may be expressed in terms of standard exponential integrals and error functions, as defined in Jahnke-Emde, as follows:[5]

$$\begin{aligned}K(\gamma)&=1-\gamma-\gamma^2e^\gamma\,\mathrm{Ei}(-\gamma),\\ L(\gamma)&=1-2\gamma+2\pi^{\frac{1}{2}}\gamma^{\frac{3}{2}}e^\gamma[1-\Phi(\gamma^{\frac{1}{2}})].\end{aligned} \tag{6}$$

The usual weak-field approximations to Eqs. (6) neglect terms higher than the first degree in γ. Since magnetoresistance calculations involve differences, adequate accuracy is realized in this approximation only when $\gamma<0.001$. It is therefore of very limited validity in cases where high mobilities are encountered. A somewhat better approximation, involving Euler's constant, is

$$\begin{aligned}K(\gamma)&\cong1-\gamma-\gamma^2(0.577+\log_e\gamma), &\gamma<0.025;\\ L(\gamma)&\cong1-2\gamma+2\pi^{\frac{1}{2}}\gamma^{\frac{3}{2}}, &\gamma<0.01.\end{aligned} \tag{7}$$

For strong magnetic fields, say $\gamma>25$, the following asymptotic expansions can be used:

$$\begin{aligned}K(\gamma)&\approx2!/\gamma-3!/\gamma^2+\cdots,\\ L(\gamma)&\approx1\cdot3/2\gamma-1\cdot3\cdot5/2^2\gamma^2+\cdots,\end{aligned}\quad(\gamma>25). \tag{8}$$

The Hall coefficient and magnetoresistance are defined for $j_y=0$ as follows:

$$R\equiv E_y/j_xH_z,\quad \Delta\rho/\rho_H\equiv1-(\sigma/\sigma_0)\equiv1-(j_x/j_x^0). \tag{9}$$

For the case of p-type germanium, let us denote the electrons by subscript 1, the ordinary holes by 2, and the high-mobility holes by 3. Then, the preceding equations can be shown to yield:

$$R=-\frac{3\pi}{8|e|n_2}\times\frac{abL_1-L_2-cdL_3}{[aK_1+K_2+cK_3]^2+\frac{1}{4}\pi\gamma_2[abL_1-L_2-cdL_3]^2}, \tag{10}$$

$$\Delta\rho/\rho_H=1-\frac{[aK_1+K_2+cK_3]^2+\frac{1}{4}\pi\gamma_2(abL_1-L_2-cdL_3)^2}{[a+1+c][aK_1+K_2+cK_3]}, \tag{11}$$

where

$$a=\sigma_1^0/\sigma_2^0,\quad b=\mu_1^0/\mu_2^0,\quad c=\sigma_3^0/\sigma_2^0,\quad d=\mu_3^0/\mu_2^0. \tag{12}$$

In the numerator of the expression for the Hall coefficient, the contribution of the term involving the high-mobility holes in the case of weak fields varies linearly with concentration, but as the square of the mobility. Thus, in the extrinsic region ($a=0$), we see that a very small percentage of high-mobility carriers can influence strongly the magnitude of the weak-field Hall coefficient. For example, in the extrinsic region for

sufficiently weak fields, Eq. (10) reduces to

$$R_{H\to 0}=\frac{3\pi}{8|e|n_2}\frac{1+(n_3/n_2)(\mu_3^0/\mu_2^0)^2}{[1+(n_3/n_2)(\mu_3^0/\mu_2^0)]^2}. \tag{13}$$

Thus, if the density of the high mobility holes is only 2 percent that of the others, and their mobility is 8 times as large,[14] then they will increase the Hall coefficient by a factor of 1.7. If, however, data are taken for the limit of strong magnetic fields, then it can be seen from Eqs. (8) and (10) that

$$R_{H\to\infty}=1/|e|(n_2+n_3). \tag{14}$$

In this case, the extrinsic Hall coefficient is related only to the total extrinsic carrier concentration and involves no weighting factors containing mobilities.

A region of special interest is where the Hall coefficient is zero. The relationship valid in these regions for the limiting case of very weak magnetic fields is:

$$R_{H\to 0}=0,\quad (n_1/n_2)(\mu_1^0/\mu_2^0)^2=1+(n_3/n_2)(\mu_3^0/\mu_2^0)^2. \tag{15}$$

For the ordinary two-band model ($n_3=0$), the above equation reduces to the familiar expression used to determine the electron-hole mobility ratios from data on p-type specimens. If the fast holes are neglected in a germanium specimen where 2 percent of the holes have a mobility of 8 times as large, the value obtained for μ_1^0/μ_2^0 is 1.3.[15] If, however, use is made of the complete expression in Eq. (15), which takes the fast holes into account, then one obtains a value of μ_1^0/μ_2^0 equal to 2.0, a result in agreement with that obtained from analysis of conductivity data alone.[8] The above values are intended merely to be illustrative, since Eq. (15) applies only in the limit of zero magnetic field. In analyzing actual data, the magnetic-field parameters must be considered. Also, the analysis of the conductivity data should be modified to distinguish the contributions due to the fast holes. This would lower the value attributed to μ_2^0 and, hence, increase μ_1^0/μ_2^0 slightly.

In the limit of very strong fields, one obtains:

$$R_{H\to\infty}=0,\quad n_1=n_2+n_3. \tag{16}$$

Here, the μ^0 factors are absent and, hence, the effect of the small number of fast holes is of much less significance.

Another interesting consequence of the additional high-mobility hole is its effect on the temperature dependence of the magnetoresistance. It has previously been pointed out how the two-carrier model predicts a maximum in the neighborhood of the Hall coefficient maximum, which is apparently absent in p-type germanium, but which is observed in indium antimonide.[6] A study of Eq. (11) shows how this can occur. If the magnetic field is not too large, it is the first term of the numerator which is significant in the region near the Hall coefficient maximum. As the temperature is lowered, from, say intrinsic, $\Delta\rho/\rho_H$ increases due to the increasing mobilities and the resulting decrease in the values of the functions represented by the K's. In the two-carrier example, in which the electrons have a greater mobility than the holes, as the temperature is further lowered, the diminishing contribution of the electrons gradually causes the denominator to offset the previous trend, and $\Delta\rho/\rho_H$ starts to decrease. At still lower temperatures, the electronic contributions are no longer significant in either member of the fraction. Then, the magnetoresistance rises again, to increase monotonically to the saturation value of 0.116. In the three-carrier case, however, the pertinent fact to note is that, unlike the electron density, the high-mobility hole density varies only slowly with temperature. The result is that the inclusion of the high-mobility holes can prevent the intermediate decrease in $\Delta\rho/\rho_H$ with decreasing temperature and preclude the occurrence of a maximum. It is to be noted, however, that the relative importance of the various terms in Eq. (11) changes with increasing magnetic field. Hence, by appropriate choice of carrier-density and mobility ratios, a maximum may be obtained in p-type germanium for certain magnetic-field strengths.

Since the general magnetoresistance equation is complicated, it is informative to examine the two limiting cases, as was done for the Hall coefficient. For weak magnetic fields, it is readily established that Eq. (11) in the extrinsic region reduces to

$$\lim_{H\to 0}\left(\frac{\Delta\rho}{\rho_H}\right)=\frac{9\pi}{16}(\mu_2^0)^2H^2\left[\frac{1+(n_3/n_2)(\mu_3^0/\mu_2^0)^3}{1+(n_3/n_2)(\mu_3^0/\mu_2^0)}-\frac{\pi}{4}\left(\frac{1+(n_3/n_2)(\mu_3^0/\mu_2^0)^2}{1+(n_3/n_2)(\mu_3^0/\mu_2^0)}\right)^2\right]. \tag{17}$$

The above relationship is a good approximation if $\gamma<0.001$. In such case the effect of the presence of high-mobility holes is quite striking. For example, if only ordinary holes of mobility μ_2^0 are present ($n_3=0$),[16] the factor in brackets has the value 0.21. If, however, 2 percent of the holes have mobilities 8 times that of the others, then the value of the factor increases to 6.7, giving an increase in magnetoresistance of over 30 times.

For the strong field limit in the extrinsic region, Eq. (11) becomes:

$$\lim_{H\to\infty}\left(\frac{\Delta\rho}{\rho_H}\right)=1-\left(\frac{9\pi}{32}\right)\times\frac{(1+n_3/n_2)^2}{[1+(n_3/n_2)(\mu_3^0/\mu_2^0)][1+(n_3/n_2)(\mu_2^0/\mu_3^0)]}. \tag{18}$$

[14] The values $n_3/n_2=0.02$ and $\mu_3^0/\mu_2^0=8$ follow from an analysis of the experimental data on p-type germanium (see following section).

[15] See, for example, Hunter, Huibregtse, and Anderson, Phys. Rev. **91**, 1315 (1953).

[16] This gives the familiar magnetoresistance equation. See, for example, Eq. (7) in G. L. Pearson and H. Suhl, Phys. Rev. **83**, 768 (1951).

A good approximation to this limit is reached when $\gamma > 100$. If one carrier only is present ($n_3 = 0$), the expression reduces to the asymptotic limit of 0.116 given by Harding[5] and others. For the special case where $n_3/n_2 = 0.02$ and $\mu_3^0/\mu_2^0 = 8$, the limit becomes 0.21, an increase of almost a factor of two.

EXPERIMENTAL PROCEDURE

The fundamental experimental data obtained to test the theoretical predictions were resistivity and Hall voltage as functions of magnetic field at several fixed temperatures, and the temperature at which the Hall voltage was zero as a function of magnetic field. Resistances were determined by the potential probe method, Hall effects from the transverse voltages in a magnetic field, and temperatures from the thermoelectric voltages of copper-constantan thermocouples attached at both ends of the specimen.

A Leeds and Northrup Type K-2 potentiometer was used to measure all of the above dc voltages. A rotating coil driven by a synchronous motor was used for measuring the magnetic field. The indicating instrument for the ac voltage was a Ballantine Model 310A Electronic Voltmeter, and the system was calibrated by means of a large, standard permanent magnet which is used for nuclear resonance studies at The Ohio State University. The specimen and its holder were enclosed in an evacuated Pyrex tube, maintained at the desired temperature by the use of a small heating coil and a thermos flask containing either liquid nitrogen or an acetone-dry ice mixture. Effects of temperature gradients (actually negligible) were eliminated by averaging the voltages obtained for sample currents in opposite directions. The small IR drop between the Hall probes was eliminated by measurements for both magnetic-field polarities. The current density through the sample, determined from the voltage drop across a precision resistor, was held constant at 6.0 milliamperes/cm². The crystal orientation, determined by A. E. Austin of the Structural Chemistry Division of Battelle Memorial Institute, was obtained from a back-reflection Laue pattern.

COMPARISON OF THEORETICAL PREDICTIONS WITH EXPERIMENT

(a) Hall Coefficient, Extrinsic Region

Measurements were made on a specimen cut from a single crystal of p-type germanium, intrinsic at room temperature. Surfaces of the sample were carefully ground in order to maximize the surface recombination velocity and minimize the effects of diffusion currents. The Hall coefficients are shown in Fig. 1. One region of primary interest is that in which the temperature is low enough that the contribution of the electrons is negligible, yet high enough that only lattice scattering need be considered. These conditions are realized at 205°K. Accordingly, the parameters in the equations

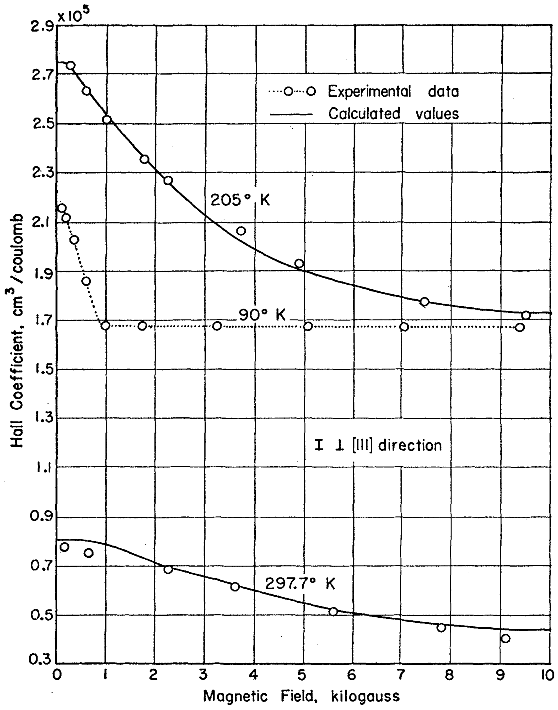

FIG. 1. Hall coefficient as a function of magnetic field for a p-type germanium specimen.

developed in the preceding section were evaluated at this temperature. In a particular calculation, the density of the slow holes is obtained in first approximation from Hall measurements taken at strong magnetic fields. An approximate mobility then follows from the conductivity data. These results, combined with the data taken at very weak fields, then give in first approximation the mobilities and densities of the fast holes. Another important consideration in determining the mobility of the fast holes is the rate of decrease of the Hall coefficient with increasing field. The values obtained by successive approximations are given in Table I for 205°K. It is seen that these values provide an excellent fit to the experimental points.

TABLE I. Tabulated values of basic parameters.

Temperature, °K	n_2,[a] cm^{-3}	n_1/n_2,[b]	n_3/n_2,[a]	μ_2^0,[c] cm²/volt-sec	μ_1^0/μ_2^0,[d]	μ_3^0/μ_2^0,[a]
205.	4.23×10^{13}	0	0.0187	4500	···	7.5
296.7	5.08×10^{13}	0.170	0.02	1900	2.2	8.0
297.7	5.16×10^{13}	0.184	0.02	1895	2.2	8.0
307.2	6.30×10^{13}	0.335	0.0200	1755	2.25	8.0
310.1	6.80×10^{13}	0.386	0.0208	1710	2.25	8.0
312.5	7.29×10^{13}	0.427	0.0206	1685	2.25	8.0
314.8	7.81×10^{13}	0.468	0.0214	1655	2.25	8.0

[a] Determined from Hall coefficients as functions of temperature and magnetic field, using other parameters as listed above, except at 296.7°K, where magnetoresistance data were used.
[b] Calculated with $n_1 n_2 = 6.25 \times 10^{26}$ cm^{-6} at $T = 300$°K; $E(0) = 0.75$ ev; see E. M. Conwell, Proc. Inst. Radio Engrs. 40, 1327 (1952).
[c] Determined at low temperatures from extrinsic data; higher temperature values computed by using $T^{-2.33}$ temperature dependence (values so obtained are within probable error of drift data (see reference 7).
[d] Drift data indicate values from 2.0 to 2.25, using limits of probable error and temperature dependences given in reference 7 over the interval from 296°K to 315°K. Our Hall data favor the higher values.

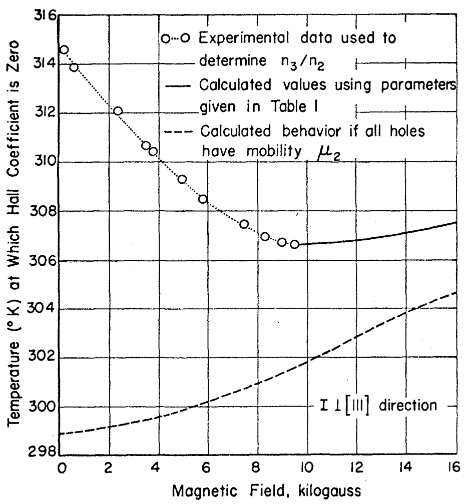

FIG. 2. Experimental data and calculated results of Hall coefficient crossover temperatures as a function of magnetic field.

A point of interest is that at 90°K, the experimental data indicate that the strong magnetic-field approximation is probably valid at 1000 gauss. Due to the acceptor concentration of $4\times10^{13}/\text{cm}^3$, impurity scattering becomes significant and thus complicates[17] the analysis of the 90°K curve. Quantitative calculations were therefore not carried out. However, it is readily seen that the preceding equations predict a large slope at weak fields and the asymptotic behavior above 1000 gauss.

(b) Hall Coefficient, Transition Region

The data near room temperature (297.7°K) also are shown in Fig. 1. At this temperature, all three types of carriers must be considered. The number of parameters is therefore too large to be satisfactorily determined from observations taken at a single temperature. However, the electron concentration at a given temperature can be obtained from the extrinsic carrier density and the intrinsic carrier concentration product. The contributions of the fast holes are then obtained from measurements in the neighborhood of the Hall coefficient crossover.

Data taken in the neighborhood of the Hall coefficient zeros are especially desirable for computational purposes due to the simplification in the equations. The temperatures of the Hall coefficient zeros as a function of magnetic field are shown by the experimental points in Fig. 2. In practice, the magnetic-field parameters were computed for the points at 0, 2, 4, and 8 kilogauss. The electron-current parameters were evaluated for the corresponding temperatures of 314.8, 312.5, 310.1, and 307.2°K in the manner previously described. These data must satisfy the equation

$$(n_1/n_2)(\mu_1^0/\mu_2^0)^2L_1=L_2+(n_3/n_2)(\mu_3^0/\mu_2^0)^2L_3. \quad (19)$$

[17] For a treatment of the types of integrals involved, see reference 13.

The magnitude of $(n_3/n_2)(\mu_3^0/\mu_2^0)^2$ which satisfies the above equation is strongly dependent on the value of μ_1^0/μ_2^0. The requirement that the fast-hole parameters be relatively independent of temperature in the region from 297 to 315°K allows μ_1^0/μ_2^0 to be evaluated. The additional constraints on the fast-hole parameters given by the dependence of Hall coefficient on magnetic field at any given temperature permits a determination of μ_3^0/μ_2^0. The results of such calculations are summarized in Table I.

An extremely interesting calculation which can now be made by use of the results in Table I is the determination of the magnetic-field dependence of the temperature at which the Hall coefficient is zero for larger fields. The results of such calculations are shown in Fig. 2. It will be noted that a minimum is predicted for the temperature at which the Hall coefficient is zero. For comparison purposes, the results of similar calculations for a two-carrier model (electrons and slow holes only) are indicated by the dashed curve. The difference in slope between the two curves, which is so conspicuous at low fields, is seen to disappear as higher field strengths are reached.

Using the values of the parameters tabulated in Table I, the magnetic-field dependence of the Hall coefficient has been calculated at four different temperatures for which experimental data were available. Results are presented in Fig. 3.

(c) Transverse Magnetoresistance Effects

Results of previous calculations of magnetoresistance effects have left much to be desired.[18] In many cases, the magnitudes have been too low by a factor of ten or more. Logical improvements have been suggested

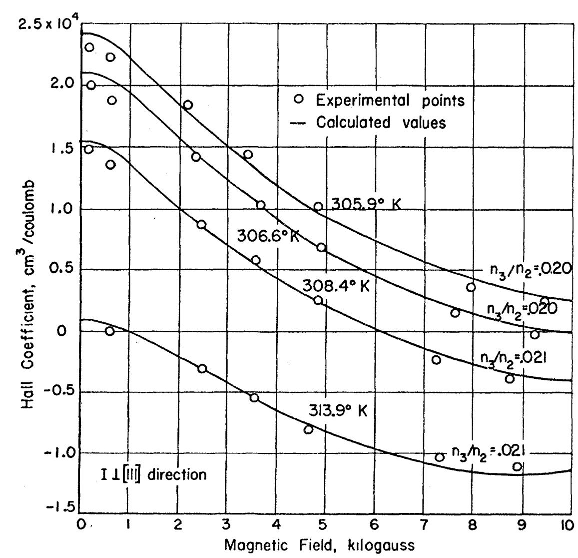

FIG. 3. Dependence of Hall coefficient on magnetic field for a p-type germanium specimen.

[18] See discussion in references 13 and 16.

by several investigators, but these have involved the introduction of anisotropies (e.g., nonspherical energy surfaces or anisotropic mean-free paths) with the result that the exact formulations had to be restricted to very general relationships. Due to mathematical difficulties exact solutions in most cases were not possible.

It was shown in a preceding section how the presence of a small number of fast holes could increase the magnetoresistance by an order of magnitude. It is therefore of great interest to calculate the magnetoresistance effect as a function of magnetic field from Eq. (11), using the values of the parameters previously determined from the Hall coefficient measurements (see Table I), and then to compare these results with experiment. This has been done in Fig. 4 for the extrinsic region (205°K). The agreement is good throughout the whole range. We have chosen to plot the quantity $\Delta\rho/\rho_H H^2$, a representation which emphasizes the behavior at weak fields. It is seen that quadratic dependence of $\Delta\rho/\rho_H$ on H is valid only below, say, 100 gauss. In order to emphasize the extreme influence of the fast holes on magnetoresistance effects, especially at weak fields, Eq. (11) was also evaluated for the case where no fast holes were present. Results are shown by the dashed curve in Fig. 4.

Another determination of magnetoresistance has been carried out in the transition region where all three types of carriers are important. The results of the calculations using the parameters listed in Table I, along with the experimental points, are shown in Fig. 5.

The possibility that transverse magnetoresistance effects in p-type germanium can be accurately predicted without introducing the severe complications of anisotropy raises the question as to whether undue emphasis has not been placed on the importance of nonspherical

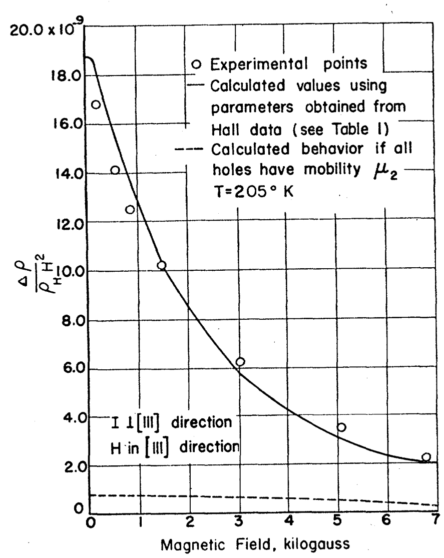

FIG. 4. $\Delta\rho/\rho_H H^2$ *versus* H for a p-type germanium specimen.

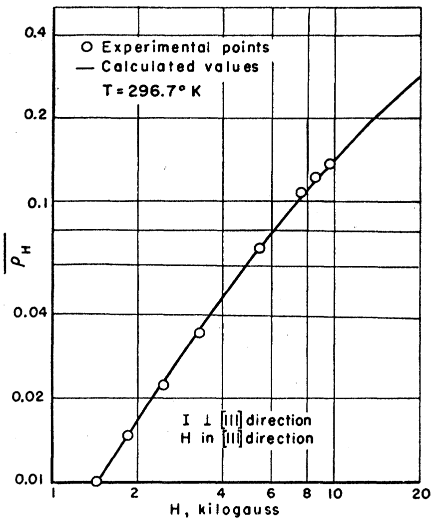

FIG. 5. Magnetoresistance as a function of magnetic field in p-type germanium.

energy surfaces in connection with galvanomagnetic effects in p-type germanium.

DISCUSSION

A question of great interest is the temperature dependency of n_3/n_2. Unfortunately, the precision of determination of this quantity is hardly adequate unless the other parameters are known quite accurately. Improved results might also be obtained by extending measurements to include much larger magnetic-field strengths. In addition, very precise temperature measurements are desirable, especially in the neighborhood of the Hall coefficient zeros. However, if more precise quantitative interpretations are to be made of calculations in the region where electron currents are significant, it will probably be necessary to take into account the pronounced anisotropy of the conduction band.[19]

Another point of interest is the magnitude of n_3/n_2, inasmuch as it gives information on the relative densities of states and hence, on the effective mass ratios of the two types of holes. In the approximation of spherical energy surfaces, density of states ratios give the following relationship:

$$n_3/n_2=(g_3/g_2)(m_3/m_2)^{\frac{3}{2}}, \tag{20}$$

where g_3 and g_2 are weighting factors dependent on the multiplicity of sources of carriers associated with masses m_3 and m_2 respectively. If significant splitting occurs, due to spin-orbit coupling, for example, then the weighting factor will be temperature dependent in those regions where the energy difference is not negligible compared with kT. With g_3 taken as unity and

[19] See, for example, Lax, Zeiger, Dexter, and Rosenblum, Phys. Rev. **93**, 1418 (1954); B. Abeles and S. Meiboom, Phys. Rev. **95**, 31 (1954); M. Shibuya, J. Phys. Soc. Japan **9**, 134 (1954).

m_3 as[20] $0.04m$, our value of 0.02 for n_3/n_2 leads to a value[21] of $0.54m$ for $g_2^{\frac{2}{3}}\, m_2$.

The question of nonspherical energy surfaces has often been raised in connection with the valence band of germanium. Large anisotropies have been postulated, because a number of electrical properties could not be explained on the basis of simple theory. These included the large discrepancies among values of hole mobilities as calculated from Hall data, conductivity and drift experiments, and transverse magnetoresistance effects. It was shown in the previous sections, however, that all these discrepancies could be reconciled, within the limits of accuracy of present measurements, by a theory which does not consider deviations from spherical energy surfaces,[22] but which does take into account the contributions of the high-mobility holes.

The Hall mobility, if calculated by the relationship $\mu_H = R\sigma$, needs to be interpreted with great care. With the above definition it is readily seen that in the case of spherical energy surfaces the ratio μ_H/μ is $3\pi/8$ for zero magnetic field, classical statistics, and negligible impurity scattering; is unity for zero-magnetic field and degenerate statistics; and in practice, is magnetic-field dependent for high-mobility materials. Furthermore, where more than one carrier is significant, the quantity $R\sigma$ has no obvious interpretation. For example, in extrinsic p-type germanium, values of μ_H/μ (i.e., $R\sigma/\mu$) of 1.17 at 90°K, 1.47 at 205°K, and 1.8 at 300°K are reported.[23] The magnitude and temperature dependence of this ratio has been interpreted as having significance with regard to the shape and number of surfaces of minimum energy in the Brillouin zone. However, analysis of these values by the methods developed in this paper shows that they can be accounted for by simple theory, assuming spherical energy surfaces but taking into account the fast holes and the value of the magnetic fields. Let us assume that a constant field of 4000 gauss was used in the measurements. At 90°K, the magnetic-field parameter γ is greater than four, even for the slow holes, so that the large magnetic-field approximation is fairly good. The factor $3\pi/8$ should, therefore, be replaced by approximately unity. The additional 17 percent results from the increase in σ due to the contributions of the fast holes. At 205°K, the Hall coefficient has been increased by 25 percent, due to the influence of the fast holes at the lower values of γ. Hence, $R\sigma/\mu$ becomes 1.47. At 300°K, γ is still smaller, so that R has been increased by 55 percent and $R\sigma/\mu$ is now 1.8.

A practical consequence of the developments presented in this paper is the emphasis placed on the value of extending experimental data to include measurements below 1000 gauss and above 10 000 gauss. The importance of specifying actual magnetic-field strengths used in obtaining galvanomagnetic data is obvious.

A pertinent consideration is the possibility of measuring the mobility of the fast holes by drift techniques. In addition to the obvious problem arising from their small numbers, the lifetimes of injected high-mobility holes may be so small as to cause difficulty. It has been pointed out by Burstein[24] that interband transitions could greatly reduce the times the fast holes remain in such states and thus make their direct observation virtually impossible.

In order for the mathematics to remain tractable and allow calculations to be readily performed, simplified equations have been applied in this initial treatment of our proposed model. Probably the point most deserving of immediate attention is the effect of interband transitions. The fact that the temperature variation of hole mobilities is significantly different from $T^{-1.5}$ certainly suggests that the scattering processes need to be re-examined, inasmuch as recent evidence on the valence-band structure in germanium renders it difficult to attribute these deviations to non-spherical or non-centrally located energy surfaces. Another argument for re-examination of the scattering processes is the inapplicability of the relationship[25] $\mu_3/\mu_2 \sim (m_2/m_3)^{5/2}$, since it would predict too large a value for the mobility ratio. In view of these considerations it is quite remarkable how consistently well the elementary theory when applied to the model postulated can account for the observed Hall and transverse magnetoresistance characteristics. Hence, it may be expected that a more rigorous analysis, taking into account the concepts discussed here, should not only yield more accurate values of the basic parameters, but would also provide a very effective means for establishing band structure from electrical properties.

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[20] In this calculation, we assume in going from 4°K to room temperature that the variation in m_3 is small compared to that in m_2, due to the greater isotropy of the surfaces corresponding to mass m_3. See Dexter, Zeiger, and Lax, Phys. Rev. **95**, 557 (1954); Dresselhaus, Kip, and Kittel, Phys. Rev. **95**, 568 (1954).

[21] From their thermoelectric power data, T. H. Geballe and G. W. Hull [Phys. Rev. **94**, 1134 (1954)] obtain a value of 0.75 $\pm 0.2m$ for the "density of states" effective mass parameter for holes.

[22] We do not mean to imply that slight anisotropies do not exist in the valence band of germanium, but, rather, to suggest that their effect on Hall coefficient and transverse magnetoresistance is small. In this connection, the papers referenced in footnote 20 are of interest.

[23] F. J. Morin, Phys. Rev. **93**, 62 (1954), and F. J. Morin and J. P. Maita, Phys. Rev. **94**, 1525 (1954).

[24] E. Burstein (private discussion).

[25] See, for example, F. Seitz, Phys. Rev. **73**, 549 (1948).