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Fourth Order Corrections to Meson-Nucleon Scattering in Pseudoscalar Meson Theory*†

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The low-energy S- and P-wave phase shifts for meson-nucleon scattering in the relativistic pseudoscalar coupling theory are calculated in perturbation theory to fourth order in the coupling constant. A comparison is made of the complete relativistic theory and the nonrelativistic approximation to it, obtained by use of the Foldy-Wouthuysen transformation, via a comparison of the low-energy scattering predicted by both theories. It is concluded that the S-wave scattering is similar in the two theories but that the P-wave scattering is very different in the two theories in fourth order. An analysis of the sources of the differences between the two theories is made by calculating the fourth order scattering in a version of the relativistic theory in which all effects due to the production of pairs are eliminated.

An investigation is made of the nature and possible validity of the Tamm-Dancoff approximation. Two possible viewpoints with respect to this approximation are developed, and the extent to which the Tamm-Dancoff method is applicable to the pseudoscalar coupling theory according to these two points of view is determined, insofar as the low-energy calculations of this paper allow.

I. INTRODUCTION

IN recent years considerable experimental evidence has accumulated on the nature of the π meson and the character of its interaction with nucleons. The experiments indicate that the π meson is a pseudoscalar particle and that the interaction is, to a good approximation, charge independent. In addition to these simple facts, a large amount of detailed information on the nature of the interaction is now available.

A great deal of effort has gone into attempts to understand this meson nucleon interaction on the basis of meson theory. In this approach one starts with a free meson field ϕ_α and a free nucleon field ψ and then couples the two with an interaction term consistent with the pseudoscalar character of ϕ_α and the charge independence of the interaction. Two simple types of coupling have been suggested and extensively studied. In the pseudoscalar coupling theory the interaction term in the Lagrangian density is

$$\mathcal{L}_I = -ig\bar{\psi}\gamma_5\tau_\alpha\phi_\alpha\psi, \tag{1}$$

and in the pseudovector coupling theory the interaction term is

$$\mathcal{L}_I = -(ig/2m)\bar{\psi}\gamma_5\gamma_\mu\tau_\alpha(\partial\phi_\alpha/\partial x_\mu)\psi. \tag{2}$$

Here m is the nucleon mass, and g is the dimensionless coupling constant, the value of which determines the strength of the interaction. If the interactions are written as in Eqs. (1) and (2), then in a certain very approximate sense to be explained below, g in Eq. (1) can be taken to be equivalent to g in Eq. (2). The interaction (1) leads to a theory from which all divergences can be removed by renormalization, while Eq. (2) leads to a theory which is still divergent after renormalization. This has led some physicists to believe that the interaction (1) is much more likely to be correct than the interaction (2), although this point is still in dispute. In any event, with the help of the renormalization procedure calculations based on the interaction (1) can be performed in an unambiguous manner, at least in perturbation theory. On the other hand, if one wishes to go beyond the lowest order in perturbation theory in calculating with the interaction (2), one must introduce some sort of cutoff procedure in order to get finite results.

In order to simplify the mathematical problem of computing with these theories, use has frequently been made of the so-called static approximation, in which all effects of nucleon recoil and all effects due to pair production are eliminated at the outset, so that the nucleon is simply regarded as a source of the meson field. In the case of the pseudovector interaction (2) this leads to a Hamiltonian,

$$H = m + H_\phi + (g/2m)\boldsymbol{\sigma}\cdot\nabla\tau_\alpha\phi_\alpha(\mathbf{x}), \tag{3}$$

for a problem with only one nucleon at the point $\mathbf{x}$. H_ϕ is the Hamiltonian of the free meson field.

If one makes the static approximation to the pseudoscalar coupling theory, the entire interaction term goes to zero. One can, however, perform a series of unitary transformations on the Hamiltonian according to the method of Foldy and Wouthuysen[1] to eliminate from the Hamiltonian, to successive orders in $1/m$, terms in which the "large" components of the field ψ are coupled to the "small" components. The static approximation is then obtained by carrying out the Foldy-Wouthuysen transformation to order m^0 only. To get any interaction term at all for the pseudoscalar coupling theory, one must carry out the transformation to order $1/m$. Up

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[1] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950); see also L. L. Foldy, Phys. Rev. **84**, 168 (1951) and G. Wentzel, Phys. Rev. **86**, 802 (1952).

to terms of order $1/m$ one finds

$$H=H_\phi+\int d^3x\psi^\dagger(\mathbf{x})\left(m-\frac{\nabla^2}{2m}\right)\psi(\mathbf{x})+\frac{g}{2m}\int d^3x\psi^\dagger(\mathbf{x})\boldsymbol{\sigma}\cdot\nabla\tau_\alpha\phi_\alpha(\mathbf{x})\psi(\mathbf{x})+\frac{g^2}{2m}\int d^3x\psi^\dagger(\mathbf{x})\phi_\alpha{}^2(\mathbf{x})\psi(\mathbf{x}). \quad (4)$$

Here the "small" components of $\psi(\mathbf{x})$ have been dropped, so that $\psi(\mathbf{x})$ has two spin components. Examination of this Hamiltonian indicates that the unitary transformation has effectively made a non-relativistic approximation to the nucleon motion and at the same time approximated by the last term in Eq. (4) that part of the interaction in which one virtual pair is produced. The third term in Eq. (4) is essentially equivalent to the interaction term in Eq. (3). Note in particular that the coupling constants have been chosen in such a way that the coefficients of these two terms are the same. In this sense only g from Eq. (1) and g from Eq. (2) are equivalent. The Hamiltonian (4), as it stands, contains a nucleon recoil term. This term helps to make calculations based on Eq. (4) more convergent than they otherwise would be, but the recoil term does not provide enough convergence factors to make a theory based on Eq. (4) finite after renormalization. Furthermore, the cutoff factors provided by the recoil term in the Hamiltonian (4) are not like those provided by the relativistic theory. For these reasons we prefer to simplify Eq. (4) still further by ignoring the nucleon recoil term. This gives us a Hamiltonian,

$$H=m+H_\phi+\frac{g}{2m}\boldsymbol{\sigma}\cdot\nabla\tau_\alpha\phi_\alpha(\mathbf{x})+\frac{g^2}{2m}\phi_\alpha{}^2(\mathbf{x}). \quad (5)$$

One must of course use a cutoff in calculating with either Eq. (3) or (5). The interaction term in Eq. (3) couples only P state mesons with the nucleon. The interaction term $(g^2/2m)\phi_\alpha{}^2(\mathbf{x})$ in Eq. (5) couples only S state mesons with the nucleon.

One of the objectives of this paper is a comparison of the full relativistic pseudoscalar coupling theory, the interaction term of which is given in Eq. (1), and the approximation to the full relativistic theory embodied in Eq. (5). This comparison is made by calculating the low-energy meson nucleon scattering to fourth order in the coupling constant for the full relativistic theory and comparing with the calculations of the fourth order scattering for the Hamiltonian (3) already published by Blair and Chew.[2] The results of Blair and Chew to fourth order and the S state scattering to fourth order as given by the Hamiltonian (5) are reviewed in Sec. II. In Sec. III the scattering to fourth order predicted by the complete relativistic pseudoscalar coupling theory is given.

As will be seen, these two theories, while very similar in second order in g, are quite different in fourth order. In order to facilitate the analysis of the differences between the two theories, it will be convenient to develop some approximations to the full relativistic theory which are less drastic than the Hamiltonian (5). First we shall eliminate from the complete relativistic theory all pair production phenomena. This gives us a theory which is a relativistic generalization of the theory studied by Chew, i.e., the Hamiltonian (3). If we write the Hamiltonian of our theory as

$$H=H_0+H_I=H_\phi+H_\psi+H_I, \quad (6)$$

where H_ϕ and H_ψ are the Hamiltonians of the free meson field and the free nucleon field respectively, and H_I is the interaction Hamiltonian,

$$H_I=ig\int d^3x\bar{\psi}(\mathbf{x})\gamma_5\tau_\alpha\phi_\alpha(\mathbf{x})\psi(\mathbf{x}), \quad (7)$$

then our problem is

$$(E-H_0)\Psi=H_I\Psi. \quad (8)$$

Let $\Psi_{\mathbf{p}si}$ be a vector in Fock space[3] representing a single nucleon with momentum $\mathbf{p}$ in spin state s and isotopic spin state i. We introduce the wave function,

$$\Psi_{si}(\mathbf{p})=(\Psi_{\mathbf{p}si},\Psi). \quad (9)$$

Note that $\Psi_{si}(\mathbf{p})$ is still a state vector with respect to the meson occupation numbers. We now take the scalar product of Eq. (8) with $\Psi_{\mathbf{p}si}$. Ignoring completely the production of pairs, we get

$$(E-H_\phi-E_\mathbf{p})\Psi_{si}(\mathbf{p})=\sum_{s'i'}\sum_{\mathbf{p}'}(\Psi_{\mathbf{p}si},H_I\Psi_{\mathbf{p}'s'i'})\Psi_{s'i'}(\mathbf{p}'), \quad (10)$$

where[4] $E_\mathbf{p}=(\mathbf{p}^2+m^2)^{\frac{1}{2}}$ and we use discrete normalization in momentum space. It is not difficult to work out the matrix element $(\Psi_{\mathbf{p}si},H_I\Psi_{\mathbf{p}'s'i'})$, and if this is done one gets, in matrix notation for the spin and isotopic spin indexes,

$$(E-E_\mathbf{p}-H_\phi)\Psi(\mathbf{p})=\frac{1}{V^{\frac{1}{2}}}\sum_{\mathbf{p}'}ig\gamma_5(\mathbf{p},\mathbf{p}')\cdot\tau_\alpha(a_{\mathbf{p}-\mathbf{p}',\alpha}+a^\dagger{}_{\mathbf{p}'-\mathbf{p},\alpha})\Psi(\mathbf{p}'). \quad (11)$$

Here V is the normalization volume and

$$\gamma_5(\mathbf{p},\mathbf{p}')=\frac{1}{(2\omega_{\mathbf{p}-\mathbf{p}'})^{\frac{1}{2}}}\times\left\{\frac{(E_{\mathbf{p}'}+m)\boldsymbol{\sigma}\cdot\mathbf{p}-(E_\mathbf{p}+m)\boldsymbol{\sigma}\cdot\mathbf{p}'}{2[E_\mathbf{p}E_{\mathbf{p}'}(E_\mathbf{p}+m)(E_{\mathbf{p}'}+m)]^{\frac{1}{2}}}\right\}, \quad (12)$$

[2] J. S. Blair and G. F. Chew, Phys. Rev. **90**, 1065 (1953).

[3] V. Fock, Z. Physik **75**, 622 (1932).

[4] We use always units such that $\hbar=c=1$.

where $\omega_{\mathbf{p}}=(\mathbf{p}^2+\mu^2)^{\frac{1}{2}}$, μ being the meson mass. Except for the factor $(2\omega_{\mathbf{p}-\mathbf{p}'})^{\frac{1}{2}}$, $\gamma_5(\mathbf{p},\mathbf{p}')$ is the matrix element of the Dirac matrix γ_5 between two positive energy nucleon states. $a_{\mathbf{p}-\mathbf{p}',\alpha}$ and $a^\dagger{}_{p'-p,\alpha}$ are destruction and creation operators for mesons. If we neglect recoil in Eq. (12), we find

$$\gamma_5(\mathbf{p},\mathbf{p}')\rightarrow\frac{1}{(2\omega_{\mathbf{p}-\mathbf{p}'})^{\frac{1}{2}}}\frac{\boldsymbol{\sigma}\cdot(\mathbf{p}-\mathbf{p}')}{2m}. \tag{13}$$

With this approximation for $\gamma_5(\mathbf{p},\mathbf{p}')$ Eq. (11) is identical with the corresponding Schrödinger equation derived from Eq. (3). Equation (11) is just the relativistic generalization of Chew's theory or the P state scattering part of the Hamiltonian (5).

We can also find a relativistic generalization of the S state scattering term in the Hamiltonian (5). To do this we take account only of the production of one pair at a time. We introduce in addition to $\Psi_{\mathbf{p}si}$ another vector in Fock space describing one nucleon and one pair. Let $\Psi_{\mathbf{p}'rj,\,\mathbf{p}''ql;\,\mathbf{p}'''tm}$ represent a state with two nucleons in states designated by $\mathbf{p}'rj$ and $\mathbf{p}''ql$ and an antinucleon in a state designated by $\mathbf{p}'''tm$. We now take the scalar product of Eq. (8) with $\Psi_{\mathbf{p}si}$ and $\Psi_{\mathbf{p}'rj,\,\mathbf{p}''ql;\,\mathbf{p}'''tm}$. Taking account only of the production of one pair at a time, we find

$$(E-H_\phi-E_{\mathbf{p}})\Psi_{si}(\mathbf{p})=\sum_{rqt}\sum_{jlm}\sum_{\mathbf{p}',\mathbf{p}'',\mathbf{p}'''}\times(\Psi_{\mathbf{p}si},H_I\Psi_{\mathbf{p}'rj,\,\mathbf{p}''ql;\,\mathbf{p}'''tm})(\Psi_{\mathbf{p}'rj,\,\mathbf{p}''ql;\,\mathbf{p}'''tm},\Psi), \tag{14}$$

and

$$(E-H_\phi-E_{\mathbf{p}'}-E_{\mathbf{p}''}-E_{\mathbf{p}'''})(\Psi_{\mathbf{p}'rj,\,\mathbf{p}''ql;\,\mathbf{p}'''tm},\Psi)=\sum_{s'i'}\sum_{\mathbf{p}^{iv}}(\Psi_{\mathbf{p}'rj,\,\mathbf{p}''ql;\,\mathbf{p}'''tm},H_I\Psi_{\mathbf{p}^{iv}s'i'})\Psi_{s'i'}(\mathbf{p}^{iv}). \tag{15}$$

Eliminating the one nucleon, one pair amplitude between Eqs. (14) and (15) we get

$$(E-E_{\mathbf{p}}-H_\phi)\Psi_{si}(\mathbf{p})=\sum_{s'i'}\sum_{\mathbf{p}^{iv}}\langle\mathbf{p}si|H_I|\mathbf{p}^{iv}s'i'\rangle\Psi_{s'i'}(\mathbf{p}^{iv}), \tag{16}$$

where

$$\langle\mathbf{p}si|H_I|\mathbf{p}^{iv}s'i'\rangle=\sum_{r,q,t}\sum_{j,l,m}\sum_{\mathbf{p}',\mathbf{p}'',\mathbf{p}'''}(\Psi_{\mathbf{p}si},H_I\Psi_{\mathbf{p}'rj,\,\mathbf{p}''ql;\,\mathbf{p}'''tm})\cdot\frac{1}{E-E_{\mathbf{p}'}-E_{\mathbf{p}''}-E_{\mathbf{p}'''}-H_\phi}\times(\Psi_{\mathbf{p}'rj,\,\mathbf{p}''ql;\,\mathbf{p}'''tm},H_I\Psi_{\mathbf{p}^{iv}s'i'}). \tag{17}$$

The matrix elements in Eq. (17) can be worked out without great difficulty. As it stands $\langle\mathbf{p}si|H_I|\mathbf{p}^{iv}s'i'\rangle$ contains terms which correspond to the formation of closed loops, in Feynman language. We drop these terms. Then Eq. (16) becomes, in matrix notation,

$$(E-E_{\mathbf{p}}-H_\phi)\Psi(\mathbf{p})=\frac{1}{V^{\frac{1}{2}}}\sum_{\mathbf{p}'}\Bigg[\frac{1}{V^{\frac{1}{2}}}\sum_{\mathbf{p}''}ig\gamma_5'(\mathbf{p},\mathbf{p}')\times ig\gamma_5'(\mathbf{p}'',\mathbf{p}')\sum_{\alpha,\alpha'}(a_{\mathbf{p}''-\mathbf{p}',\alpha}+a^\dagger{}_{\mathbf{p}'-\mathbf{p}'',\alpha})\times\frac{\tau_{\alpha'}\tau_\alpha}{E-E_{\mathbf{p}}-E_{\mathbf{p}'}-E_{\mathbf{p}''}-H_\phi}\times(a_{\mathbf{p}-\mathbf{p}'',\alpha'}+a^\dagger{}_{\mathbf{p}''-\mathbf{p},\alpha'})\Bigg]\Psi(\mathbf{p}'), \tag{18}$$

where

$$\gamma_5'(\mathbf{p},\mathbf{p}')=\frac{1}{(2\omega_{\mathbf{p}-\mathbf{p}'})^{\frac{1}{2}}}\times\left\{\frac{(E_{\mathbf{p}}+m)(E_{\mathbf{p}'}+m)+\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}'}{2[E_{\mathbf{p}}E_{\mathbf{p}'}(E_{\mathbf{p}}+m)(E_{\mathbf{p}'}+m)]^{\frac{1}{2}}}\right\}. \tag{19}$$

Except for the factor $(2\omega_{\mathbf{p}-\mathbf{p}'})^{\frac{1}{2}}$, $\gamma_5'(\mathbf{p},\mathbf{p}')$ is the matrix element of the Dirac matrix γ_5 between a positive energy nucleon state and a negative energy nucleon state. If we drop all recoil terms in Eq. (19), we find

$$\gamma_5'(\mathbf{p},\mathbf{p}')\rightarrow 1/(2\omega_{\mathbf{p}-\mathbf{p}'})^{\frac{1}{2}}. \tag{20}$$

If, in addition, we make the crude approximation,

$$E-E_{\mathbf{p}}-E_{\mathbf{p}'}-E_{\mathbf{p}''}-H_\phi\rightarrow -2m \tag{21}$$

in the energy denominator in Eq. (18), we obtain an equation which is precisely the same as what would be obtained from the Hamiltonian (5) if only the pair term in the interaction were kept. So the interaction term of Eq. (18) is just the relativistic generalization of the S state scattering part of the Hamiltonian (5). At this point it becomes apparent that the factors $E_{\mathbf{p}}$, $E_{\mathbf{p}'}$, etc., which appear in the denominators of $\gamma_5(\mathbf{p},\mathbf{p}')$ and $\gamma_5'(\mathbf{p},\mathbf{p}')$ provide the cutoff which makes calculations based on the interaction (1) finite after renormalization. Since $E_{\mathbf{p}}$ begins to become appreciably larger than m in the neighborhood of $|\mathbf{p}|=m$, the relativistic theory is cut off in the neighborhood of $|\mathbf{p}|=m$. Thus insofar as we wish to simulate the complete relativistic theory by the simplified Hamiltonian (5), we should use a cutoff of the order of m in calculating with Eq. (5). In Sec. IV of this paper the differences between the fourth order scattering in the complete relativistic theory and the fourth order scattering according to the Hamiltonian (5) will be analyzed with the help of Eqs. (11) and (18).

It is a well known fact that the coupling constant g of meson theory is sufficiently large so that perturbation theory, or the weak coupling approximation, cannot be used in calculations which are intended to be compared with experimental results. In view of this failure of perturbation theory, a new method of calculation known

as the Tamm-Dancoff (TD) approximation[5] has been seriously investigated recently. The nature of this approximation can be explained in various ways. Dancoff, for example, takes the following point of view. Let us write the true state vector of the meson-nucleon system as a superposition of vectors in Fock space referring to various numbers of particles in various free-particle momentum states. For the meson-nucleons scattering problem we can write

$$\begin{aligned}\Psi=&\sum_{\mathbf{p},\iota} a_\iota(\mathbf{p})\Psi_{\mathbf{p}\iota}+\sum_{\mathbf{p},\mathbf{k},\iota,\alpha} b_{\iota;\alpha}(\mathbf{p};\mathbf{k})\Psi_{\mathbf{p}\iota;\mathbf{k}\alpha}\\&+\sum_{\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\iota,\alpha,\beta} c_{\iota;\alpha,\beta}(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2)\Psi_{\mathbf{p}\iota;\mathbf{k}_1\alpha,\mathbf{k}_2\beta}+\cdots\\&+\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\iota,\kappa,\lambda} d_{\iota,\kappa;\lambda}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_3)\Psi_{\mathbf{p}_1\iota,\mathbf{p}_2\kappa;\mathbf{p}_3\lambda}\\&+\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{k},\iota,\kappa,\lambda,\alpha} f_{\iota,\kappa;\lambda;\alpha}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_3;\mathbf{k})\\&\qquad\qquad\times\Psi_{\mathbf{p}_1\iota,\mathbf{p}_2\kappa;\mathbf{p}_3\lambda;\mathbf{k}\alpha}+\cdots.\end{aligned}\tag{22}$$

Here $\Psi_{\mathbf{p}\iota}$ is a vector in Fock space describing a single nucleon in a momentum state $\mathbf{p}$ and spin and isotopic spin state ι; $\Psi_{\mathbf{p}\iota;\mathbf{k}\alpha}$ describes a nucleon in state $\mathbf{p}\iota$ and a meson in state $\mathbf{k}\alpha$; $\Psi_{\mathbf{p}_1\iota,\mathbf{p}_2\kappa;\mathbf{p}_3\lambda;\mathbf{k}\alpha}$ describes two nucleons in state $\mathbf{p}_1\iota$ and $\mathbf{p}_2\kappa$, an antinucleon in state $\mathbf{p}_3\lambda$, and a meson in state $\mathbf{k}\alpha$. The TD approximation consists of keeping in this expansion of Ψ only a certain set of the simpler terms, the terms kept depending to a certain extent on which processes one deems to be important. Once this approximation has been made, an attempt is made to solve to the remainder of the problem rigorously. The approximate expansion of Ψ is substituted into Schrödinger's equation, yielding a set of integral equations for the amplitudes $a_\iota(\mathbf{p})$ $b_{\iota;\alpha}(\mathbf{p};\mathbf{k})$, $\cdots$. For the scattering problem the amplitudes other than $b_{\iota;\alpha}(\mathbf{p};\mathbf{k})$ would be eliminated by substitution from the integral equations to yield an integral equation for $b_{\iota;\alpha}(\mathbf{p},\mathbf{k})$.

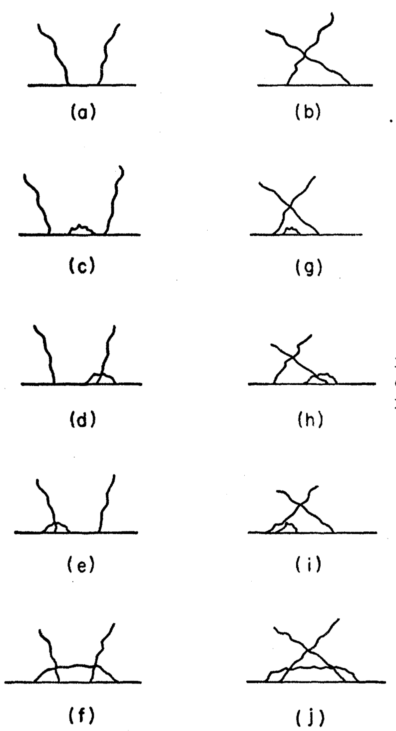

FIG. 1. Second- and fourth-order Feynman diagrams for meson nucleon scattering.

In the case of the scattering problem the amplitudes usually retained are those describing two, one, or no mesons. Amplitudes referring to states with three or more mesons are arbitrarily dropped. What is done with the pair terms depends to a large extent on which theory is used. For the static pseudovector coupling theory, Eq. (3), there are no pairs. If a relativistic but non-covariant approach, e.g., Eq. (22), is used, one might restrict oneself to terms obtainable from the one meson, one nucleon state by a single action of the interaction Hamiltonian (1). The TD approximation can also be written out in covariant Feynman notation. In this case one obtains equations of the form of the Bethe-Salpeter[6] equation. If this is done, all of the pair terms with the exception of the closed loops are taken care of by a one nucleon amplitude, since the one nucleon can go "backward in time."

Proceeding from a certain point of view, we have defined the TD approximation mathematically. Let us now see what physical significance we can attach to this point of view. Keeping only those terms involving two or fewer mesons in the field at a time will evidently be a good approximation if processes involving the emission of more than two virtual mesons are strongly inhibited. We have a qualitative reason for supposing that this might actually be the case. Terms associated with the emission of many virtual mesons will necessarily involve large energy denominators, which will tend to make the contributions from these terms smaller than the contributions from those terms involving the emission of only one or two virtual mesons. More physically, according to the Heisenberg uncertainty relation for time and energy, the system will spend only short periods of time in those states involving many virtual mesons.

Let us now consider the meson-nucleon scattering problem and compare the TD method with a straightforward perturbation theory approach. In Fig. 1 the second and fourth order Feynman diagrams for meson-nucleon scattering are enumerated. It is apparent that the approximation of keeping only terms referring to states with two or fewer mesons in the field will include both second order diagrams and all iterations and cross iterations of them, so that diagrams (*c*), (*d*), (*e*), (*f*) will be included. Diagrams (*g*), (*h*), (*i*), (*j*), however, will not be included, because they involve three mesons at a time in the field. Note that some of the energy

[5] S. M. Dancoff, Phys. Rev. **78**, 382 (1950); I. Tamm, J. Phys. (U.S.S.R.) **9**, 449 (1945); see also M. M. Levy, Phys. Rev. **88**, 72 (1952).

[6] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

denominators in diagrams (*c*), (*d*), (*e*), (*f*) may vanish for certain values of the momenta of the virtual mesons, whereas for diagrams (*g*), (*h*), (*i*), (*j*) the energy denominators are always at least as large as μ, the meson mass. One can hope to test the validity of the TD approximation according to the ideas we have developed simply by calculating the contributions to the fourth order scattering from diagrams (*g*), (*h*), (*i*), (*j*) and comparing with the contributions from diagrams (*c*), (*d*), (*e*), (*f*). If the contributions of diagrams (*g*), (*h*), (*i*), (*j*) should turn out to be small compared to the contributions of diagrams (*c*), (*d*), (*e*), (*f*), one would have some confidence in the validity of the TD method. Blair and Chew[2] have attempted to justify the use of the TD approximation for the static pseudovector coupling theory on the basis of the sort of argument outlined here. The results of the calculations of the fourth order scattering in the relativistic pseudoscalar coupling theory, given in Sec. III of this paper, show very definitely that this sort of argument cannot be used to justify the use of the TD method for the pseudoscalar coupling theory.

The TD method can also be considered from a somewhat different point of view. We shall explain this new point of view in terms of the relativistic TD or Bethe-Salpeter approximation. We consider only the meson nucleon scattering problem. The treatment of the meson-nucleon scattering problem in terms of covariant integral equations was first considered by Gell-Mann and Goldberger[7] and has recently been discussed in detail by Levy.[8] All conceivable Feynman diagrams for meson-nucleon scattering can be divided into two classes. In class one are all diagrams in which the incoming nucleon line appears completely bare of virtual or real mesons at some point in the middle of the diagram. Diagrams (*a*), (*c*), (*d*), (*e*), of Fig. 1 are diagrams of this class. In class two are all other scattering diagrams. All diagrams of class one are represented in the contribution,

$$S_1 = ig\Gamma_5(P-k', P)\tau_\alpha S_F'(P) ig\Gamma_5(P, P-k)\tau_\beta, \quad (23)$$

to the S matrix. Here k, k' are the initial and final meson four-momenta, and P is the total four-momentum of the system. Γ_5 is the modified vertex operator, and S_F' is the modified nucleon propagation function. Note that S_1 contributes to the scattering only for states in which both the isotopic spin and the angular momentum are 1/2. The second class of Feynman diagrams describing scattering can be generated by an integral equation of the form,

$$\psi_\alpha(k) = \phi_{\alpha i}(k) + D_F'(k^2) S_F'(P-k) \times \frac{1}{(2\pi)^4} \sum_\beta \int d^4q G_{\alpha\beta}(k,q)\psi_\beta(q). \quad (24)$$

[7] M. Gell-Mann and M. L. Goldberger (unpublished).
[8] M. M. Levy, Phys. Rev. **94**, 460 (1954).

$\phi_{\alpha i}(k)$ in Eq. (24) represents the incoming plane wave. D_F' is the modified meson propagation function. The kernel $G_{\alpha\beta}(k,q)$ contains a term for each Feynman diagram of class two which is not itself the iteration or cross iteration of simpler diagrams of class two. The iteration of simple scattering diagrams is taken care of by the integral Eq. (24) itself. In terms of the solution of Eq. (24), a matrix element of that part of the S matrix coming from diagrams of class two is given by

$$(S_2)_{fi} = \int d^4k' \bar{\phi}_{\alpha f}(k') i(\mu^2+k^2) \times [i\gamma\cdot(P-k')+m]\psi_\alpha(k'), \quad (25)$$

where $\phi_{\alpha f}(k')$ is the final plane wave state.

All of the diagrams contributing to $G_{\alpha\beta}(k,q)$ can be generated from irreducible scattering diagrams by substituting for the bare nucleon and meson lines and simple vertices the complete nucleon and meson self energy parts and the complete vertex parts. The same division of effects can of course be made in the terms of $G_{\alpha\beta}(k,q)$ themselves, so that the operators S_F', D_F' and Γ_5 can be clearly separated out in $G_{\alpha\beta}(k,q)$. Renormalization of these equations is now an easy matter, at least symbolically. One simply replaces S_F', D_F', and Γ_5, wherever they occur in Eqs. (24) and (25), by the renormalized propagation functions and vertex operators and at the same time changes the coupling constant g to the renormalized coupling constant. The division of diagrams into two classes is necessary if the equations are to be renormalized in a simple fashion.

$G_{\alpha\beta}(k,q)$ is known only as a power series in g. To second order in g we have

$$G_{\alpha\beta}(k,q) = ig\gamma_5\tau_\beta \frac{1}{i\gamma\cdot(P-k-q)+m} ig\gamma_5\tau_\alpha, \quad (26)$$

corresponding to the second order diagram (*b*) of Fig. 1. The fourth order corrections to $G_{\alpha\beta}(k,q)$ are represented by the fourth order diagrams (*g*), (*h*), (*i*), (*j*). Now, Eq. (24) is a sort of Schrödinger equation in momentum space for the wave function $\psi_\alpha(k)$, and $G_{\alpha\beta}(k,q)$ is the potential energy of this Schrödinger equation. It is a complicated energy- and momentum-dependent potential energy, but still, in a formal way at least, it can be thought of as a potential energy. It is not difficult to see that the TD approximation essentially amounts to cutting off the power series development of $G_{\alpha\beta}(k,q)$ at some arbitrary power of g. This definition of the TD approximation is not quite identical with the definition previously given. But certainly the spirit of the two approximations is the same. From this point of view we see that the TD approximation will be a good approximation when the coupling constant is sufficiently small so that only the first term in Eq. (26) of the power series development of $G_{\alpha\beta}(k,q)$ need be kept. Weak-coupling theory will be a valid approximation when g is sufficiently smaller still so that the potential

$G_{\alpha\beta}(k,q)$ can be treated in Born approximation. A test of the validity of the TD approximation according to this point of view would be quite complicated. The fourth order potential would have to be calculated and compared with the second order potential in Eq. (26) for the whole range of the variables k and q, g being taken as whatever value was found necessary to fit the experimental data. To carry out this calculation completely would be an extremely laborious job. In this paper the fourth order scattering has been computed only for very low energies, where the S-wave phase shifts vary as the first power of the momentum and the P-wave phase shifts as the third power of the momentum. These calculations will enable us to compare the zero-energy matrix element of the fourth order potential with the zero-energy matrix element of the second order potential. This is done for Chew's theory in Sec. II and for the relativistic pseudoscalar coupling theory in Sec. III.

II. CHEW'S THEORY

Recently, Chew has investigated in great detail the predictions of a theory based on the Hamiltonian (3). We shall be interested in Chew's results for the scattering problem as an approximation to the P-wave scattering of the relativistic pseudoscalar coupling theory. It must be emphasized that Chew has never considered his theory as an approximation to the relativistic theory. He regards his theory as an independent nonrelativistic form of meson theory. For the sake of convenience, however, we shall often refer to Chew's theory as an approximation to the relativistic theory in this paper. Blair and Chew[2] have calculated the second and fourth order scattering with the Hamiltonian (3). They find for the phase shifts

$$\begin{aligned}\delta_{11}&=-4y(1-4\Delta_--2\Delta_+),\\ \delta_{13}=\delta_{31}&=-y(1-\Delta_-+\Delta_+),\\ \delta_{33}&=2y(1+2\Delta_-+\Delta_+),\end{aligned}\tag{27}$$

where

$$y=\frac{1}{6}\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{m}\right)^2\left(\frac{k}{\mu}\right)^3,$$

$$\Delta_\pm=\frac{1}{6\pi}\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{m}\right)\frac{1}{m}\int_0^K dq\frac{q^4}{\omega_q^3}\frac{1}{\omega_q\pm\mu}.\tag{28}$$

We shall always use the notation δ_{2I} for S-wave phase shifts and $\delta_{2I,2J}$ for P-wave phase shifts, I and J being the isotopic spin and angular momentum respectively. Here we have restricted ourselves to very low energies and expanded the phase shifts in powers of the momentum k, keeping only the lowest order term. K is the cutoff momentum. We indicated in the Introduction that we should take $K=m$ if we wish to regard the Hamiltonian (3) as an approximation to the P-state scattering part of the pseudoscalar coupling theory. Actually, the calculations of Sec. IV will indicate that the effective value of K is somewhat smaller than m. For future reference the contributions to the phase shifts from the various fourth order diagrams for the case $K=m$ have been evaluated numerically and are given in Table I.

The integral Δ_- comes from the TD diagrams (c), (d), (e), (f) in Fig. 1. Δ_+ comes from the diagrams which are ignored in the TD approximation, (g), (h), (i), (j). Because of the vanishing energy denominator in Δ_-, Δ_- is always larger than Δ_+. How much larger it is depends on the cutoff K. As K increases, the significance of the vanishing denominator in Δ_- decreases, and Δ_- approaches Δ_+. For $K=m$, $\mu/m=0.147$, we find

$$\Delta_+/\Delta_-=0.54,\quad K=m.\tag{29}$$

If the cutoff K is decreased, Δ_+/Δ_- decreases. For $K=\mu$ we find

$$\Delta_+/\Delta_-=0.11,\quad K=\mu.\tag{30}$$

Note also that to lowest order in μ/m for $K=m$,

$$\Delta_+=\Delta_-=\frac{1}{6\pi}\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{m}\right).\tag{31}$$

Thus the difference between Δ_+ and Δ_- is a μ/m correction, a large one, however, according to Eq. (29). For small cutoff then, $K<m$, the TD approximation can be justified for the Hamiltonian (3) simply on the basis that the contribution of the TD terms to the fourth order scattering is considerably larger than the contribution of the terms ignored in the TD method. Precisely this argument has been used by Blair and Chew, who at the time their paper was written were investigating the Hamiltonian (3) with a small cutoff. For $K=m$ this sort of argument is considerably weaker as can be seen from Eq. (29).

Consider now the Hamiltonian (5), which is supposed to be a rough approximation to the complete relativistic pseudoscalar coupling theory. This Hamiltonian contains, in addition to the terms of the static pseudovector coupling theory, a new term, which gives rise to S-wave scattering. Examination of the diagrams involved shows that to fourth order in g the pair term in Eq. (5) contributes nothing to the P-state scattering, so that the P-state scattering due to the Hamiltonian (5) is identical with that in Chew's theory, as given in Eqs. (27) and Table I. The S wave scattering to fourth order with the Hamiltonian (5) turns out to be

$$\delta_1=\delta_3=-\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{m}\right)\left(\frac{k}{\mu}\right)\times\left[1-\frac{1}{\pi}\frac{g^2}{4\pi}\left\{2\frac{K}{m}-\frac{4}{3}\frac{1}{m^2}\int_0^K\frac{q^4dq}{\omega_q^3}\right\}\right].\tag{32}$$

The first term in the curly bracket here comes from the pair term in Eq. (5). The last term is a charge renormalization term, which must be put in if the

TABLE I. Fourth order phase shifts in Chew's theory with $K=m$. The letters in the first column refer to the diagrams in Fig. 1.

	$\frac{\delta_{11}}{1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3}$	$\frac{\delta_{31}}{1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3}$	$\frac{\delta_{13}}{1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3}$	$\frac{\delta_{33}}{1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3}$
(c)	0.58			
(d)=(e)	−0.064			
(f)	0.0072	0.0288	0.0288	0.115
(g)	0.034	−0.068	−0.068	0.136
(h)=(i)	−0.0038	0.0076	0.0076	−0.0152
(j)	0.096	0.038	0.038	0.015

coupling constant g of Eq. (32) is to be the same as the coupling constant used by Chew in Eqs. (27). If we evaluate the integral only to lowest order in μ/m, we find for $K=m$,

$$\delta_1=\delta_3=-\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{m}\right)\left(\frac{k}{\mu}\right)\left[1-\frac{4}{3\pi}\frac{g^2}{4\pi}\right]. \quad (33)$$

We can also test the TD method for Chew's theory according to point of view that the TD method represents an expansion of the potential energy in powers of g. To do this we simply find the ratio of the sum of the contributions of the diagrams (g), (h), (i), (j) of Fig. 1 to the contribution of diagram (b) in the formulas (27). The contribution of diagram (b) to δ_{11} is $y/2$, so we find for the ratios of the matrix elements of the fourth order potentials to the second order potentials the values,

$$\left(\frac{V_4}{V_2}\right)_{11}=16\Delta_+,\quad \left(\frac{V_4}{V_2}\right)_{13}=\left(\frac{V_4}{V_2}\right)_{31}=\Delta_+,$$
$$\left(\frac{V_4}{V_2}\right)_{33}=4\Delta_+. \quad (34)$$

Recent indications are that $g^2/4\pi$ is of the order of 10. Using this value we find $\Delta_+=0.043$ for $K=m$. With this value of Δ_+, Eq. (34) tells us that the fourth order potential is small compared to the second order potential for the states (13), (31), and (33). For the (11) state the fourth order potential is not very small compared to the second order potential.[9] Note that the figures given here are for zero energy only. As the energy increases Δ_+ increases according to the formula

$$\Delta_+(k)=\frac{1}{6\pi}\left(\frac{g^2}{4\pi}\right)\left(\frac{\omega_k}{m}\right)\frac{1}{m}\int_0^K\frac{q^4dq}{\omega_q^3}\frac{1}{\omega_q+\omega_k}. \quad (28a)$$

There is one more point worth mentioning in connection with the non-relativistic Hamiltonian (5). Examination of Eqs. (27), (31), and (33) shows that for S states the ratio of the fourth order contributions to the second order contributions is of order $g^2/4\pi$, while for P states it is of order $(g^2/4\pi)(\mu/m)$. This might lead one to hope that for P-wave scattering there is an effective coupling constant, i.e., that the expansion parameter of perturbation theory is effectively $(g^2/4\pi)(\mu/m)$ rather than $g^2/4\pi$. Actually this is not the case. The factor μ/m appears only once, in going from second order to fourth order. Except for certain special cases, the ratio of a sixth order contribution to a fourth order contribution is $g^2/4\pi$ rather than $(g^2/4\pi)(\mu/m)$. The situation is best described by saying that the second order contributions to the phase shifts are anomalously large by one power of m/μ. This has the interesting consequence that for very small μ/m perturbation theory would become valid at low energies, all of the radiative corrections becoming small compared to the contributions of diagrams (a) and (b). These remarks also have some bearing on our test of the validity of the TD method according to the idea that this method represents an expansion of the potential energy as a power series in g. The anomalous factor μ/m, which appears in the ratio of the fourth order scattering to the second order scattering, also appears in Δ_+, which determines the ratio of the fourth and second order potentials according to Eq. (34). This factor will not appear, however, in the ratio of the sixth order potential to the fourth order potential. So the rate of convergence of the potential series which one would estimate from Eq. (34) is entirely false. Note also that, according to Eq. (28a), the factor μ/m in Δ_+ disappears as the energy increases.

It is not difficult to show that for low energies the radiative corrections are smaller by one power of μ/m than the contributions of the diagrams (a) and (b) of Fig. 1. Consider first the set of diagrams which we called class one diagrams in the Introduction. The contribution of these diagrams to the transition or reaction matrices is given in Chew's theory by an equation of the same form as Eq. (23). For these diagrams our theorem follows directly from the general theory of renormalization developed by Chew for his theory.[10] Chew has renormalized the nucleon propagation functions and vertex operators in such a way that for zero-energy scattering the contributions from all the radiative corrections in Eq. (23) are smaller by one power

[9] It should be noted that if we were to count the sum of the contributions of diagrams (a) and (b) as the second order potential, we would get $(V_4/V_2)_{11}=-2\Delta_+$ instead of the value given in Eq. (34).

[10] G. F. Chew, Phys. Rev. 94, 1748 (1954).

of μ/m than the contribution of the irreducible diagram (a). Note, however, that, except for certain special cases, the higher order radiative corrections are not smaller by successive powers of $(g^2/4\pi)(\mu/m)$. The factor μ/m comes in only once in going from second order to fourth order. In considering the diagrams of class two, one cannot prove anything from the renormalization procedure alone. However, the dependence on m of the integral representing an arbitrary nth order radiative correction of class two can be ascertained simply by counting the powers of the integration variables in the numerator and denominator of the integrand and using the fact that the cutoff is of order m. One concludes in this way that for zero energy scattering all of the radiative corrections of class two are smaller by one power of μ/m than the contribution of the second order diagram (b) of Fig. 1. This happens simply because for the second order diagram the energy denominator has the value μ, while for the higher order diagrams the energy denominators depend on momenta which are integrated over up to a value of the order of m for cutoff $K=m$. For the S-wave scattering this is not the case, because the energy denominator for the second order scattering is $2m$, because of the formation of the pair.

III. THE RELATIVISTIC PSEUDOSCALAR COUPLING THEORY

In an attempt to understand something about the nature of the complete relativistic pseudoscalar coupling theory and the possible validity of the Tamm-Dancoff approximation as applied to it, the low-energy S- and P-wave phase shifts have been calculated rigorously to fourth order in this theory using the Feynman Dyson techniques[11] and the renormalization theory developed by Dyson, Ward,[12] *et al.* The interaction term in this theory is given by Eq. (1).

The phase shift for the state of total angular momentum J, parity $(-1)^{J\pm\frac{1}{2}}$, and isotopic spin I at momentum k in the center-of-mass system is given by the formula

$$\frac{\exp[2i\delta_{I,J,(-1)^{J\pm\frac{1}{2}}}(k)]-1}{2i}=\frac{\pi^2 kE_k}{(2\pi)^4(E_k+\omega_k)}\int d\Omega_{\mathbf{k}}\int d\Omega_{\mathbf{k}'}$$
$$\times\phi^*_{I,n}\psi^*_{J,m,(-1)^{J\pm\frac{1}{2}}}(\mathbf{k}')\beta S\psi_{J,m,(-1)^{J\pm\frac{1}{2}}}(\mathbf{k})\phi_{I,n}. \quad (35)$$

Here $\phi_{I,n}$ is the isotopic spin function for total isotopic spin I and z component n; and the $\psi_{J,m,(-1)^{J\pm\frac{1}{2}}}$ are the normalized relativistic angular momentum functions for a spin 1/2 particle:

$$\psi_{J,m,(-1)^{J-\frac{1}{2}}}(\mathbf{k})=\frac{-\boldsymbol{\alpha}\cdot\mathbf{k}+\beta m+E_k}{[2E_k(E_k+m)]^{\frac{1}{2}}}\left\{\left(\frac{J+m}{2J}\right)^{\frac{1}{2}}Y_{J-\frac{1}{2}}^{m-\frac{1}{2}}(\mathbf{k})\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\left(\frac{J-m}{2J}\right)^{\frac{1}{2}}Y_{J-\frac{1}{2}}^{m+\frac{1}{2}}(\mathbf{k})\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\},$$

$$\psi_{J,m,(-1)^{J+\frac{1}{2}}}(\mathbf{k})=\frac{-\boldsymbol{\alpha}\cdot\mathbf{k}+\beta m+E_k}{[2E_k(E_k+m)]^{\frac{1}{2}}}\left\{\left[\frac{J+1-m}{2(J+1)}\right]^{\frac{1}{2}}Y_{J+\frac{1}{2}}^{m-\frac{1}{2}}(\mathbf{k})\begin{pmatrix}1\\0\\0\\0\end{pmatrix}-\left[\frac{J+1+m}{2(J+1)}\right]^{\frac{1}{2}}Y_{J+\frac{1}{2}}^{m+\frac{1}{2}}(\mathbf{k})\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}. \quad (36)$$

These are eigenfunctions of J^2 and J_z and positive-energy eigenfunctions of the Dirac Hamiltonian,

$$H=-\boldsymbol{\alpha}\cdot\mathbf{k}+\beta m. \quad (37)$$

The minus sign comes in because we have regarded $\mathbf{k}$ as the meson momentum in the center-of-mass system. After integration over the angles of $\mathbf{k}$ and $\mathbf{k}'$ in Eq. (35) we set $|\mathbf{k}|=|\mathbf{k}'|=k$. S is the S matrix calculated according to the Feynman-Dyson methods with meson propagation function,

$$D_F(k^2)=1/i(\mu^2+k^2), \quad (38)$$

nucleon propagation function

$$S_F(p)=1/(i\gamma\cdot p+m), \quad (39)$$

and interaction operator

$$ig\gamma_5\tau_\alpha. \quad (40)$$

In order to calculate diagrams (c) and (g) of Fig. 1, one must first find the renormalized nucleon propagation function to second order. This has been done by Brueckner, Gell-Mann, and Goldberger.[13] The result of their calculation is that to second order in g the renormalized propagation function is

$$S_{F'}'(p)=\frac{1}{i\gamma\cdot p+m}+\frac{1}{i\gamma\cdot p+m}\sum_r(p)\frac{1}{i\gamma\cdot p+m}, \quad (41)$$

[11] R. P. Feynman, Phys. Rev. **76**, 749 (1949) and **76**, 769 (1949); F. J. Dyson, Phys. Rev. **75**, 486 (1949) and **75**, 1736 (1949).

[12] F. J. Dyson, Phys. Rev. **75**, 1736 (1949); J. C. Ward, Phys. Rev. **84**, 897 (1951).

[13] Brueckner, Gell-Mann, and Goldberger, Phys. Rev. **90**, 476 (1953).

where

$$\sum\nolimits_r(p)=\frac{3g^2}{16\pi^2}\int_0^1 dx\bigg[\{(i\gamma\cdot p+m)(1-x)+mx\}\times\log\frac{p^2x(1-x)+m^2x+\mu^2(1-x)}{m^2x^2+\mu^2(1-x)}-(i\gamma\cdot p+m)\frac{2m^2x^2(1-x)}{m^2x^2+\mu^2(1-x)}\bigg]. \quad (42)$$

In order to calculate diagrams (d), (e), (h), (i), one must renormalize the vertex operator to second order in g. This has been done in the following way: If p, p' are the nucleon four-momenta to the left and right of the vertex, the vertex operator has the form

$$\Gamma_5(p,p')=\gamma_5+Y_5(p,p'). \quad (43)$$

On grounds of covariance $Y_5(p,p')$ can be written in the form[14]

$$Y_5(p,p')=F(p^2,\, p'^2,\, p\cdot p')\gamma_5+G(p^2,\, p'^2,\, p\cdot p')i\gamma\cdot p\gamma_5+H(p^2,\, p'^2,\, p\cdot p')\gamma_5 i\gamma\cdot p'+K(p^2,\, p'^2,\, p\cdot p')i\gamma\cdot p\gamma_5 i\gamma\cdot p'. \quad (44)$$

We renormalize by taking

$$\Gamma_{5r}(p,p')=\gamma_5+Y_5(p,p')-Y_5(p,p')|_{p=p',\, i\gamma\cdot p=-m,\, i\gamma\cdot p'=-m}. \quad (45)$$

The symbol $Y_5(p,p')|_{p=p',\, i\gamma\cdot p=-m,\, i\gamma\cdot p'=-m}$ means

$$Y_5(p,p')|_{p=p',\, i\gamma\cdot p=-m,\, i\gamma\cdot p'=-m}=F(-m^2,\,-m^2,\,-m^2)\gamma_5+G(-m^2,\,-m^2,\,-m^2)(-m)\gamma_5+H(-m^2,\,-m^2,\,-m^2)\times\gamma_5(-m)+K(-m^2,\,-m^2,\,-m^2)(-m)\gamma_5(-m). \quad (46)$$

Note carefully that $i\gamma\cdot p$ is commuted through to the left of γ_5 and $i\gamma\cdot p'$ to the right of γ_5 before they are evaluated at $(-m)$.[15] By using this procedure the renormalized vertex operator given below was calculated:

$$\Gamma_{5r}(p,p')=\gamma_5+\frac{g^2}{16\pi^2}\gamma_5\int_0^1 dx\int_0^1 xdy\bigg[\{\mu^2(1-x)+m^2(1+x)-i\gamma\cdot p i\gamma\cdot p'(1-x)+m(-i\gamma\cdot p+i\gamma\cdot p')\}\times\frac{1}{\Lambda(x,y)}-\frac{\mu^2(1-x)}{m^2x^2+\mu^2(1-x)}-2\log\frac{\Lambda(x,y)}{m^2x^2+\mu^2(1-x)}\bigg], \quad (47)$$

where

$$\Lambda(x,y)=\mu^2(1-x)+m^2x+p^2xy(1-x)+p'^2x(1-x)(1-y)+(p-p')^2x^2y(1-y).$$

The renormalization procedure given by Eqs. (45) and (46) is not the only possible one. It was chosen for a definite reason. Brueckner, Gell-Mann, and Goldberger showed in their paper[13] that at low energies for $P_{\frac{1}{2}}$ states the second order correction in Eq. (41) is of order $(g^2/4\pi)(\mu/m)$ times the unmodified nucleon propagation function for terms of the form of Eq. (23). For $S_{\frac{1}{2}}$ states, on the other hand, the second order correction in Eq. (41) is of order $g^2/4\pi$ times the unmodified nucleon propagation function. For other states the scattering due to terms of the form of Eq. (23) vanishes. These statements can be generalized. It can be shown by an argument similar to that to be given below for the vertex operator that for $P_{\frac{1}{2}}$ states all of the radiative corrections in the complete renormalized nucleon propagation function are smaller by one power of μ/m than the unmodified propagation function for terms of the form of Eq. (23). The same situation obtains for the vertex operator with the form of renormalization defined by Eqs. (45) and (46). This is not difficult to see in general. Suppose we consider the vertex operator $\Gamma_{5r}(P,\, P-k)$ of Eq. (23). Then $P-k$ is a free nucleon momentum, so that $i\gamma\cdot(P-k)=-m$, $(P-k)^2=-m^2$. For scattering at zero energy, $P=[0,0,0,i(m+\mu)]$ in the center-of-mass system. At low energies Eq. (36) gives for the $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ wave functions:

$$\psi_{S\frac{1}{2}}(\mathbf{k})=\frac{1}{(4\pi)^{\frac{1}{2}}}\begin{bmatrix}1\\0\\0\\0\end{bmatrix},$$

$$\psi_{P\frac{1}{2}}(\mathbf{k})=\left(\frac{-\boldsymbol{\alpha}\cdot\mathbf{k}}{k}+\frac{k}{2m}\right)\frac{1}{(4\pi)^{\frac{1}{2}}}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}. \quad (48)$$

For the part of the S matrix in Eq. (23) the term involving $\boldsymbol{\alpha}\cdot\mathbf{k}$ drops out on integration over the angles of $\mathbf{k}$ to get a phase shift. Using these facts, we get for the $P_{\frac{1}{2}}$ state

$$\Gamma_{5r}(P,\, P-k)\to\gamma_5+F(-(m+\mu)^2,\,-m^2,\,-m(m+\mu))\gamma_5+G(-(m+\mu)^2,\,-m^2,\,-m(m+\mu))(-(m+\mu))\gamma_5+H(-(m+\mu)^2,\,-m^2,\,-m(m+\mu))\gamma_5(-m)+K(-(m+\mu)^2,\,-m^2,\,-m(m+\mu))(-(m+\mu))\times\gamma_5(-m)-Y_5(p,p')|_{p=p',\, i\gamma\cdot p=-m,\, i\gamma\cdot p'=-m}, \quad (49)$$

where the last term is given by Eq. (46). If we expand Eq. (49) in powers of μ/m, it becomes apparent that

[14] The invariance of the theory with respect to charge conjugation implies that $F(p^2,p'^2,p\cdot p')=F(p'^2,p^2,p'\cdot p)$, $K(p^2,p'^2,p\cdot p')=K(p'^2,p^2,p'\cdot p)$, $G(p^2,p'^2,p\cdot p')=H(p'^2,p^2,p'\cdot p)$. We will not need these relations for our purposes, however.

[15] This is the same renormalization prescription as used by N. Kroll and M. Ruderman, Phys. Rev. 93, 233 (1954).

the correction term to γ_5 is of order $(\mu/m)\gamma_5$. For the $S_{\frac{1}{2}}$ state $i\gamma\cdot P\rightarrow m+\mu$, instead of $-(m+\mu)$, and the correction term to γ_5 is of order γ_5. So the renormalization procedure defined by Eqs. (45) and (46) is a good one for $P_{\frac{1}{2}}$ states in the sense that it tends to make the radiative corrections small. Note that with the form of renormalization used here the whole term Eq. (23) reduces to lowest order in μ/m to just the second order contribution represented by diagram (a) of Fig. 1 for P-state scattering. This is exactly the same situation as we encountered in Chew's theory.

The phase shift calculations have been done only for very low energies. Only the lowest order terms in an expansion in powers of k have been kept, so that the S-wave phase shifts vary as k and the P-wave phase shifts as k^3. We shall signify the contributions of various diagrams by symbols such as $\delta_{11}(a)$. Here the (a) refers to diagram (a) in Fig. 1. Also we let $\alpha=\mu/m$, the ratio of the mass of the meson to the mass of the nucleon. The second order phase shifts for diagram (a) are

$$\delta_1(a)=-3\frac{g^2}{4\pi}\frac{\alpha}{(1+\alpha)(2+\alpha)}\left(\frac{k}{\mu}\right),$$
$$\delta_{11}(a)=-\frac{3}{4}\frac{g^2}{4\pi}\frac{\alpha^2}{1+\alpha}\left(\frac{k}{\mu}\right)^3. \qquad (50)$$

All other phase shifts vanish for diagram (a). For diagram (b) we find

$$\delta_1(b)=\frac{g^2}{4\pi}\frac{\alpha}{(1+\alpha)(2-\alpha)}\left(\frac{k}{\mu}\right),$$
$$\delta_3(b)=-2\frac{g^2}{4\pi}\frac{\alpha}{(1+\alpha)(2-\alpha)}\left(\frac{k}{\mu}\right),$$
$$\delta_{11}(b)=\frac{1}{12}\frac{g^2}{4\pi}\frac{\alpha^2(4-3\alpha^2)}{(1+\alpha)(2-\alpha)^2}\left(\frac{k}{\mu}\right)^3,$$
$$\delta_{31}(b)=-\frac{1}{6}\frac{g^2}{4\pi}\frac{\alpha^2(4-3\alpha^2)}{(1+\alpha)(2-\alpha)^2}\left(\frac{k}{\mu}\right)^3, \qquad (51)$$
$$\delta_{13}(b)=-\frac{2}{3}\frac{g^2}{4\pi}\frac{\alpha^2}{(1+\alpha)(2-\alpha)^2}\left(\frac{k}{\mu}\right)^3,$$
$$\delta_{33}(b)=\frac{4}{3}\frac{g^2}{4\pi}\frac{\alpha^2}{(1+\alpha)(2-\alpha)^2}\left(\frac{k}{\mu}\right)^3.$$

The fourth order phase shifts have been calculated analytically also, but the formulas are much too long to reproduce here. In Table II are given the lowest order terms in an expansion in powers of μ/m of the analytical formulas. In Table III the formulas in Table II have been evaluated numerically for $\alpha=0.147$. In Table IV the results of evaluating the complete analytical formulas for the phase shifts for $\alpha=0.147$ are tabu-

TABLE II. Lowest order terms in expansion in powers of μ/m of the fourth order phase shifts for the pseudoscalar coupling theory.

	δ_1 $1/\pi(g^2/4\pi)^2(\mu/m)(k/\mu)$	δ_3 $1/\pi(g^2/4\pi)^2(\mu/m)(k/\mu)$	δ_{11} $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	δ_{31} $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	δ_{13} $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	δ_{33} $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$
(c)	$\frac{9}{8}$		$\frac{9}{32}$			
(d)=(e)	$-\frac{3}{4}$		$-\frac{\alpha}{16}\left(\log\frac{1}{\alpha}-\frac{1}{6}-\frac{\pi}{2}\right)$			
(f)	$\frac{1}{4}$	1	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{9}$
(g)	$-\frac{3}{8}$	$\frac{3}{4}$	$\frac{1}{4}\left(\log\frac{1}{\alpha}-\frac{15}{8}\right)$	$-\frac{1}{2}\left(\log\frac{1}{\alpha}-\frac{15}{8}\right)$	$\frac{1}{4}\left(\log\frac{1}{\alpha}-3\right)$	$-\frac{1}{2}\left(\log\frac{1}{\alpha}-3\right)$
(h)=(i)	$\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{12}\left(\log\frac{1}{\alpha}-1\right)$	$\frac{1}{6}\left(\log\frac{1}{\alpha}-1\right)$	$-\frac{1}{12}\left(\log\frac{1}{\alpha}-\frac{5}{2}\right)$	$\frac{1}{6}\left(\log\frac{1}{\alpha}-\frac{5}{2}\right)$
(j)	$\frac{5}{4}$	$\frac{1}{2}$	$-\frac{5}{12}\left(\log\frac{1}{\alpha}-\frac{1}{3}\right)$	$-\frac{1}{6}\left(\log\frac{1}{\alpha}-\frac{1}{3}\right)$	$-\frac{5}{12}\left(\log\frac{1}{\alpha}-\frac{11}{6}\right)$	$-\frac{1}{6}\left(\log\frac{1}{\alpha}-\frac{11}{6}\right)$

TABLE III. The formulas of Table II evaluated numerically with $\alpha=0.147$.

	δ_1	δ_3	δ_{11}	δ_{31}	δ_{13}	δ_{33}
	$1/\pi(g^2/4\pi)^2(\mu/m)(k/\mu)$	$1/\pi(g^2/4\pi)^2(\mu/m)(k/\mu)$	$1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	$1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	$1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	$1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$
(c)	1.13		0.281			
$(d)=(e)$	−0.75		−0.0017			
(f)	0.25	1.00	0.028	0.111	0.028	0.111
(g)	−0.38	0.75	0.01	−0.02	−0.27	0.54
$(h)=(i)$	0.25	−0.50	−0.077	0.154	0.048	−0.096
(j)	1.25	0.50	−0.663	−0.265	−0.04	−0.014

TABLE IV. The rigorous fourth order phase shifts for the pseudoscalar coupling theory.

	δ_1	δ_3	δ_{11}	δ_{31}	δ_{13}	δ_{33}
	$1/\pi(g^2/4\pi)^2(\mu/m)(k/\mu)$	$1/\pi(g^2/4\pi)^2(\mu/m)(k/\mu)$	$1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	$1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	$1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	$1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$
(c)	1.36		0.230			
$(d)=(e)$	−0.83		−0.0078			
(f)	0.32	1.27	0.026	0.103	0.027	0.108
(g)	−0.17	0.33	0.202	−0.404	0.057	−0.114
$(h)=(i)$	0.15	−0.30	−0.112	0.224	−0.035	0.070
(j)	0.64	0.26	−0.731	−0.293	−0.317	−0.127

lated. Comparison of Tables III and IV indicates that for P states the approximation of expanding in powers of μ/m and keeping only the lowest order terms is a very bad approximation. Note that the phase shifts $\delta_{13}(g)$, $\delta_{33}(g)$, $\delta_{13}(h)=\delta_{13}(i)$, and $\delta_{33}(h)=\delta_{33}(i)$ actually change sign in going from Table III to Table IV. For the S states the agreement between Tables III and IV is considerably better. Observe that the phase shift $\delta_{11}(d)=\delta_{11}(e)$ shows a somewhat anomalous behavior in Table II, being smaller by one power of α than the other phase shifts. An analysis of the differences between the results of the relativistic theory given in Tables II, III, and IV and the results of Chew's theory will be given in Sec. IV.

Examination of Tables II, III, and IV reveals a number of interesting points with respect to the possible validity of the TD approximation applied to the relativistic theory. Table II shows that for P states the diagrams with three meson in the field at a time, i.e., the diagrams which are not included in the TD approximation, all contain to lowest order in α terms varying as $\log(1/\alpha)$. On the other hand, the TD diagrams have no logarithmic terms to lowest order in α. If α were very very small, the diagrams ignored by the TD approximation would make much larger contributions to the phase shifts than the diagrams retained. Of course $\alpha=0.147$ is not very small. Consider then Table IV, which gives the rigourous results of the pseudoscalar coupling theory with $\alpha=0.147$. Except for a few cases in the S-state scattering, the terms ignored by the TD approximation are as large as or larger than the corresponding terms kept. If one adds up the figures in the columns of Table IV to get the total fourth order contributions to the phase shifts, one finds

$$\begin{aligned}
\delta_1{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu}{m}\right)\left(\frac{k}{\mu}\right)(0.02+0.77),\\
\delta_3{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu}{m}\right)\left(\frac{k}{\mu}\right)(1.27-0.01),\\
\delta_{11}{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu}{m}\right)^3\left(\frac{k}{\mu}\right)^3(0.240-0.753),\\
\delta_{31}{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu}{m}\right)^3\left(\frac{k}{\mu}\right)^3(0.103-0.249),\\
\delta_{13}{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu}{m}\right)^3\left(\frac{k}{\mu}\right)^3(0.027-0.330),\\
\delta_{33}{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu}{m}\right)^3\left(\frac{k}{\mu}\right)^3(0.108-0.101).
\end{aligned}\tag{52}$$

In each case here the first number in parentheses gives the sum of the contributions from the TD diagrams, and the second number gives the sum of the contribution from the three meson diagrams.

It is quite apparent from Eqs. (52) that one cannot justify the use of the TD approximation in the pseudoscalar coupling theory by the sort of argument Chew has used to justify the TD method for the static pseudovector coupling theory with a small cutoff. Comparison of Tables I and IV and Eqs. (27) and (52) indicates that the three meson diagrams make a relatively much larger contribution to the fourth order phase shifts in the pseudoscalar coupling theory than in the static pseudovector coupling theory, even when a large cutoff $K=m$ is used in the latter.

It will be recalled that in Chew's theory the TD approximation could be regarded as a large μ/m approximation, i.e., an approximation which became better and better as $\mu/m=\mu/K$ increased. In order to ascertain whether or not this was also the case for the relativistic pseudoscalar coupling theory, the fourth order phase shifts were computed for the relativistic theory in the case $\alpha=\mu/m=1$. We find

$$\begin{aligned}
\delta_1{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{k}{\mu}\right)(-0.14+0.04),\\
\delta_3{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{k}{\mu}\right)(1.72-0.20),\\
\delta_{11}{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{k}{\mu}\right)^3(0.052-0.247),\\
\delta_{31}{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{k}{\mu}\right)^3(0.060-0.044),\\
\delta_{13}{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{k}{\mu}\right)^3(0.022-0.087),\\
\delta_{33}{}^{(4)}&=\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{k}{\mu}\right)^3(0.088+0.020),
\end{aligned}\tag{53}$$

where the same conventions are used as in Eq. (52). Comparison of Eqs. (52) and (53) indicates that increasing the value of α tends to make the three meson terms less important, although the effect is certainly not as pronounced as in Chew's theory.

The situation with respect to the validity of the TD method looks somewhat better if we examine this approximation from the point of view that it represents an expansion of the potential energy in powers of g. The ratios of the zero-energy matrix elements of the fourth and second order potentials can be easily calculated from Eqs. (50), (51), and (52), and we obtain,

for $g^2/4\pi=10$,

$$\begin{aligned}
&(V_4/V_2)_1=5.2,\\
&(V_4/V_2)_3=0.034,\\
&(V_4/V_2)_{11}=-4.2,^{16}\\
&(V_4/V_2)_{31}=0.73,\\
&(V_4/V_2)_{13}=0.91,\\
&(V_4/V_2)_{33}=-0.14.
\end{aligned} \tag{54}$$

Comparison of these figures with Eqs. (34) indicates that the ratios (V_4/V_2) are larger for the relativistic theory than they are for Chew's theory. Even so, the figures given in Eqs. (54), taken by themselves, are rather encouraging for the prospects of the TD method for P-wave scattering, especially for the most important state (33). However, it should be recalled that in Chew's theory the ratios (V_4/V_2) were increasing functions of the energy, and this may well be the case for the relativistic theory also. Moreover, as we shall show below, the ratios of the sixth order potentials to the fourth order potentials must be expected to be larger than the ratios V_4/V_2 by one power of m/μ.

Comparison of Eqs. (50) and (51) with Table II indicates that for P states the ratio of the fourth order phase shifts to the second order phase shifts is of order $(g^2/4\pi)(\mu/m)$ except for the $\log(1/\alpha)$ terms. For S states this ratio is of order $g^2/4\pi$. The situation here is almost identical with that in the non-relativistic approximation to the pseudoscalar coupling theory discussed in Sec. II. We have already proved quite generally—see the discussion following Eq. (47)—that for all class one diagrams, represented in Eq. (23), the radiative corrections are smaller by one power of μ/m than the irreducible diagram (a) of Fig. 1. Actually, according to Table II, the second order correction to the vertex operator of diagrams (d) and (e) is a somewhat anomalous case in that it is smaller than the primitive vertex operator γ_5 by a factor $(\mu/m)^2$. It is not easy to see how to prove generally and rigorously a corresponding theorem for the diagrams of class two, although the fourth order results of Table II indicate that among the diagrams of class two the radiative corrections are smaller except for logarithmic factors by one power of μ/m than the contribution of the second order diagram (b) of Fig. 1. Just as for Chew's theory, comparison of the second and fourth order results might lead one to hope that the expansion parameter of perturbation theory is $(g^2/4\pi)(\mu/m)$ rather $g^2/4\pi$ for P states, i.e., that there is an effective coupling constant $(g^2/4\pi)(\mu/m)$. This point was checked by calculating the sixth-order iteration of diagram (b) of Fig. 1 to lowest order in μ/m for a $P_{\frac{3}{2}}$ state. The phase shift turned out to have the form

$$\delta=N(g^2/4\pi)^6\alpha^3(k/\mu)^3,$$

where N was a nonvanishing number independent of α. Comparison with Table II indicates that there is no effective coupling constant. It would seem that, just as for the non-relativistic theory, the second order diagrams make anomalously large contributions by one power of m/μ for P-wave scattering at low energies. These remarks have the same implications for our test of the TD method according to the idea that this method is an expansion of the potential in powers of g as they did in the case of Chew's theory. The ratios in Eq. (54) must be regarded as anomalously small by one power of μ/m, since they contain the factor μ/m which appears only in going from second order to fourth order.

Gell-Mann and Goldberger[17] have proved an interesting theorem in connection with the S-state scattering. They have shown that to lowest order in expansion in powers of μ/m the low-energy S-state scattering is isotopic spin independent to all orders in the coupling constant g. That this is true to fourth order can be seen from Eqs. (50), (51), and Table II. We find, to lowest order in μ/m,

$$\delta_1=\delta_3=-\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{m}\right)\left(\frac{k}{\mu}\right)\left[1-\frac{5}{4\pi}\frac{g^2}{4\pi}\right]. \tag{55}$$

The effect on this theorem to fourth order in g of the μ/m corrections can be determined from Eqs. (50), (51), and Table IV. We find including the μ/m corrections

$$\begin{aligned}
\delta_1&=-\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{m}\right)\left(\frac{k}{\mu}\right)\left[0.75-\frac{0.79}{\pi}\frac{g^2}{4\pi}\right],\\
\delta_3&=-\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{m}\right)\left(\frac{k}{\mu}\right)\left[0.94-\frac{1.26}{\pi}\frac{g^2}{4\pi}\right].
\end{aligned} \tag{56}$$

The μ/m corrections lead to a slight isotopic spin dependence. However, to fourth order at least the μ/m corrections are not anomalously large. Compare, for example, the small μ/m corrections in going from Eq. (55) to Eq. (56) with the large μ/m corrections in going from the P wave scattering in Table III to the P wave scattering in Table IV. This would seem to indicate that the theorem of Gell-Mann and Goldberger is a meaningful theorem, which must be taken seriously even though it has been proved only to lowest order in μ/m. The indications are that the low-energy S-wave scattering is approximately isotopic-spin-independent. Gell-Mann and Goldberger's theorem is proved by balancing the contribution of a given diagram in which the incoming and outgoing meson lines are not crossed against the contribution of the corresponding diagram in which the external meson lines are crossed. Referring to Table II, we see that diagrams (c) and (g) taken together give an isotopic-spin-independent contribution, and similarly for diagrams (d) and (h), etc. If the TD approximation is used, this isotopic spin inde-

[16] If the sum of the contributions of diagrams (a) and (b) is used to determine the second order potential energy instead of just the contribution of diagram (b) alone, we find $(V_4/V_2)_{11}=0.53$, instead of the value given in Eq. (54).

[17] M. Gell-Mann and M. L. Goldberger (unpublished).

pendence of the theory will be destroyed. The total contribution of all the TD diagrams in Table II, for example, is not isotopic-spin-independent. Since the indications are that the pseudoscalar coupling theory gives an S-state scattering which is approximately isotopic-spin-independent, while on the other hand the TD approximation destroys this isotopic spin independence, there would seem to be very little justification for using this approximation to discuss the S-state scattering. Equations (52) and (54) also indicate that there is little hope for the TD approximation in the case of the S-wave scattering.

IV. INTERPRETATION OF RESULTS

Since the Hamiltonian (5) was derived, using the Foldy-Wouthuysen transformation, as an approximation to the Hamiltonian of the complete relativistic pseudoscalar coupling theory, we would expect *a priori* that the results of the calculations in Sec. II based on the Hamiltonian (5) should bear a close relationship to the results of the calculations in Sec. III based on the complete relativistic theory. It will be the purpose of this section to analyze, partially at least, the similarities and differences between these two sets of calculations. Some of the differences between these two theories arise from complicated effects of one sort or another which appear as higher order μ/m corrections in the calculations based on the relativistic theory. It is to be expected that many of these effects would not appear in the calculations based on the Hamiltonian (5), because this Hamiltonian was obtained by a Foldy-Wouthuysen transformation carried out only to order $1/m$. On the other hand, some higher order μ/m corrections, e.g., the difference between Δ_+ and Δ_-, are retained in the transformed Hamiltonian (5). We shall concentrate on trying to understand the differences between these two theories to lowest order in μ/m.

Consider first the S-wave scattering. The results of the relativistic theory are given in Eq. (55). The results of the non-relativistic theory are given in Eq. (33). Comparing Eq. (55) with Eq. (33), we see that to second order in g the phase shifts are identical and that in fourth order they differ only very slightly. Since the coefficient of the fourth order correction in Eq. (33) depends on the precise value of the cutoff, this was to be expected. The simplified Hamiltonian (5) evidently gives a pretty fair approximation to the S-wave scattering to fourth order in the relativistic theory. In the relativistic theory the only isotopic spin dependence arises from μ/m corrections. In the nonrelativistic theory the isotopic spin dependence can arise only from higher order terms in the Foldy-Wouthuysen transformation than those we have kept.

The similarity between the two theories does not extend to the P-wave scattering. Indeed, to second order in g the two theories agree according to Eqs. (27), (50), and (51). However, comparison of Tables II, III, IV with Table I indicates that there are drastic differences between the predictions of the two theories with respect to the fourth order P-wave scattering. The most obvious difference is the appearance of the $\log(1/\alpha)$ terms in Table II. According to Eq. (31) there are no logarithm terms in Chew's theory to lowest order in μ/m. Also, all of the P-wave phase shifts due to diagram (j) of Fig. 1. are negative in the relativistic theory, whereas they are positive in Chew's theory. According to Table II the $\log(1/\alpha)$ terms determine the signs of these phase shifts, so that if we can find the origion of these logarithmic terms, we will presumably understand the reason for the sign change in going from Chew's theory to the relativistic theory. Note that all of the logarithmic terms in Table II are independent of whether the state involved is $P_{\frac{1}{2}}$ or $P_{\frac{3}{2}}$ as long as the isotopic spin is the same. This suggests that the logarithmic terms come from a spin independent recoil effect.

In an attempt to understand some of these effects, some calculations were made with Eqs. (11) and (18) derived in the Introduction. It will be recalled that the interaction terms of Eqs. (11) and (18) are relativistic generalizations of the interaction terms $\boldsymbol{\sigma}\cdot\nabla\tau_\alpha\phi_\alpha$ and $\phi_\alpha^2/2m$ of the Hamiltonian (5). With the non-covariant approach of Eqs. (11) and (18) it is easier to separate the calculations into terms which have a simple physical significance than with the Feynman-Dyson methods used in Sec. III. The phase shift $\delta_{L,J,I}(k)$ at momentum k for the state with orbital angular momentum L, total angular momentum J, and isotopic spin I is related to an element of the reaction matrix K calculated from Eqs. (11) or (18) by the formula

$$\tan\delta_{L,J,I}(k)=\frac{-\pi k}{(2\pi)^3}\frac{E_k\omega_k}{E_k+\omega_k}\int d\Omega_{\mathbf{k}'}\int d\Omega_{\mathbf{k}} \times\phi^*_{I,n}\psi^*_{L,J,m}(\mathbf{k}')K\psi_{L,J,m}(\mathbf{k})\phi_{I,n}, \quad (57)$$

where the $\psi_{L,J,m}$ are non-relativistic angular momentum functions.

Consider first the relativistic generalization of Chew's theory, Eq. (11). Recall that this theory was derived from the complete relativistic theory with the single approximation that all terms involving the production of pairs were dropped. The S-wave scattering vanishes in this theory in the approximation that we keep only terms proportional to k in the S-wave phase shifts. The second order P-wave phase shifts agree to lowest order in μ/m with those given by the relativistic theory in Eqs. (50) and (51). The fourth order P-wave scattering has been calculated to lowest order in μ/m and is given in Table V. For comparison purposes the results of Chew's theory to lowest order in μ/m are also given in Table V. Table V should be compared with Tables II and III, which give the results of the complete relativistic theory, which includes the effects of pair production. It will be observed that the $\log(1/\alpha)$ terms

TABLE V. The top numbers in each row give the fourth order P-wave scattering calculated to lowest order in μ/m from Eq. (11). The bottom numbers in each row are the results of Chew's theory to lowest order in μ/m for cutoff $K=m$.

	δ_{11} / $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	δ_{31} / $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	δ_{13} / $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	δ_{33} / $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$
(c)	0.422 0.563			
$(d)=(e)$	−0.030 −0.063			
(f)	0.0041 0.0069	0.0164 0.028	0.012 0.028	0.048 0.111
(g)	0.047 0.063	−0.094 −0.125	−0.094 −0.125	0.188 0.250
$(h)=(l)$	−0.0033 −0.0069	0.0067 0.014	0.0067 0.014	−0.0134 −0.028
(j)	0.073 0.174	0.029 0.070	0.033 0.070	0.013 0.028

have all disappeared in Table V. These terms are associated with pair production in the relativistic theory. Examination of Table V indicates that that part of the relativistic theory left when pair production phenomena are eliminated shows much the same behavior as Chew's theory except that the results of the calculations based on Eq. (11) seem to be smaller than the results of Chew's theory by a factor of the order of 2. As explained below, this factor of 2 arises from the differences in the cutoff procedures in the two theories. There has been some speculation recently, based on the results of a calculation of the S-wave scattering in the non-relativistic theory (5) by Wentzel,[18] that the terms in the relativistic theory involving pair production may be highly damped by higher order radiative corrections. If this should turn out to be true, the results of Table V would indicate that the relativistic theory and Chew's theory should show much the same behavior for the P-wave scattering calculated so as to include the radiative corrections. It must be emphasized, however, that the proposed damping of the effects of pair production can only be speculated at present, particularly in the case of P-wave scattering.

The vertex operator $\gamma_5(\mathbf{p},\mathbf{p}')$, Eq. (12), of this theory from which the pair effects have been eliminated does not in general have quite the same form as the vertex operator of Chew's theory. However, examination of the cases which arise in calculating the low-energy P-wave scattering in fourth order shows that for these cases the vertex operator $\gamma_5(\mathbf{p},\mathbf{p}')$ reduces approximately to an operator which does have the same form as Chew's vertex operator except for the presence of momentum-dependent cutoff factors. It is the differences between the cutoff procedures in the two cases which give rise to the almost constant ratio between the results of calculations based on Eq. (11) and the results of Chew's theory as given in Table V. If a cutoff smaller than m were used in calculating with Chew's theory, the results of his theory could be brought into closer agreement with the calculations based on Eq. (11). It is perhaps significant that Chew actually uses a cutoff somewhat smaller than m in fitting his theory to the experiments.[19]

The renormalization of the vertex operator and of the nucleon propagation function which was necessary in order to carry out the calculations reported in Table V was done in almost exactly the same way as Chew did the renormalization of his theory.[10] Only one difficulty arises. On writing out Chew's renormalization scheme for Eq. (11) it appears at first sight that the self-mass and the renormalization constants Z_1 and Z_2 depend on the momenta. On closer examination it is seen that these quantities are all given by divergent integrals, the most divergent parts of which are independent of the momenta. This is perhaps not very satisfactory, but it is probably the best that can be done in the way of renormalizing Eq. (11).

We have now come to the conclusion that the very large differences between the low-energy P-wave scattering in the relativistic pseudoscalar coupling theory and in Chew's theory arise from the intrusion of effects due to the production of pairs. This situation is to be contrasted with that in the non-relativistic theory based on the Hamiltonian (5). There the pair term did not enter at all in calculating the fourth-order P-wave scattering. For illustrative purposes we shall now report the results of some calculations with Eq. (18). It will be recalled that the interaction term of Eq. (18) is a relativistic generalization of the pair term $\phi_\alpha^2/2m$ of the non-relativistic theory. In deriving Eq. (18) only the effects due to the production of one pair at a time were kept. No attempt was made to renormalize the calculations based on Eq. (18), so only the results for the finite diagrams will be reported in fourth order.

[18] G. Wentzel, Phys. Rev. 86, 802 (1952).

[19] G. F. Chew, Phys. Rev. 95, 285 (1954).

In second order the S-wave scattering calculated from Eq. (11) agrees to lowest order in μ/m with the S-wave scattering calculated in the complete relativistic theory, as given in Eqs. (50) and (51). The second order P-wave phase shifts are smaller by one power of μ/m than those calculated from Eq. (11). In fourth order, however, the pair-forming part of the interaction contributes to the P-wave scattering to the same order in μ/m as that part of the interaction in which no pairs are formed. In Table VI will be found the results of some fourth order phase shift calculations with Eq. (18) for the finite diagrams. These calculations have been done only to lowest order in μ/m. The integrals giving the contributions of diagram (l) in Fig. 2 were very complicated, and only the coefficients of the $\log(1/\alpha)$ terms were computed. A numerical term independent of α must be added to the $\log(1/\alpha)$ terms in Table VI in order to get the whole contribution to lowest order in μ/m for diagram (l). Diagrams (m) and (n) in Fig. 2 represent a cross-product term between the interaction terms of Eqs. (11) and (18). Comparison of Tables V and VI shows that terms involving the production of pairs make large contributions to the P-wave scattering in fourth order and cannot be neglected in a perturbation-theoretic calculation. Comparison of Tables VI and II shows that the coefficients of the $\log(1/\alpha)$ terms for diagram (l) in Table VI are the same as the coefficients of the $\log(1/\alpha)$ terms for diagram (j) in Table II. For the finite diagram (j) the $\log(1/\alpha)$ terms in the relativistic theory all come from that component of the relativistic Feynman diagram (j) which is represented by diagram (l) of Fig. 2.

If one writes down according to noncovariant methods like those used in Eqs. (11) and (18) all the various components which are collected together in the one relativistic Feynman diagram (j), one can locate all of the log $(1/\alpha)$ terms simply by investigating whether or not the integrals diverge as $\alpha\rightarrow 0$. An investigation of this sort indicates that one can get a $\log(1/\alpha)$ term only from that component of diagram (j) which is represented by diagram (l) of Fig. 2. Even in this term the $\log(1/\alpha)$ comes from a recoil effect. If the initial and final mesons have momenta $\mathbf{k}$, $\mathbf{k}'$ and the virtual meson has momentum $\mathbf{q}$ in diagram (l), the energy denominator after the emission of the virtual meson is

$$\frac{1}{E-E_{\mathbf{k}+\mathbf{k}'+\mathbf{q}}-\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{q}}}=\frac{1}{E-E_{\mathbf{q}}-2\mu-\omega_{\mathbf{q}}}\times\left[1+\frac{\mathbf{k}\cdot\mathbf{k}'}{E_{\mathbf{q}}(E-E_{\mathbf{q}}-2\mu-\omega_{\mathbf{q}})}+\cdots\right]. \quad (58)$$

The term involving $\mathbf{k}\cdot\mathbf{k}'$ in Eq. (58) leads to the $\log(1/\alpha)$ terms in the phase shifts. In diagram (k) of Fig. 2, where the initial and final meson lines are not crossed, the $\mathbf{k}\cdot\mathbf{k}'$ term of Eq. (58) is absent, and hence there are no $\log(1/\alpha)$ terms. Examination of the details of the calculations shows that the virtual meson is emitted and absorbed as an S-wave meson in the case of this term which leads to the $\log(1/\alpha)$ terms in the phase shifts. Presumably the $\log(1/\alpha)$ terms which appear in the contributions from the fourth order diagrams which must be renormalized arise from recoil effects similar to those which caused the appearance of the $\log(1/\alpha)$ terms in the finite diagram.

As one final point, note the excellent agreement between the S-wave scattering terms as far as they have been calculated in Table VI and the corresponding terms in Table II. Apparently the contributions to the S-wave scattering of diagrams in which a second pair is produced before the first pair is annihilated are quite small. In other words, the contributions of the term $(\phi_\alpha^2/2m)^2$, obtained when the Foldy-Wouthuysen transformation is carried out to order $1/m^2$, are negligibly small.

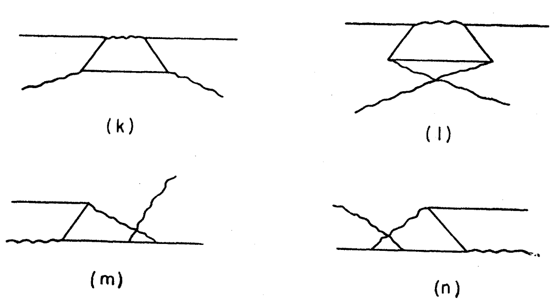

FIG. 2. Some scattering diagrams, the contributions of which are given in Table VI. In this figure a nucleon line going backward indicates the production of a pair.

V. CONCLUSIONS AND COMPARISON WITH EXPERIMENT

In this section we want to summarize our arguments and discuss the relation of our calculations to the experiments on meson-nucleon scattering. When the coupling constant $g^2/4\pi$ is as large as 10, the results of a perturbation theory calculation cannot be compared directly with experiment. However, we can speculate a little on what our fourth order results indicate about the nature of a more complete calculation and how the results of this more complete calculation might agree or disagree with the experiments.

We consider first the S-wave scattering. We have presented arguments in Sec. IV to the effect that the S-wave scattering in the complete relativistic pseudoscalar coupling theory is probably well approximated by the non-relativistic approximation to the relativistic theory obtained by performing the Foldy-Wouthuysen transformation to order $1/m$ only. In the non-relativistic theory the S-wave scattering arises from a pair term $\phi_\alpha^2/2m$ in the Hamiltonian. A higher order term $(\phi_\alpha^2/2m)^2$ in the Foldy-Wouthuysen transformed Hamiltonian seems to be negligible. The S-wave scattering in

the non-relativistic theory is isotopic-spin-independent. According to the arguments of Sec. III, this is probably approximately true for the relativistic theory also. Insofar as these arguments are valid, there is disagreement between the theory and the experiments for S-wave scattering. Experimentally, at low energies δ_1 is large and positive and δ_3 is very small and probably negative according to the latest analysis.[20] The phase shifts given by Wentzel's calculations[18] with the pair term including the damping have about the right magnitude, but the phase shifts are both negative and equal. Furthermore, the energy dependence predicted by Wentzel's calculations is in disagreement with the experiments. It is conceivable that some queer effect due to the μ/m corrections in higher-order processes, perhaps the meson-meson interaction, could lead to the observed isotopic spin dependence, but there is no indication of this to fourth order.

In any event the arguments given at the end of Sec. III make it appear unlikely that the TD approximation will be a useful one in discussing the S-wave scattering.

The situation for the P-wave scattering is somewhat more complicated. The calculations of Secs. II and III show that the nonrelativistic theory gives a very bad approximation to the P-wave scattering of the relativistic theory in fourth order. One might be inclined to infer from this that a rigorous calculation of the P-wave scattering in the relativistic theory to all orders in the coupling would show no relation to a corresponding calculation in Chew's theory. However, we have shown in Sec. IV that the differences between the relativistic theory and Chew's theory in fourth order arise from the pair effects which are present in the relativistic theory. With a suitably chosen cutoff, Chew's theory can be brought into quite good agreement with that part of the relativistic theory left when the production of pairs is eliminated. Now it may be, as we have mentioned before, that when the calculation is carried out to all orders in $g^2/4\pi$ the effects due to the production of pairs are highly damped. If this happens, Chew's theory may give a reasonably good description of the P-wave scattering in the relativistic theory.

As for the TD approximation, the calculations of Sec. III indicate that this method is of somewhat dubious validity as applied to the complete relativistic theory. On the other hand, if the pair effects are damped out by higher order processes, the relativistic theory is reduced essentially to Chew's theory, for which the TD method has a much greater chance of success. A TD calculation in the relativistic theory including pair effects might indeed show this damping of the pair effects if it exists.

Experimentally δ_{33} is large and positive and may go through 90° in the neighborhood of 200 Mev. The other

[20] H. A. Bethe and F. de Hoffman, *Meson Fields* (Row, Peterson, and Company, Evanston, to be published), Vol. II.

TABLE VI. Some phase shifts calculated according to Eqs. (11) and (18). The letters in the first column refer to the diagrams in Fig. 2.

	δ_1 $1/\pi(g^2/4\pi)^2(\mu/m)(k/\mu)$	δ_3 $1/\pi(g^2/4\pi)^2(\mu/m)(k/\mu)$	δ_{11} $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/m)^3$	δ_{31} $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	δ_{13} $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$	δ_{33} $1/\pi(g^2/4\pi)^2(\mu/m)^3(k/\mu)^3$
(*k*)	0.24	0.96	0.010	0.040	0.0033	0.0132
(*l*)	1.20	0.48	$-\frac{5}{12}\log\frac{1}{\alpha}$	$-\frac{1}{6}\log\frac{1}{\alpha}$	$-\frac{5}{12}\log\frac{1}{\alpha}$	$-\frac{1}{6}\log\frac{1}{\alpha}$
(*m*)=(*n*)			0.0038	0.0142	0.0048	0.0192

P-wave phase shifts seem to be very small.[20] The fourth order phase shifts in Chew's theory give some indication that this sort of behavior might be predicted by a more complete calculation, the fourth order corrections reinforcing the second order phase shift for δ_{33} and detracting from it for δ_{11} and $\delta_{13}=\delta_{31}$. And, indeed, a TD calculation by Chew[19] has succeeded in reproducing the experimental phase shifts as far as they are known. The results of the calculations for the relativistic theory given in Eq. (52) are not nearly so encouraging. For the relativistic theory the fourth order correction to δ_{33} is very very small. If we invoke the damping of the pair effects again, the phase shifts in the relativistic theory will reduce essentially to those given by Chew's theory, and things will look more encouraging.

The chances both for the success of the TD method as applied to the pseudoscalar coupling theory and for the success of this theory in agreeing with experiment would be considerably improved if it could be unambiguously shown that the effects due to pair production are damped out by higher order radiative corrections. Unfortunately, this has not been done as yet.

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Renormalization of a Neutral Vector Meson Interaction

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A proof is given that the vector interaction of neutral vector mesons and fermions can be renormalized, to replace a fallacious proof by Matthews. A new interaction representation, formally different from the usual one, is used. The basic idea, not a new one, is that the objectionable part of the interaction can be identified with a certain scalar meson interaction, known to be illusory.

1. INTRODUCTION

In general, vector and pseudovector meson theories cannot be renormalized by present-day methods, because derivatives of the Δ function in the free-field commutation rules give rise to an unlimited number of primitive divergent graphs in S-matrix calculations.[1] One exception to this rule is the vector interaction of neutral vector mesons with fermions. Matthews[2] has given a fallacious proof for this, using the formulation in the interaction representation of Miyamoto[3] and Stückelberg.[4]

We prove renormalizability by transforming to a new interaction representation, formally different from the usual one. The proof is based essentially upon identifying the objectionable part of the interaction with a certain scalar meson interaction, which is known to be illusory, and to this extent our work is simply a refinement of that of Matthews.

The fallacy in Matthews' work lay in the mathematical tools he used. He had previously proposed[5] a generalization of the Tomonaga-Schwinger theory to the Schrödinger representation,

$$i\hbar\delta\Psi_s[\sigma]/\delta\sigma(x)=H_s(\sigma,x)\Psi_s[\sigma] \tag{1.1}$$

starting from a generalized form of the Heisenberg field equations,

$$i\hbar\delta\phi(x)/\delta\sigma(x')=[\phi(x),H(x)']. \tag{1.2}$$

This generalization he assumed in his proof of renormalizability. However, it has been shown by the author (in an unpublished work) that Eq. (1.2) is not integrable in cases of physical interest, and that a direct transformation $U[\sigma]$ from the Heisenberg representation to this generalized Schrödinger representation cannot be found, since it would be defined by

$$i\hbar\delta U[\sigma]/\delta\sigma(x)=U[\sigma]H(x), \tag{1.3}$$

(in the Heisenberg representation) which is not integrable either. Nor can an analogous transformation be found from the interaction representation. Thus, even if such a generalized formulation can be made in the Schrödinger representation in a self-consistent way, it cannot be equivalent to the usual theory.

2. STUCKELBERG'S FORMULATION

Stückelberg[4] has shown how to express the Proca field $\phi_\mu(x)$, with rest mass $\hbar\kappa$, in terms of a vector field $A_\mu(x)$ and a scalar field $C(x)$, using the Lagrangian,

$$L=-\tfrac{1}{2}(A_{\mu\nu}A_{\mu\nu}+\kappa^2A_\mu A_\mu+C_\nu C_\nu+\kappa^2C^2)+L', \tag{2.1}$$

where L' describes any other fields, and their interactions with the A_μ and C fields. C and the four components A_μ are quantized in the usual way as inde-

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[3] Y. Miyamoto, Progr. Theoret. Phys. **3**, 124 (1948).
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