

the deuterons from B<sup>10</sup>( $\alpha, d$ )C<sup>12</sup> and the protons from B<sup>10</sup>( $\alpha, p$ )C<sup>12</sup>\* ( $Q=0.981$  Mev and  $Q=0.391$  Mev) were hardly resolved in the backward direction,  $\theta_{\text{c.m.}} > 90^\circ$ . In Fig. 2 the theoretical angular distributions calculated from a simple stripping equation derived by Bhatia *et al.*<sup>3</sup> are compared with the experiment, the theory being assumed to be applicable to the ( $\alpha, d$ ) reaction. We have calculated only  $|j_2(KR)|^2$ , considering the form factor nearly constant. In order to fit the calculated functions to the experiment, it is necessary to assume  $l_d=2$ ,  $R=5.4 \times 10^{-13}$  cm. The value required for  $R$  is a reasonable one that is used for interpreting the ( $d, p$ ) stripping reaction.<sup>4</sup> The fact that good agreement is found between the calculated distribution and the experimental one in the forward direction provides strong support for a direct process. The increase of intensity in the backward direction suggests that heavy-particle stripping may exist. Although the absolute differential cross-section measurements are not highly precise, it is to be noted that their magnitudes are fairly large and comparable with the largest values in B<sup>10</sup>( $\alpha, p$ )C<sup>12</sup> reactions.

The angular distribution of the inverse reaction, C<sup>12</sup>( $d, \alpha$ )B<sup>10</sup>, has been measured at  $\theta < 60^\circ$  by El Bedewi

\* Bhatia, Huang, Huby, and Newns, *Phil. Mag.* **43**, 485 (1953).

<sup>4</sup> R. Huby, in *Progress in Nuclear Physics*, edited by O. R. Frisch (Academic Press, New York, 1953), Vol. 3, p. 206.

and Hussein<sup>5</sup> at relatively high-deuteron bombarding energy of 8.9 Mev. The forward peak can approximately be fitted to  $|j_2(KR)|^2$  with  $R \approx 8 \times 10^{-13}$  cm, which is somewhat larger than the value employed for the B<sup>10</sup>( $\alpha, d$ )C<sup>12</sup> reaction. The difference in the values of  $R$  between the two reactions may be due to the difference in the bombarding energies employed and to incompleteness of the calculation.

In conclusion, the results obtained in the present work indicate that the B<sup>10</sup>( $\alpha, d$ )C<sup>12</sup> reaction at our relatively low bombarding energy proceeds mostly by a direct process as in the case of ( $\alpha, p$ ) and ( $\alpha, \alpha'$ ) reactions at high bombarding energy. The results also suggest that the probabilities of finding a deuteron and an alpha particle at the nuclear radii in B<sup>10</sup> and C<sup>12</sup>, respectively, are fairly large. These features are very interesting in terms of a nuclear model, especially a cluster model in a light nucleus.

A more detailed report is in preparation and will be published in the *Journal of the Physical Society of Japan*.

#### ACKNOWLEDGMENTS

The authors wish to thank Professor T. Muto and his co-workers for many valuable discussions.

<sup>5</sup> F. A. El Bedewi and I. Hussein, *Proc. Phys. Soc. (London)* **A70**, 233 (1957).

## Stability of the Adiabatic Motion of Charged Particles in the Earth's Field\*

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The motion of charged particles in a magnetic field such as that of the earth or that of a magnetic mirror machine is discussed. It is shown that during the motion and drift of a relativistic particle, not only the magnetic moment, but also a longitudinal invariant and an additional flux invariant are adiabatically conserved. These conservation laws lead to retention of the particles in the field. The derivation of the adiabatic invariants leads to a set of equations of motion which describe the average drift of the particles from one force line to the other, and which also describe the changes that occur in the energies and periods associated with the motion. In the absence of scattering, loss of particles from the magnetic field will be due to the violation of the adiabatic laws.

### I. THE PROBLEM

MOTION of charged particles outside the atmosphere in the geomagnetic field has received recently increased attention because of the discovery of the Van Allen radiation belts and also because of the artificial temporary generation of exceedingly low intensity belts of this kind by small nuclear explosions.<sup>1</sup>

\* Work was performed under auspices of the U. S. Atomic Energy Commission.

<sup>1</sup> These experiments have become known under the code name, Argus. For description and results see, for example: N. C. Christofilos, University of California Radiation Laboratory Report UCRL-5548 (to be published). Also, see the Proceedings of the

It follows from the simplest considerations of the motion of particles in magnetic fields that many charged particles will oscillate between the north and south polar regions along magnetic lines and that they will be reflected by the mirrors formed by the stronger magnetic fields in high latitudes. It is also well known that due to the inhomogeneity of the earth's magnetic field electrons will drift from west to east and positive ions from east to west, giving rise in this manner to a corpuscular radiation belt.

Argus Symposium held at the National Academy of Sciences, April 29, 1959 (to be published).



FIG. 1. Lines of force in a magnetic mirror machine.

If one assumes that the earth's magnetic field possesses azimuthal symmetry and is independent of time, then it is obvious that after a circuit of the earth each particle returns to its original magnetic field line and will therefore not get lost by drifting away from the earth or else by drifting toward the atmosphere where it would be absorbed. In fact, however, the earth's magnetic field is not symmetrical about any axis. Furthermore since the field varies with time, the reason for the continued existence of a radiation belt is less obvious.

Similar questions arise in connection with the mirror machines which have been used in an attempt to confine plasmas for the purpose of generating controlled thermonuclear power. The lines of force of a mirror machine having azimuthal symmetry about  $Z$  are shown in Fig. 1. Long containment times have been found<sup>2</sup> for charged particles in such a laboratory-size mirror machine, and this containment time does not change when small azimuthal asymmetries are introduced.

We shall show in this paper that long containment times are indeed to be expected, provided that the variation of the magnetic field with position and time is sufficiently slow.

## II. THE ADIABATIC MOTION OF CHARGED PARTICLES

The concepts of guiding center motion and of adiabatic invariants are very useful in predicting the motion of a particle in a slowly varying field. In a magnetic field with time and space variations small compared to the period and radius of gyration of the particle, the particle moves approximately in a circle with a center moving rapidly along a line of force and drifting slowly at right angles to the line.

The equations for this guiding center motion have been given by Spitzer<sup>3</sup> and by Alfvén<sup>4</sup> and are written here in a form which remains valid for particles with relativistic<sup>5</sup> energy. The rapid motion along the line is given by

$$\frac{dP_{11}}{dt} = -\frac{M}{\gamma} \frac{\partial B}{\partial s} + e\mathbf{E} \cdot \mathbf{n}, \quad (1)$$

where  $\mathbf{B}$  and  $\mathbf{E}$  are the magnetic and electric fields,  $M = P_{11}^2/2m_0B$  is the well-known magnetic moment,  $P_{11}$  and  $P_{11}$  are the components of the particle's rela-

tivistic momentum  $\mathbf{P}$  perpendicular and parallel to  $\mathbf{B}$ , and  $m_0$  is the rest mass. The quantity  $\gamma$  equals total mass divided by  $m_0$ ;  $\mathbf{n}$  is the unit vector  $\mathbf{B}/B$  along the line of force, and  $s$  is the distance along the line of force.

The drift velocity  $\mathbf{u}_d$  which moves the guiding center to a neighboring line is given by

$$\mathbf{u}_d = -\frac{\mathbf{n}}{B} \times \left( -c\mathbf{E} + \frac{Mc}{\gamma e} \nabla B + \frac{c}{\gamma e} \frac{P_{11}^2}{m_0} \frac{\partial \mathbf{n}}{\partial s} \right). \quad (2)$$

The first term of  $\mathbf{u}_d$  is the velocity of a frame of reference in which the component of the electric field perpendicular to  $\mathbf{B}$  is eliminated. The second term comes from the variation in the magnetic field over the circle of gyration. The third term is the drift due to the centrifugal force affecting a particle of velocity  $\mathbf{n}v_{11}$ .

For Eq. (1) to be valid, the parallel electric force  $e\mathbf{E} \cdot \mathbf{n}$  must not dominate the magnetic force term  $(M/\gamma)(\partial B/\partial s)$ . If the parallel electric force is not small, the change in the magnetic field in one period of gyration is large and the guiding center concept is not valid. In addition, the magnetic moment  $M$ , which has been extensively studied by Kruskal,<sup>6</sup> will not be an adiabatic invariant. Also the derivation of (1) requires that the component of  $\mathbf{E}$  perpendicular to  $\mathbf{B}$  be small.

Equation (2) is valid only if the three terms on the right-hand side are small compared to  $v$ , the velocity of the particle. If the first term is not small, the guiding center concept and invariance of  $M$  are still valid, but there are additional drift terms coming, for example, from the acceleration  $(d/dt)(c\mathbf{E} \times \mathbf{n}/B)$ , which are comparable to the  $\nabla B$  and  $\partial \mathbf{n}/\partial s$  terms. In this paper we assume that  $\mathbf{E}$  is small.<sup>7</sup>

Invariance of  $M$  predicts that, at a field of magnitude  $B_T = P^2/2Mm_0$ ,  $P_{11}$  will vanish and that the particle will be reflected. If there are no electric fields, kinetic energy, hence  $P^2$ , are constants of the motion and the particle will always reflect at the same magnitude of magnetic field,  $B = B_T$ . The surfaces of constant  $B$  for the earth's field have the general shape shown in Fig. 2. The field is intentionally shown as nonazimuthally symmetric.

The statements made so far (conservation of  $M$  and  $P^2$ , constancy of  $B_T$ ) do not lead to the conclusion that a particle, after drifting around the earth, must return to the line of force from which it started. Actually

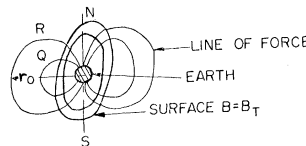


FIG. 2. Surfaces of constant magnetic field strength and lines of force about the earth.

<sup>2</sup> G. Gibson and E. Lauer, *Bull. Am. Phys. Soc.* **3**, 412 (1958).

<sup>3</sup> L. Spitzer, *Astrophys. J.* **116**, 299 (1952).

<sup>4</sup> H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, 1950), Chap. II.

<sup>5</sup> If electric fields are absent, particle energy is constant, and the trajectory of a relativistic particle can be obtained from the nonrelativistic equation of motion for a particle of the same velocity and same total mass.

<sup>6</sup> M. Kruskal, Princeton University, Project Matterhorn Report PM-S-33 (NYO-7903), March, 1958 (unpublished).

<sup>7</sup> Or following Kruskal,  $\mathbf{E}$  is assumed to go to zero as  $m_0/e \rightarrow \infty$ .

in the absence of azimuthal symmetry, there are field gradients and components of line curvature in the azimuthal direction, and by Eq. (2) these give drifts in a generally radial direction. In a static field it is nevertheless true that the particle returns to its original line so long as the second or longitudinal adiabatic invariant,<sup>8</sup>

$$J = \oint P_{\parallel} ds, \quad (3)$$

is conserved. Here  $ds$  is the element of length of the line of force. The integral is taken over a complete oscillation along the line. In the next section we will prove that  $J$  is an adiabatic invariant if during a period of oscillation  $T$  the effects of the drift  $\mathbf{u}_d$  and the fractional change of  $B$  due to time dependence are small. The quantity  $J$  is the action variable for the parallel equation of motion (1) and it seems plausible that the number of quanta of action should be conserved in a slow process. But because of the slow drift off the line there is no strict analogy with one-dimensional motion and the proof in the next section seems necessary. Also the proof gives insight into the mechanism by which the particle drifts conserve  $J$  and supplies us with equations of motion for the average drift. But first we shall discuss consequences of the invariance of  $J$ .

In a static field invariance of  $J$  makes it unnecessary to integrate the guiding center equations of motion (1) and (2) to locate the particle after it has drifted once around the earth. If a particle starting from an interior line  $Q$  returned to line  $R$  of Fig. 2, it would have a larger  $J$  than if it returned to  $Q$ . In a dipole field  $J$  increases faster than the first power of  $r_0$ , where  $r_0$  is the distance at the equator from the dipole to the line of force. The first power of  $r_0$  comes from the scaling of  $ds$  in  $\oint P_{\parallel} ds$ . The "faster than" arises because  $P_{\parallel}$  is somewhat larger on  $R$  than on  $Q$  at a given latitude, since in the absence of electric fields,  $P_{\parallel} = P(1 - B/B_T)^{1/2}$  and  $B$  is less on  $R$ . For the actual nonazimuthally symmetric field a qualitatively similar situation is encountered. Therefore, as has been pointed out previously,<sup>9</sup> the particle must return to line  $Q$  after a circuit of the earth. As the particle drifts in longitude it sweeps out a "longitudinal invariant surface." Such an invariant surface is sketched in Fig. 3.

The five quantities  $P_{\perp}$ ,  $P_{\parallel}$ , and the coordinates of position  $\mathbf{r}$  at some time  $t$  are sufficient to specify the motion of a guiding center and therefore to specify the invariant surface on which it moves. The perpendicular momentum  $P_{\perp}$  can be replaced by  $M$ , and  $J$  can be used in place of  $P_{\parallel}$ . Therefore  $J$ ,  $M$ , and  $\mathbf{r}$  are also sufficient to specify a surface. In specifying a surface the position of the particle between reflection points on a given line

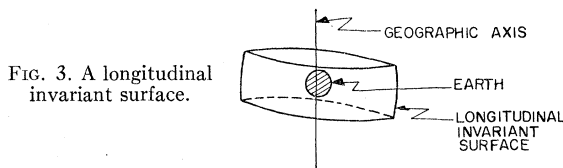


FIG. 3. A longitudinal invariant surface.

is not of interest, nor is the particular line on the surface. Thus two of the five quantities are unnecessary and we expect the invariant surfaces to form in general a three-parameter family, two of the parameters being  $J$  and  $M$ .

In static fields, the total energy  $K = (P^2 c^2 + m_0^2 c^4)^{1/2} + e\phi$  is constant and constitutes the third parameter,  $\phi$  being the electrostatic potential. Then the longitudinal invariant is given by

$$J = \oint \left[ \left( \frac{K - e\phi}{c} \right)^2 - m_0^2 c^2 - 2Mm_0 B \right]^{1/2} ds. \quad (4)$$

The three constants of the motion  $J$ ,  $M$ , and  $K$  are then the three parameters which specify an invariant surface.

If there are no electric fields, the system of surfaces is degenerate. For if  $\phi = 0$ , the momentum  $P$  is constant and (4) reduces to

$$J/P = \oint (1 - B/B_T)^{1/2} ds. \quad (5)$$

The two parameters  $J/P$  and  $B_T = P^2/2Mm_0$  are then sufficient to specify a surface. Varying  $P^2$  while holding  $P^2/M$  and  $J/P$  constant changes the speed with which the particle traverses the same surface. For in the absence of electric fields, Eq. (2) can be written

$$\mathbf{u}_d = \frac{\mathbf{n}}{B} \times \frac{P^2 c}{2\gamma e m_0} \left[ \frac{\nabla B}{B_T} + 2 \left( 1 - \frac{B}{B_T} \right) \frac{\partial \mathbf{n}}{\partial s} \right]. \quad (6)$$

It is apparent that the drift velocity is proportional to  $P^2/\gamma$  for a given  $B_T$ .

In the presence of static fields an infinite number of invariant surfaces intersect along a finite length of a line of force. Consider a particle as it rapidly oscillates between reflection points and drifts slowly at right angles to the line with velocity  $\mathbf{u}_d$ . The time average of the drift over a period  $T$  gives the adjacent line on which the particle is to be found at the end of the period. In the next section we prove that this time average of the drift conserves the longitudinal invariant  $J$ . If two particles with the same  $M$  and  $K$  are started at different points on the same line, they will be on the same adjacent line one period later, but not at times in between. For only after one complete period have both particles experienced the same drifts (although in different time sequence). They have the same average drift and by Eq. (4) they have the same  $J$ . But suppose

<sup>8</sup> According to Chew, Goldberger, and Low, Los Alamos Scientific Laboratory Report LA-2055 (unpublished), the existence of  $J$  was initially suggested by Rosenbluth.

<sup>9</sup> E. Teller, University of California Radiation Laboratory Report UCRL-5257, July 3, 1958 (unpublished).

the two particles have different  $M$  or  $K$ . They then have different reflection points and different periods of oscillation. They do not experience the same drifts and their average drifts do not carry them to the same adjacent line. They therefore must be on different invariant surfaces. This conclusion is again in agreement with (4), since the  $J$  integral along a given line is a function of  $M$  and  $K$ . After each particle has drifted all the way around the earth, it will return to its original line.

If a collection of particles with a distribution of  $M$  and  $K$  is injected along a line of force by an Argus-type explosion, then when the particles have drifted around on their respective surfaces there will be a layer of zero thickness at the injection longitude, but of greater thickness at other longitudes. We have estimated the maximum layer thickness to be of the order of the radius of the earth times the fractional azimuthal asymmetry of the magnetic field, or approximately 300 km.

To treat the case of the time-dependent field, the third or flux invariant  $\Phi$  is needed, where  $\Phi$  is the flux of  $\mathbf{B}$  inside the invariant surface on which the particle is located. In Sec. V it will be proved that if the field varies slowly compared to the time for the particle to drift around the invariant surface then  $d\Phi/dt=0$ . Although  $J$  and  $M$  are also constants, their invariance is not sufficient to prescribe the particle motion, because  $K$  is no longer a constant. However if the variation is slow enough,  $\Phi$  replaces  $K$  as a constant.

To illustrate the use of the third invariant, consider an initially static field which undergoes slow changes and then at some later time returns to its original configuration. All the magnetic surfaces obviously return to their original geometry and any particle will be back on its original surface provided its  $K$  returns to its original value. But unless  $K$  has returned to its original value,  $\Phi$  will be different since  $\Phi$  is a function of  $J$ ,  $M$ , and  $K$ . An example is furnished by the earth's rotation coupled with the azimuthal asymmetry of the field about the geographic axis. In a nonrotating frame an observer sees a time-dependent  $\mathbf{B}$  field and an  $\mathbf{E}$  field due to  $\partial\mathbf{B}/\partial t$ . The time scale of the variation is  $\sim 24$  hours. A particle which drifts around the earth in a fraction of an hour might satisfy the requirement for the invariance of  $\Phi$ . The particle will then appear to move rapidly around a surface like that of Fig. 3, and the surface rotates slowly and rigidly with the earth.

If time fluctuations are comparable to the drift time around the earth, but slow compared to  $T$ , then  $\Phi$  is lost as an invariant, but  $J$  and  $M$  are retained and may furnish useful information. If the fluctuations are comparable to  $T$  but slow compared to the gyration frequency, only  $M$  is invariant. One would therefore expect that, of the three invariants,  $M$  should be the most difficult to destroy.

### III. THE LONGITUDINAL INVARIANT, $J = \oint P_{11} ds$

The particle (i.e., guiding center) motion has a component  $\mathbf{n}v_{11}$  along the line of force on which the particle is instantaneously located, and a perpendicular drift  $\mathbf{u}_d$  towards an adjacent line. Because  $J$  is an integral along the line, it is not changed by the parallel motion, but is changed by  $\mathbf{u}_d$ . It will be shown that  $dJ/dt$  does not in general vanish, but that the quantity

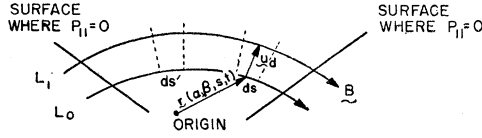
$$\left\langle \frac{dJ}{dt} \right\rangle = \frac{1}{T} \oint \frac{ds}{v_{11}} \frac{dJ}{dt}$$

does vanish,<sup>10</sup> where the integral is to be evaluated along the line of force.

In Fig. 4 is shown the line of force  $L_0$  on which the particle is located at some instant of time. The particle is assumed to be on the arc element  $ds$  and drifting towards the adjacent line  $L_1$  with velocity  $\mathbf{u}_d$ . On  $L_1$  the element of arc which is opposite  $ds$  will have a different length than  $ds$  because of the curvature. Also  $P_{11}$  will be different on the adjacent arc element because of  $\nabla B$  and because of electric fields. The gradient of  $B$  changes the distribution of  $P^2$  between  $P_{11}^2$  and  $P_{1\perp}^2$  without changing  $P^2$  itself during the drift to the adjacent line. Electric fields change  $P^2$ . Both the change in  $P_{11}$  and in  $ds$  affect  $J$ . Since  $J$  on the adjacent line is an integral along that line, one must calculate the variation in  $P_{11}$  and arc length not only for  $ds$ , but for all other arcs  $ds'$  on  $L_0$  between the reflection points. At any other arc  $ds'$  let  $\mathbf{V}(s, s')$  be the velocity which carries a point from  $L_0$  to  $L_1$  in the same time that the actual particle on  $ds$  goes from  $L_0$  to  $L_1$ . It is this velocity  $\mathbf{V}$ , not the drift velocity  $\mathbf{u}_d'$  at  $s'$ , that is needed to compute  $dJ/dt$  at the instant the particle is at  $s$ . The velocities  $\mathbf{V}(s, s')$  and  $\mathbf{u}_d'$  are not even in the same direction, except for the special case where the particle always drifts towards the same adjacent line at all points of its rapid motion along  $L_0$ . When the particle actually arrives at  $s'$ , it will not be drifting towards  $L_1$ , but towards some other line  $L_2$ . However in the following analysis it will be shown that: *The change in  $J$  due to  $ds'$  while the particle is on  $ds$  and drifting towards  $L_1$  just cancels the change in  $J$  due to  $ds$  while the particle is on  $ds'$  and drifting towards  $L_2$ .* This cancellation applies to *all* pairs of arc elements on  $L_0$  and is the detailed mechanism by which the drifts make the net change in  $J$  between reflections vanish.

There is a convenient way to describe the divergence-free field  $\mathbf{B}$  and its vector potential  $\mathbf{A}$ . One sets  $\mathbf{A} = \alpha \nabla \beta$ , where  $\alpha$  and  $\beta$  are two appropriate functions of  $\mathbf{r}$  and  $t$ . Then  $\mathbf{B} = \nabla \times \mathbf{A} = \nabla \alpha \times \nabla \beta$ . The flux of  $\mathbf{B}$  through a surface is  $\oint \mathbf{A} \cdot d\mathbf{l}$  around the boundary of the surface,

<sup>10</sup> What is actually proved is that  $\langle dJ/dt \rangle = 0 + O(\epsilon^2)$ , so that times of order  $1/\epsilon^2$  are required for  $J$  to change. Then  $J$  is constant for times of order  $1/\epsilon$ , which is the time to drift around the earth. If  $\mathbf{E}_1$  did not go to zero as  $\epsilon$ , the drift off the line would not be negligible and  $T\langle dJ/dt \rangle$  would not approximate the change in  $J$  per period.

FIG. 4. Particle at  $ds$  on line of force  $L_0$  drifts towards  $L_1$ .

and this becomes  $\oint \alpha \nabla \beta \cdot d\mathbf{l} = \oint \alpha d\beta$ . This can also be written as  $\int \int d\alpha d\beta$  over the surface, and  $d\alpha d\beta$  then represents the flux through a surface element.

In order to determine the effect of the motion of the particle at  $s$  on the contribution at  $s'$  to the  $J$  integral, it is convenient to use a quantity which is conserved by the motion along a line of force. If the field is static, the total energy  $K$  is such a quantity. We shall show that in the nonstatic case  $K$  can be generalized to

$$K = (P^2 c^2 + m_0^2 c^4)^{1/2} + e(\phi + \psi), \quad (7)$$

where  $\psi = (\alpha/c)(\partial\beta/\partial t)$ . To verify that this is the suitable generalization, we calculate the rate of change of  $K$  due to the guiding center motion

$$\dot{K} = e(v_{11}\mathbf{n} + \mathbf{u}_d) \cdot \mathbf{E} + \frac{M}{\gamma} \frac{\partial B}{\partial t} + e(v_{11}\mathbf{n} + \mathbf{u}_d) \cdot \nabla(\phi + \psi) + e \frac{\partial}{\partial t}(\phi + \psi). \quad (8)$$

The first term in (8) is the change in the energy term  $(P^2 c^2 + m_0^2 c^4)^{1/2}$  due to work done by the electric field on the guiding center. A static magnetic field has no effect on the energy; however, the induction effect of a time-dependent field gives rise to the second term, which is proportional to  $\partial B/\partial t$  and is due to the curl  $\mathbf{E}$  acting on the gyrating particle. The last two terms are the total rate of change of  $e(\phi + \psi)$ . The two terms containing  $v_{11}$  in (8) cancel, since

$$\begin{aligned} \mathbf{n} \cdot \mathbf{E} &= -\mathbf{n} \cdot \left( \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla \phi \right) \\ &= -\frac{\partial}{\partial s} \left( \frac{\alpha}{c} \frac{\partial \beta}{\partial t} + \phi \right) = -\frac{\partial}{\partial s}(\phi + \psi). \end{aligned} \quad (9)$$

Because the  $v_{11}$  terms cancel, we conclude that  $\partial K/\partial s$  must be zero. Thus  $K$  is not affected by the rapid particle motion along the line. Actually  $K$  can be considered as the energy integral of the parallel equation of motion (1). After cancellation of the  $v_{11}$  terms,  $\dot{K}$  becomes

$$\dot{K} = e \frac{\partial}{\partial t} \left( \phi + \psi + \frac{M}{\gamma e} B \right) + \frac{e}{c} \mathbf{u}_d \cdot \left( \frac{\partial \beta}{\partial t} \nabla \alpha - \frac{\partial \alpha}{\partial t} \nabla \beta \right). \quad (10)$$

The quantity  $[(\partial\beta/\partial t)\nabla\alpha - (\partial\alpha/\partial t)\nabla\beta]$  will appear frequently and will be denoted by  $\mathbf{w}$ .

With the generalized definition of  $K$  in (7),

$J = J(\alpha, \beta, K, M, t)$  is given by

$$J = \oint \left\{ \left[ \frac{K - e(\phi + \psi)}{c} \right]^2 - m_0^2 c^2 - 2Mm_0 B \right\}^{1/2} ds', \quad (11)$$

where the radical is  $P_{11}$ . At the instant the particle is at  $s$ ,  $dJ/dt$  is

$$dJ/dt = \oint [\dot{P}_{11}(s') ds' + P_{11}(s') d\dot{s}'], \quad (12)$$

where the dots mean the time derivative including terms due to the velocity  $\mathbf{V}(s, s')$ . It is easily seen geometrically that if  $\delta x$  is a displacement along the radius of curvature of the field line, then the change in arc length is

$$\delta(ds') = -\delta x(ds'/R), \quad (13)$$

where  $R$  is the radius of curvature and equals  $|\partial\mathbf{n}/\partial s|^{-1}$ . Therefore

$$d\dot{s}' = -\mathbf{V} \cdot (\partial\mathbf{n}/\partial s') ds'. \quad (14)$$

In obtaining  $\dot{P}_{11}(s')$ , the value of  $\dot{K}$  at  $s$  must be used. If we solve (7) for  $P_{11}^2$ , replace  $P_{11}^2$  by  $2Mm_0 B$ , and differentiate with respect to time, we obtain

$$\begin{aligned} \dot{P}_{11}(s') &= \frac{1}{P_{11}'} \left\{ \frac{K - e(\phi + \psi)'}{c^2} \right. \\ &\quad \left. \times [\dot{K}(s) - e(\phi + \psi)'] - Mm_0 \dot{B}' \right\}, \end{aligned} \quad (15)$$

where the primes mean evaluated at  $s'$ . Here  $\dot{B}' = (\partial B/\partial t)_{s'} + \mathbf{V} \cdot (\nabla B)_{s'}$ , and similarly for  $\psi'$  and  $\phi'$ . Now  $[K - e(\phi + \psi)']/c^2 = m_0 \gamma'$ , so that  $[K - e(\phi + \psi)']/P_{11}' c^2 = 1/v_{11}'$ . Substitution of (14) and (15) into (12) gives

$$\begin{aligned} \frac{dJ}{dt} &= \oint \frac{ds'}{v_{11}'} \left\{ \dot{K}(s) - e \frac{\partial}{\partial t} \left( \phi + \psi + \frac{MB}{\gamma e} \right)' \right. \\ &\quad \left. - \mathbf{V} \cdot \left[ e \nabla \left( \phi + \psi + \frac{MB}{\gamma e} \right) + v_{11} P_{11} \frac{\partial \mathbf{n}}{\partial s} \right]' \right\}, \end{aligned} \quad (16)$$

where the prime on any quantity means evaluated at  $s'$ . The vector  $\mathbf{V}$  must now be evaluated explicitly. Since  $\mathbf{V}$  is perpendicular to  $\mathbf{n}$  and is defined so that  $\dot{\alpha}$  and  $\dot{\beta}$  at  $s'$  are the same as at  $s$ , we have

$$\begin{aligned} \dot{\alpha}(s) &= (\partial\alpha/\partial t + \mathbf{u}_d \cdot \nabla \alpha)_s = (\partial\alpha/\partial t + \mathbf{V} \cdot \nabla \alpha)_{s'}, \\ \dot{\beta}(s) &= (\partial\beta/\partial t + \mathbf{u}_d \cdot \nabla \beta)_s = (\partial\beta/\partial t + \mathbf{V} \cdot \nabla \beta)_{s'}, \\ 0 &= \mathbf{n}' \cdot \mathbf{V}. \end{aligned} \quad (17)$$

We will now verify that the solution of (17) for  $\mathbf{V}$  is

$$\mathbf{V} = [(\dot{\alpha} \nabla \beta' - \dot{\beta} \nabla \alpha') + \mathbf{w}'] \times \mathbf{n}' / B'. \quad (18)$$

The scalar product of (18) with  $\nabla \alpha'$  is [after substituting  $\mathbf{w}' = (\partial\beta'/\partial t)\nabla \alpha' - (\partial\alpha'/\partial t)\nabla \beta']$

$$\mathbf{V} \cdot \nabla \alpha' = \left( \dot{\alpha} - \frac{\partial \alpha'}{\partial t} \right) \frac{\nabla \beta' \times \mathbf{n}'}{B'} \cdot \nabla \alpha'. \quad (19)$$

But because  $\mathbf{B}' = \nabla\alpha' \times \nabla\beta'$ , the factor  $(\nabla\beta' \times \mathbf{n}'/B') \cdot \nabla\alpha' = 1$ , and the first equation of (17) is verified. Similarly, multiplication by  $\nabla\beta'$  verifies the second equation (17). The third equation is satisfied since the right-hand side of (18) is perpendicular to  $\mathbf{n}'$ .

Let us now return to expression (16) for  $dJ/dt$  and eliminate  $\nabla(\phi + \psi)$  in terms of  $\mathbf{E}$ .

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c} \frac{\partial}{\partial t} (\alpha \nabla \beta) - \nabla \phi \\ &= -\nabla(\phi + \psi) + \frac{\mathbf{w}}{c}. \end{aligned} \quad (20)$$

Using (20) to eliminate  $\nabla(\phi + \psi)$  from (16) gives

$$\begin{aligned} \frac{dJ}{dt} &= \oint \frac{ds'}{v_{11}'} \left\{ \dot{K}(s) - e \frac{\partial}{\partial t} \left( \phi + \psi + \frac{M}{\gamma e} \right)' \right. \\ &\quad \left. - \mathbf{V} \cdot \left[ -c \mathbf{E} + \frac{Mc}{\gamma e} \nabla B + \frac{c}{e} v_{11} P_{11} \frac{\partial \mathbf{n}}{\partial s} + \mathbf{w} \right] \right\}. \end{aligned} \quad (21)$$

The first three terms in the square bracket occur in the drift velocity (2), and when expression (18) for  $\mathbf{V}$  is substituted and the dot and cross interchanged, one obtains

$$\begin{aligned} \frac{dJ}{dt} &= \oint \frac{ds'}{v_{11}'} \left\{ \dot{K}(s) - e \frac{\partial}{\partial t} \left( \phi + \psi + \frac{M}{\gamma e} \right)' \right. \\ &\quad \left. - \frac{e}{c} (\dot{\alpha} \nabla \beta' - \dot{\beta} \nabla \alpha' + \mathbf{w}') \cdot \left( \mathbf{u}_d + \frac{\mathbf{n} \times \mathbf{w}}{B} \right)' \right\}. \end{aligned}$$

Then by use of  $\nabla\alpha \times \nabla\beta = \mathbf{B}$  and the definition of  $\mathbf{w}$  the result is

$$\begin{aligned} \frac{dJ}{dt} &= \oint \frac{ds'}{v_{11}'} \left\{ \dot{K}(s) - \left[ e \frac{\partial}{\partial t} \left( \phi + \psi + \frac{M}{\gamma e} \right) + \frac{e}{c} \mathbf{u}_d \cdot \mathbf{w} \right]' \right. \\ &\quad \left. - \frac{e}{c} \dot{\alpha} \left( \mathbf{u}_d \cdot \nabla \beta + \frac{\partial \beta}{\partial t} \right)' + \frac{e}{c} \dot{\beta} \left( \mathbf{u}_d \cdot \nabla \alpha + \frac{\partial \alpha}{\partial t} \right)' \right\}. \end{aligned} \quad (22)$$

Here the quantities  $\dot{\alpha}$  and  $\dot{\beta}$  are evaluated at  $s$ . These quantities are multiplied by factors which contain the drift velocities and therefore these factors are  $\dot{\alpha}$  and  $\dot{\beta}$  evaluated at  $s'$ . The expression in the brackets in (22) is according to (10) equal to  $\dot{K}(s')$ , so that

$$\begin{aligned} \frac{dJ}{dt} &= \oint \frac{ds'}{v_{11}'} \left\{ \dot{K}(s) - \dot{K}(s') \right. \\ &\quad \left. + \frac{e}{c} [\dot{\beta}(s) \dot{\alpha}(s') - \dot{\alpha}(s) \dot{\beta}(s')] \right\} \\ &= \oint dl' \left\{ \dot{K}(s) - \dot{K}(s') \right. \\ &\quad \left. + \frac{e}{c} [\dot{\beta}(s) \dot{\alpha}(s') - \dot{\alpha}(s) \dot{\beta}(s')] \right\}, \end{aligned} \quad (23)$$

where  $dl'$  is the time element  $ds'/v_{11}'$  spent in  $ds'$ . We see that the integrand is antisymmetric in  $s$  and  $s'$ . Equation (23) can also be written

$$\frac{dJ}{dt} = T [\dot{K} - \langle \dot{K} \rangle] - \frac{eT}{c} [\dot{\alpha} \langle \dot{\beta} \rangle - \dot{\beta} \langle \dot{\alpha} \rangle] \quad (24)$$

where the average  $\langle \dots \rangle$  means  $\oint ds/v_{11}(\dots)$ . There is no reason for  $dJ/dt$  to vanish in general. However,

$$\begin{aligned} \left\langle \frac{dJ}{dt} \right\rangle &= \oint \oint \frac{ds ds'}{v_{11} v_{11}'} \left\{ \dot{K}(s) - \dot{K}(s') \right. \\ &\quad \left. + \frac{e}{c} [\dot{\beta}(s) \dot{\alpha}(s') - \dot{\alpha}(s) \dot{\beta}(s')] \right\} = 0. \end{aligned} \quad (25)$$

The average rate of change of  $J$  thus vanishes because of the antisymmetry of the integrand of (25), and it is because of this antisymmetry that the contributions of  $ds$  and  $ds'$  to the change in  $J$  cancel over a period  $T$ .

#### IV. EQUATIONS OF MOTION FOR THE AVERAGE DRIFT

Equation (2) gives the instantaneous value of the drift velocity. In case the oscillation along a line of force is fast compared to the effects of the drift, one will be primarily interested in the line on which the particle finds itself and what energy it possesses. One is therefore interested in the average drift which transfers the particle from line to line (i.e., the change in  $\alpha$  and  $\beta$ ), and in the change of the kinetic energy, derivable from the quantity  $K$ . Equation (22) permits one to obtain the motion of a particle in the  $\alpha, \beta, K$  space. By differentiating  $J = J(\alpha, \beta, K, M, t)$  with respect to time we get

$$\frac{dJ}{dt} = \frac{\partial J}{\partial K} \dot{K} + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial \alpha} \dot{\alpha} + \frac{\partial J}{\partial \beta} \dot{\beta}. \quad (26)$$

Comparison of this  $dJ/dt$  with (24), which also holds at all places and times, gives

$$\begin{aligned} \langle \dot{\alpha} \rangle &= \frac{c}{eT} \frac{\partial J}{\partial \beta}(\alpha, \beta, K, M, t), \\ \langle \dot{\beta} \rangle &= -\frac{c}{eT} \frac{\partial J}{\partial \alpha}, \\ \langle \dot{K} \rangle &= -\frac{1}{T} \frac{\partial J}{\partial t}, \\ 1 &= \frac{1}{T} \frac{\partial J}{\partial K}. \end{aligned} \quad (27)$$

The last of these four is obvious from Eq. (11). The first three are the required equations of motion with the longitudinal motion eliminated.

On the average the particle drifts towards that adjacent line on which  $J$  is unchanged. In the special case of a static field,  $\dot{K}=0$ ; if in addition  $\dot{\alpha}/\dot{\beta}=\langle\dot{\alpha}\rangle/\langle\dot{\beta}\rangle$  at all points of the motion along the line, then all instantaneous drifts will be directed toward that same line and  $dJ/dt=0$ , by Eq. (24).

The equations in (27) for the rate of change of  $\alpha$  and  $\beta$  can be written in vector form. Suppose  $\langle\mathbf{V}\rangle=(1/T)\oint(ds/v_{||})\mathbf{V}(s,s')$  is calculated. Physically  $\langle\mathbf{V}\rangle$  is the average drift at  $s'$ . Substitution of (18) for  $\mathbf{V}$  into the integral defining  $\langle\mathbf{V}\rangle$  gives

$$\langle\mathbf{V}\rangle=[\langle\dot{\alpha}\rangle\nabla\beta'-\langle\dot{\beta}\rangle\nabla\alpha'+\mathbf{w}']\times\mathbf{n}'/B'. \quad (28)$$

Substituting  $\langle\dot{\alpha}\rangle$  and  $\langle\dot{\beta}\rangle$  from (27) and dropping the primes gives

$$\langle\mathbf{V}\rangle=\frac{c}{eBT}\nabla J\times\mathbf{n}+\frac{\mathbf{w}\times\mathbf{n}}{B} \quad (29)$$

as the average velocity at any point.<sup>11</sup> The vector  $\nabla J$  is to be obtained at fixed values of  $K$ ,  $M$ , and  $t$ .

The equations of motion can be expressed differently. The equation  $J=J(\alpha,\beta,K,M,t)$  can be rewritten as  $K=K(\alpha,\beta,J,M,t)$ . By implicit differentiation we obtain  $\partial J/\partial\beta=-\partial K/\partial\beta/(\partial K/\partial J)$ , etc. Then

$$\begin{aligned} \langle\dot{\alpha}\rangle &= -\frac{c}{e}\frac{\partial K}{\partial\beta}(\alpha,\beta,J,M,t), \\ \langle\dot{\beta}\rangle &= -\frac{c}{e}\frac{\partial K}{\partial\alpha}, \\ \langle\dot{K}\rangle &= \partial K/\partial t, \\ 1 &= T(\partial K/\partial J). \end{aligned} \quad (30)$$

These equations are of canonical form, where  $\alpha$  and  $\beta$  play the roles of momentum and spatial coordinate, respectively, and  $cK/e$  plays the role of Hamiltonian. The Eqs. (27) are not of canonical form because the factor  $T$  is a function of  $(\alpha,\beta,K,M,t)$ .

In terms of  $K$  the average velocity can be expressed as

$$\langle\mathbf{V}\rangle=(c/eB)\mathbf{n}\times\nabla K+(\mathbf{w}\times\mathbf{n})/B. \quad (31)$$

The second term in (31) may be considered as the velocity of the line of force. In fact if an observer moves with this velocity, then the label  $\alpha$  will change at the rate

$$d\alpha/dt=\partial\alpha/\partial t+[(\mathbf{w}\times\mathbf{n})/B]\cdot\nabla\alpha.$$

By the definition of  $\mathbf{w}$  this is zero. The same holds for  $\beta$ . If we adopt this interpretation of the second term, then the first term in (31) gives the average drift of the particle with respect to the moving line. One should

<sup>11</sup> B. B. Kadomtsev has derived the first two equations of (27) for the case of static fields. See *Plasma Physics and the Problem of Controlled Thermonuclear Reactions* (Akad. Nauk USSR, 1958), Vol. III, p. 285. In the present paper we have given a proof for the more general case of relativistic particles in nonstatic fields, and the results are contained in Eqs. (27) and (28).

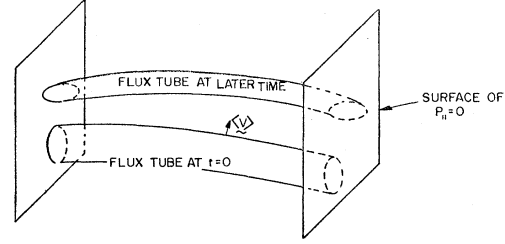


FIG. 5. Flux tubes defined by a collection of particles at two different times.

realize of course that the velocity of a line of force is arbitrary except for the requirement that the velocity field must lead to the correct fluxes<sup>12</sup> and therefore to the correct values of  $B$ .

Two flux conservation laws follow from (27) and (30). Suppose we observe a collection of particles with the same  $J$  and  $M$  distributed on a bundle of magnetic lines of force which form a finite flux tube (Fig. 5). These particles will have different  $K$ , since they will have different  $\alpha$  and  $\beta$ , and each will drift according to Eqs. (27) or (30). At any later time the particles will be found within a new flux tube. It will now be shown that the flux of  $\mathbf{B}$  is the same at the later time. The rate of change of the flux of any divergence free vector  $\mathbf{U}$  through a closed curve whose boundary moves at a velocity  $\langle\mathbf{V}\rangle$  is given by<sup>13</sup>

$$\frac{d}{dt}\int_{\text{surface}}\mathbf{U}\cdot d\mathbf{S}=\int\left[\frac{\partial\mathbf{U}}{\partial t}-\nabla\times(\langle\mathbf{V}\rangle\times\mathbf{U})\right]\cdot d\mathbf{S}. \quad (32)$$

The  $\partial\mathbf{U}/\partial t$  term gives the change in the integral due to the change with time of  $\mathbf{U}$  at all points within the loop. The second term, which comes from the distortion of the shape of the loop with time, is observed by applying Stokes' theorem to the loop integral of  $\langle\mathbf{V}\rangle\times\mathbf{U}$ . We set  $\mathbf{U}=\mathbf{B}$ , and  $\langle\mathbf{V}\rangle$  is given by (31); then (32) reduces to

$$\begin{aligned} \frac{d}{dt}\int\mathbf{B}\cdot d\mathbf{S} &= \int\left\{\frac{\partial\mathbf{B}}{\partial t} \right. \\ &\quad \left. -\nabla\times\left[\frac{c}{eB}(\mathbf{n}\times\nabla K)\times\mathbf{n}B+\frac{\mathbf{w}\times\mathbf{n}}{B}\times\mathbf{n}B\right]\right\}\cdot d\mathbf{S}, \end{aligned}$$

and since  $\mathbf{n}\cdot\nabla K=\partial K/\partial s=0$ , the integral becomes

$$\begin{aligned} \frac{d}{dt}\int\mathbf{B}\cdot d\mathbf{S} &= \int\left\{\frac{\partial\mathbf{B}}{\partial t}-\frac{c}{e}\nabla\times\nabla K \right. \\ &\quad \left. +\nabla\times\left(\frac{\partial\beta}{\partial t}\nabla\alpha-\frac{\partial\alpha}{\partial t}\nabla\beta\right)\right\}\cdot d\mathbf{S}. \end{aligned} \quad (33)$$

<sup>12</sup> W. Newcomb, *Ann. of Phys.* **3**, 347 (1958).

<sup>13</sup> M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism* (Blackie and Son, Ltd., London, 1950), second edition, p. 40.

The first and third terms of the integrand cancel because  $\mathbf{B} = \nabla\alpha \times \nabla\beta$ , and the second term is zero, thus proving the theorem.

The flux of the vector  $T\mathbf{B}$  is conserved by the motion in a static field of a collection of particles having the same magnetic moment  $M$  and energy  $K$  and distributed on a flux tube of finite size. These particles will have different  $\alpha$  and  $\beta$ , hence different  $J$ , in contrast to the case above, where they all had the same  $J$  but different  $K$ . In a static field  $K$  is a constant of the motion, so that if the particles initially have the same  $K$ , they always have. The proof is similar to that above for the flux of  $\mathbf{B}$ . However, first it must be established that  $T\mathbf{B}$  has no divergence, and that it indeed is a property of a tube of force. This is true, since  $\nabla \cdot (T\mathbf{B}) = \mathbf{B} \cdot \nabla T + T \nabla \cdot \mathbf{B}$  and  $\mathbf{B} \cdot \nabla T = B(\partial T / \partial s) = 0$ . In this case  $\mathbf{w} = 0$ , and if the velocity  $\langle \mathbf{V} \rangle$  from (29) is substituted into (32), then

$$\begin{aligned} \frac{d}{dt} \int T\mathbf{B} \cdot d\mathbf{S} &= - \int \nabla \times (\langle \mathbf{V} \rangle \times T\mathbf{B}) \cdot d\mathbf{S} \\ &= - \int \nabla \times \left[ \frac{c}{e} (\nabla J \times \mathbf{n}) \times \mathbf{n} \right] \cdot d\mathbf{S} \\ &= \frac{c}{e} \int \nabla \times \nabla J \cdot d\mathbf{S}, \end{aligned}$$

which vanishes.

A Liouville theorem exists in  $(\alpha, \beta, J, M)$  space, since the equations of motion (30) are canonical. Let  $Q(\alpha, \beta, J, M, t)$  be the particle density in this space at time  $t$ . Each point in the space represents a particle somewhere on the line  $(\alpha, \beta)$  at time  $t$  with magnetic moment  $M$  and longitudinal invariant  $J$ . The equation of continuity in this space is, since  $\dot{J}$  and  $\dot{M}$  vanish,

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial \alpha} (Q \langle \dot{\alpha} \rangle) + \frac{\partial}{\partial \beta} (Q \langle \dot{\beta} \rangle) = 0. \quad (34)$$

By (30),  $(\partial / \partial \alpha) \langle \dot{\alpha} \rangle + (\partial / \partial \beta) \langle \dot{\beta} \rangle = 0$ , so that

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \langle \dot{\alpha} \rangle \frac{\partial Q}{\partial \alpha} + \langle \dot{\beta} \rangle \frac{\partial Q}{\partial \beta} = 0, \quad (35)$$

and  $Q$  is conserved under the velocity  $\langle \dot{\alpha} \rangle, \langle \dot{\beta} \rangle$ .

Physically,  $Q d\alpha d\beta$  is the number of particles of moment  $M$  and longitudinal invariant  $J$  in the flux tube  $d\Phi = d\alpha d\beta$  at time  $t$ . Suppose now there is a steady-state particle distribution in the  $(\alpha, \beta, J, M)$  space, so that  $\partial Q / \partial t = 0$ . This will occur if we have a steady-state in configuration space, with static fields. Then (35) becomes, after eliminating  $\langle \dot{\alpha} \rangle$  and  $\langle \dot{\beta} \rangle$  by (30)

$$\partial(Q, K) / \partial(\alpha, \beta) = 0. \quad (36)$$

Since this Jacobian vanishes,  $Q$  is a function of the constants of the motion  $J, M$ , and  $K$  in the steady

state, a familiar result for a canonical system. Then  $Q$  is constant on a longitudinal invariant surface—i.e., on a surface of fixed  $J, M$ , and  $K$ .

Next let us consider the particle density in configuration space. Let  $n(\mathbf{r}, K, M, t)$  be the density at point  $\mathbf{r}$  of particles with energy  $K$  and magnetic moment  $M$ . If a steady state exists along a given line of force, we can write

$$n = X(B/v_{11}). \quad (37)$$

It is obvious that  $n$  should vary inversely as  $v_{11}$  along a line; the factor  $B$  corresponds to the inverse dependence of  $n$  on the cross-sectional area of the flux tube. The quantity  $X$ , which is independent of distance along the line, can be evaluated by integrating (37) between reflection points with respect to distance  $s$ .

$$\int \frac{n}{B} ds = X \int \frac{ds}{v_{11}} = \frac{XT}{2}. \quad (38)$$

Since  $d\alpha d\beta$  is the element of flux, we have  $d\alpha d\beta = B dV_f / ds$ , where  $dV_f$  is the volume element in the flux tube and  $dV_f / ds$  is therefore the cross-sectional area. Then (38) becomes

$$\int n dV_f = d\alpha d\beta (XT/2). \quad (39)$$

The left-hand side is the total number of particles of moment  $M$  and energy  $K$  in  $d\alpha d\beta$ . Let this total number be denoted by  $N(\alpha, \beta, K, M, t) d\alpha d\beta$ . Then  $X = 2N/T$  and (37) becomes

$$n = (2B/v_{11})(N/T). \quad (40)$$

The quantities  $N$  and  $Q$  are related by  $N dK = Q dJ$ , or  $N = Q \partial J / \partial K$ . By Eq. (27),  $T = \partial J / \partial K$ , so that  $N/T = Q$ . Then (40) becomes

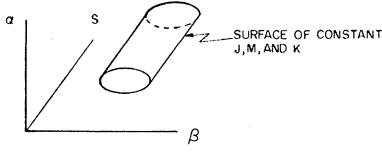
$$n = (2B/v_{11})Q. \quad (41)$$

Because  $Q$  is constant on a longitudinal invariant surface in a steady state, Eq. (41) says that in a steady state, the density  $n$  is a constant times  $B/v_{11}$  on an invariant surface. In the special case where electric fields are absent,  $v_{11} = (1/m_0\gamma) [(K^2 - m_0^2 c^4)/c^2 - 2Mm_0 B]^{1/2}$  and  $n$  becomes a function of  $B$  for a given  $J, M$ , and  $K$ . *In a steady state with no electric fields present, contours of constant  $B$  on an invariant surface are also contours of constant particle density  $n$ .*

### V. THIRD ADIABATIC THEORY AND THE THIRD OR FLUX INVARIANT $\Phi$

The equation of motion of a charged particle gives the guiding center equations of motion (1) and (2) and the adiabatic invariant  $M$  after an average has been taken over the rapid gyration around the field line. In the previous section it was shown how the guiding center equations of motion and the invariance of  $M$  lead to the equations of motion (30) in  $\alpha, \beta$ , and  $K$ , and



FIG. 6. The representation in  $(\alpha, \beta, s)$  space of a longitudinal invariant surface.

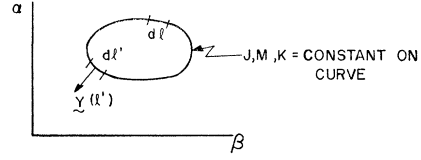
to the invariant  $J$ . An average over the rapid oscillation between reflection points was used. In this section it will be shown how the  $(\alpha, \beta, K)$  equations of motion lead to the third invariant  $\Phi$  by means of an average over rapid motion in  $\alpha$  and  $\beta$ .

This third invariant  $\Phi$  has been defined as the flux of  $\mathbf{B}$  enclosed by the invariant surface; the invariance of  $\Phi$  has been used in Sec. II. To prove its invariance, consider the representation of longitudinal invariant surfaces in  $(\alpha, \beta, s)$  space (Fig. 6). Each surface is a cylinder of finite length with elements parallel to the  $s$  axis. At any time  $t$  the three parameters  $(J, M, K)$  are needed to specify a surface. Since the line length between reflection points is a function of  $(\alpha, \beta)$ , the elements of the cylinder are not all of equal length. In a static field a particle rapidly oscillates between the ends of the cylinder and slowly drifts around it. If the field is nonstatic with a time dependence slow compared to the time to drift around the surface, the particle moves slowly from one cylinder to another characterized by the same  $J$  and  $M$ , but different  $K$ . Then  $d\Phi/dt$  can be found at each instant of the motion around the cylinder, and the time average of  $d\Phi/dt$  over one circuit of the cylinder can be shown to vanish. This is analogous to calculating  $dJ/dt$  at each instant of the lowest order motion along the line of force and then showing that  $\langle dJ/dt \rangle = 0$ .

Since the differential of flux is  $d\Phi = d\alpha d\beta$ ,  $\Phi$  is the cross-sectional area of the cylinder, and invariance of  $\Phi$  is equivalent to invariance of the cross-sectional area of the cylinder on which the particle is located. Figure 7 shows the intersection of the cylinder with the  $(\alpha, \beta)$  plane. Suppose that at some instant of time the particle is on  $dl$  and drifting slowly at right angles to it while moving rapidly around the surface. At any other arc element  $dl'$  let  $\mathbf{Y}(l')$  be the velocity which is required in order to remain on the same  $(J, M, K)$  surface as the actual particle during its slow drift off  $dl$ . The velocity  $\mathbf{Y}(l')$  is the analog of  $\mathbf{V}(s')$  for the longitudinal invariant. By using  $K = K(\alpha, \beta, J, M, t)$  we find that  $\mathbf{Y}$  must satisfy the equation

$$\langle \dot{K} \rangle_l = \nabla_{\alpha\beta} K(l') \cdot \mathbf{Y}(l') + \partial K(l') / \partial t, \quad (42)$$

where  $\nabla_{\alpha\beta}$  means the gradient in the  $\alpha, \beta$  plane. Since  $K$  is constant on the closed curve of Fig. 7, then  $\nabla_{\alpha\beta} K(l')$  is perpendicular to the line element  $dl'$ , and the rate of change of area is (assuming that  $\nabla_{\alpha\beta} K$  is towards the

FIG. 7. Cross section of a longitudinal invariant surface in  $(\alpha, \beta, s)$  space.

outside of the loop)

$$\begin{aligned} \frac{d\Phi}{dt} &= \oint dl' \frac{\mathbf{Y}(l') \cdot \nabla_{\alpha\beta} K(l')}{|\nabla_{\alpha\beta} K(l')|} \\ &= \oint \frac{dl'}{|\nabla_{\alpha\beta} K'|} \left( \langle \dot{K} \rangle - \frac{\partial K'}{\partial t} \right), \end{aligned} \quad (43)$$

where the primes mean evaluated at  $l'$ . By (30)  $\partial K'/\partial t = \langle \dot{K} \rangle'$ , and  $(c/e)|\nabla_{\alpha\beta} K'| = [\langle \dot{\alpha} \rangle^2 + \langle \dot{\beta} \rangle^2]^{1/2}$ , which is the velocity of the particle parallel to the loop at  $l'$ . Denote this velocity by  $v_{\alpha\beta}'$ . Then

$$\frac{d\Phi}{dt} = -\frac{c}{e} \oint \frac{dl'}{v_{\alpha\beta}'} (\langle \dot{K} \rangle_l - \langle \dot{K} \rangle_{l'}). \quad (44)$$

This is the analog of (24) and does not in general vanish. However,

$$\begin{aligned} \left\langle \frac{d\Phi}{dt} \right\rangle &= \oint \frac{d\Phi}{dt} \frac{dl}{v_{\alpha\beta}} \\ &= -\frac{c}{e} \oint \oint \frac{dl dl'}{v_{\alpha\beta} v_{\alpha\beta}'} (\langle \dot{K} \rangle_l - \langle \dot{K} \rangle_{l'}) = 0. \end{aligned} \quad (45)$$

Because of the antisymmetry in  $l$  and  $l'$  of the integrand in (45), it follows that the effects of  $dl$  and  $dl'$  on  $\Phi$  cancel. This is the analog of the cancellation of the effects of  $ds'$  and  $ds$  on  $J$ .

Equation (44) can be written as

$$\frac{d\Phi}{dt} = -\frac{c\tau}{e} (\langle \langle \dot{K} \rangle \rangle - \langle \dot{K} \rangle), \quad (46)$$

where  $\langle \langle \dot{K} \rangle \rangle$  is the time average of  $\langle \dot{K} \rangle$  during the motion around the surface, and  $\tau = \oint dl / v_{\alpha\beta}$  is the time to drift around the surface. Since  $\Phi = \Phi(J, M, K, t)$ ,

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial K} \langle \dot{K} \rangle + \frac{\partial \Phi}{\partial t}. \quad (47)$$

Comparison of (46) and (47) gives

$$\begin{aligned} \tau &= -\frac{e}{c} \frac{\partial \Phi}{\partial K} (J, M, K, t), \\ \langle \langle \dot{K} \rangle \rangle &= -\frac{e}{c\tau} \frac{\partial \Phi}{\partial t}, \end{aligned} \quad (48)$$

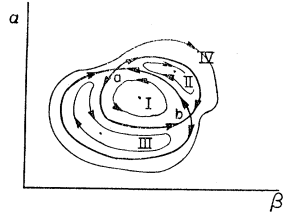


FIG. 8. Cross section of a double-leaved longitudinal invariant surface.

as the equations of motion, analogous to those in (27); they can be solved to give  $K$  and  $\tau$  as functions of time. The second equation in (48) can also be written as

$$\langle \dot{K} \rangle = \frac{\partial K(\Phi, J, M, t)}{\partial t}, \quad (49)$$

which is the analog of (30).

## VI. CONTAINMENT OF PARTICLES IN THE EARTH'S MAGNETIC FIELD

In the previous sections we have derived the equations of motion for the drift of particles from line to line and we have also discussed the flux inside the invariant surfaces in a time-dependent field. We are now prepared to discuss the question to what extent the charged spiralling particles will be retained by the earth's magnetic field.

In Sec. II we have given reasons that particles are retained provided the quantities  $M$ ,  $J$ , and  $\Phi$ , the magnetic moment, the longitudinal invariant, and the flux through the invariant surface remain time independent. Because of the rapidity of the spiralling action we must expect that  $M$  indeed is conserved in extremely good approximation.<sup>6,14</sup> Magnetic disturbances probably due to solar activity could have short enough periods to interfere with the conservation of the longitudinal invariant. On the other hand, the third invariant  $\Phi$  requires that time variation should be slow compared to the period in which the particle encircles the earth. This period lies in the vicinity of a half hour for a 1-Mev electron. One will expect that in a magnetic storm particles will diffuse from one invariant surface to another and may eventually get lost either by diffusing away from the earth or diffusing down into the atmosphere.

We shall conclude our discussion by considering other ways in which the actual situation might differ from the simple adiabatic one presented in the previous sections, since such differences might conceivably lead to a loss of particles. Thus, in the previous discussion we have assumed that each invariant surface is a single cylinder as shown in Fig. 6. This is not necessarily true. For example an invariant surface might be double, as shown in Fig. 8 in an  $(\alpha, \beta)$  plane. Neighboring constant

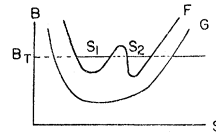


FIG. 9. Variation of field strength  $B$  as a function of  $s$ .

$J$  contours are also represented. The intersections of the two surfaces with the same  $J$  value, which occur at  $a$  and  $b$ , represent lines of stagnation on which  $\langle \dot{\alpha} \rangle$  and  $\langle \dot{\beta} \rangle$  must vanish. Because they vanish, the equations of motion (27) show that  $\partial J / \partial \alpha$  and  $\partial J / \partial \beta$  must also vanish at  $a$  and  $b$ . One possibility is that the surface  $J = J(\alpha, \beta)$  has saddle points at  $a$  and  $b$ ; the arrows showing particle motion in Fig. 8 have been drawn in a manner consistent with such a topology.

The time for a particle to approach  $a$  and  $b$  along a branch diverges logarithmically. For expansion of  $J$  about the saddle value  $J_0$  gives

$$J \cong J_0 + \frac{(\Delta\alpha)^2}{2} \frac{\partial^2 J_0}{\partial \alpha^2} + \Delta\alpha \Delta\beta \frac{\partial^2 J_0}{\partial \alpha \partial \beta} + \frac{(\Delta\beta)^2}{2} \frac{\partial^2 J_0}{\partial \beta^2}. \quad (50)$$

Then in the vicinity of  $a$  or  $b$

$$\langle \dot{\alpha} \rangle = \frac{c}{eT} \frac{\partial J}{\partial \beta} \cong \frac{c}{eT} \left( \Delta\alpha \frac{\partial^2 J_0}{\partial \alpha^2} + \Delta\beta \frac{\partial^2 J_0}{\partial \beta^2} \right). \quad (51)$$

Along the invariant surface given by  $J = J_0$ ,  $\Delta\alpha$  is proportional to  $\Delta\beta$ , as is seen from Eq. (50). Therefore on this invariant surface, Eq. (51) for  $\langle \dot{\alpha} \rangle$  takes on the form  $\langle \dot{\alpha} \rangle = k \Delta\alpha$ , where  $k$  is a constant. By integration,

$$t \cong (1/k) \ln \Delta\alpha. \quad (52)$$

This expression is approximate since higher powers in the expansion (50), as well as the variation of  $T$ , have been neglected.

Another assumption which we have tacitly made in our earlier discussion is that the field strength  $B$  has a single minimum as a function of  $s$  between the two mirror points  $B_T$ , as illustrated by curve  $G$  of Fig. 9. Suppose that a particle is initially on a line of this type, and suppose that the particle is then brought into a configuration corresponding to the line  $F$ . This can happen in one of two ways, either the magnetic field is time dependent and it happens to acquire a maximum within the original range of the longitudinal motion of the particle or else the particle drifts toward a configuration with a maximum. One will offhand suspect that when this happens the original orbit of the particle will split into two smaller segments  $s_1$  and  $s_2$  and that the original value of the longitudinal invariant  $J$  will be replaced by one of two new values  $J_1$  or  $J_2$  where  $J_1 + J_2 = J$ . If this were the case, there would clearly be a reason for a change in the longitudinal invariant. Furthermore, one will expect that the disappearance of the maximum along the magnetic line will lead to a change which is qualitatively the reverse

<sup>14</sup> A. Garren *et al.*, University of California Radiation Laboratory Report UCRL-8076, March, 1958 (unpublished); and *Proceedings of the Second United Nations International Conference on Peaceful Uses of Atomic Energy, Geneva, 1958* (United Nations, Geneva, 1958), Paper P/383.

of the change which we have discussed. There exists, however, the possibility that the drift along the two segments  $s_1$  and  $s_2$  will have carried particles to two different flux lines and that, when the maximum vanishes,  $s_1$  and  $s_2$  will join new flux lines instead of being reunited. This indeed could lead to a permanent change in  $J$  and one might expect that as a consequence a significant radial motion in the earth's magnetic field might be set up.

We shall suggest the reason why the types of processes which we discussed above may require an infinite time in the approximation which has been made throughout this paper.

When an appearance of the maximum in the magnetic field is about to sever the longitudinal oscillation into two portions, the particle will have a large period of oscillation, and in particular it will spend a long time near the maximum, at time which tends toward logarithmic infinity at the time of severance. However, for the time-dependent field case the increase of magnetic field near the maximum will by its inductive effect increase the energy of the particle. Thus a particle will not be trapped on either side if it is near the maximum, but will instead acquire enough energy to remain above the maximum. Actual trapping is likely to occur only if the particle is not near the maximum as severance is reached. However, the probability that the particle is not near the maximum decreases as severance is approached.

In the case of a static purely magnetic field where the particle drifts toward a region where its longitudinal orbit could be severed, we shall again find that during the drift the particle will spend increasing time intervals near the maximum of the magnetic field and again the time spent near the maximum will tend toward logarithmic infinity. During the proximity of the particle to the maximum, its drift due to the centrifugal force will approach zero. The drift due to the inhomogeneity of the magnetic field will persist but will be directed at right angles to the gradient of the magnetic field and move the particle at right angles to the direction of approach toward a line of severance. Again, as in the previous case, the approach is likely to depend on the periods that the particle spends away from the proximity of the maximum in

the magnetic field, and again the fraction of time that the particle spends in these regions will tend to zero as the line of severance is approached.

We expect that in a more exact and detailed theory the processes to which we have assigned infinite time in the previous two examples will actually be accomplished in finite but long times. We cannot exclude the possibility that near points  $a$  and  $b$  in Fig. 8, particles might be transferred between zones I, II, III, and IV, a possibility which does not exist according to the strictly adiabatic theory. Likewise, we must expect that in the time-dependent case, the growth of the maximum in the magnetic field such as shown in Fig. 9 will actually give rise to a severance of a longitudinal orbit. Our present purpose is only to show that a simple application of our equations of motion gives arguments against the ready occurrence of these more complex patterns of motion.

The observed radiation around the earth has a marked structure,<sup>15</sup> with maxima at 10 000 km and 22 000 km equatorial distances separated by a radiation minimum at approximately 15 000 km. It might be tempting to assume that these two radiation belts are due to some complexity of the earth's magnetic field. However, preliminary observations have shown that the particle energy spectra differ in the two belts. Thus it is likely that the two belts have a different physical origin. The discussion which we have given here indeed does not open up any simple explanation why two such belts should be due to purely kinematic causes.

## VII. ACKNOWLEDGMENTS

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<sup>15</sup> J. A. Van Allen and L. A. Frank, *Nature* **183**, 430 (1959).