Analytic Properties of Perturbation Expansions

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A method is introduced for locating and interpreting the singularities in finite-order perturbation expansions of Green's functions, in cases where the external momenta are complex vectors with real scalar products. In particular, when the external momenta form a real Euclidean set, there is a simple graphical construction, applicable to all finite orders. As an example of a case in which the external momenta form a real set in a space of signature $(+,+;-,-)$, a feature of Mandelstam's representation for scattering amplitudes is interpreted in perturbation theory. Finally, the ranges of values of momentum-transfer for which (in perturbation theory) dispersion relations certainly hold are enlarged, in particular by using the fact that the pion is pseudoscalar.

1. INTRODUCTION

UNTIL there are more powerful techniques for determining the analyticity of exact Green's functions, a study of finite-order perturbation expansions will remain a valuable guide to what dispersion relations can be expected to be true. To this end it is important, not only to locate the singularities of perturbation theory, but also to have whatever "physical" interpretation of them may be possible. The emphasis in this paper therefore is on understanding old results rather than deriving new ones. However, the new method turns out to be more powerful than existing methods in several respects.

Let the external momenta for a given Green's function be p_i and their scalar products $p_i\cdot p_j$. The analyticity of S-matrix elements, where the p_i are real, is completely understood from the work of Dyson and Eden.[1] The singularities are determined by the unitarity of S, and they correspond to physical thresholds. Mathematically, these singularities come from displaced poles; and the simplicity of their structure is, as we shall see, connected with the signature of the Lorentz metric.

Dispersion relation theory, however, requires Green's functions for which the p_i are not real. Like previous workers, we restrict ourselves to the greatly simplified case in which the $p_i\cdot p_j$ *are* real. It is then always possible to choose a set of real vectors, p_i', in a 4-dimensional space, L', of some signature, such that $p_i'\cdot p_j'=p_i\cdot p_j$. The basic method of this paper is to consider integrals with exactly the same form as ordinary Feynman perturbation theory integrals, but with all the vectors belonging to L'; and then to investigate the analyticity of these integrals by a generalization of the idea of displaced poles. (There is no unitary condition in L' so these displaced poles do not correspond in any obvious way to physical thresholds.)

The integrals in L' are relevant for the following reason. If, as a result of the standard Feynman procedure, a perturbation theory integral takes the form

$$\int d^4k[k^2-F(p_i\cdot p_j,u)+i\epsilon]^{-r} =\frac{1}{2}\frac{1}{r(r-1)}i\pi^2(F-i\epsilon)^{2-r}, \quad (1)$$

then the corresponding integral in L' would be

$$\int d^4k'[k'^2-F(p_i'\cdot p_j',u)+i\epsilon]^{-r} =\frac{1}{2}\frac{1}{r(r-1)}i^m\pi^2(F-i\epsilon)^{2-r}. \quad (2)$$

Here the u's are a set of Feynman parameters (later to be integrated over), k' is a vector in L', and m is the number of $+$ signs occurring in the signature of L'. Thus the analytic properties, which are determined by the zeros of F, are the same in the two cases, apart from the extra factor i^{m-1}. Further, it is the right-hand side of (1) which defines the Green's function when the p_i are not real. The "Feynman" integral of which (2) is a reduction, therefore, affords a representation of essentially the same Green's function; but it is a representation in which the structure of the "Feynman" denominators is not obscured by complex vectors.

In Sec. 2, the method just sketched is employed to interpret the singularities found by Karplus, Sommerfield, and Wichmann.[2] This class of singularities is probably already pretty well understood [see reference 2 around Eq. (17)] but we re-examine it in detail, partly to show the similarity to the less well known class introduced in Sec. 4.

In Sec. 3, the method of Sec. 1 is generalized so as to provide a simple geometrical construction for locating singularities, to any finite order, in cases where L' is Euclidian. This construction is a generalization of one already given, without explanation, by Karplus, Sommerfield, and Wichmann[3] in an Appendix.

[1] F. J. Dyson, Phys. Rev. **75**, 1736 (1949); R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952).

[2] Karplus, Sommerfield, and Wichmann, Phys. Rev. **111**, 1187 (1958).

[3] Karplus, Sommerfield, and Wichmann, Phys. Rev. **114**, 376 (1959).

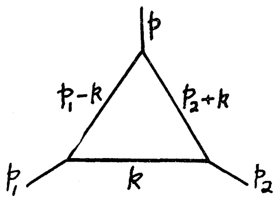

FIG. 1. Feynman graph for a vertex-part Green function.

Section 4 gives an example where an L' with signature $(+,+;-,-)$ is appropriate. This is two-particle scattering with nonzero momentum-transfer in fourth order perturbation theory. Here, our analysis points to the existence of singularities on a certain curve in the energy-momentum-transfer plane. This curve proves to be the boundary of the region in which the spectral function of Mandelstam's[4] representation does not vanish. The present method, therefore, adds a little to our understanding of this curve.

Finally, the methods developed are applied to determine the maximum momentum transfer below which dispersion relations can be proved to hold for finite-order perturbation expansions. The work of Nambu[5] and Symanzik[6] is simply rederived, and their limits are improved upon. In particular, it is possible to take account of the pseudoscalar nature of the pion.

2. ABNORMAL SINGULARITIES IN VERTEX PARTS

Consider a vertex part Green's function in third order perturbation theory, with the Feynman graph shown in Fig. 1. For simplicity, suppose the masses on the internal lines to be each m, and put $p_1^2=p_2^2=M^2$, $p^2=W^2$. For $W<2M$, p_1, p_2, and p cannot each be taken real; but a Euclidean set, p_i', does exist. As a matter of convenience, we shall take L' to have signature $(+,+;-,-)$ and each of the p_i' to have their space components zero. Thus put

$$p_1'=(\tfrac{1}{2}W,q;0,0),\qquad p=(W,0;0,0),$$
$$p_2'=(\tfrac{1}{2}W,-q;0,0),\quad k=(x,y;k_1,k_2),$$

where $q^2=M^2-\frac{1}{4}W^2$.

The "Feynman" denominators of the L'-space integral have zeros on three circles, C_1, C_2, and C_3, in the x-y plane, with centers at (0,0), $(\frac{1}{2}W,q)$, and $(-\frac{1}{2}W,q)$, each of radius $R=(k_1^2+k_2^2+m^2)^{\frac{1}{2}}>m$. The analytic properties of the function are entirely determined by the relative positions of these three circles. Because of the (assumed) stability condition $M<2m$, C_1 and C_2 always overlap, and so do C_1 and C_3. Otherwise, the possible configurations are easily enumerated as follows.

(a) $M^2<2m^2$. (i) $W^2<4m^2$: C_1, C_2, and C_3 contain a common point for all R.

(ii) $W^2>4m^2$: either $R<\frac{1}{2}W$ and C_2 and C_3 do not overlap, or $R>\frac{1}{2}W$ and C_1, C_2, and C_3 contain a common point.

(b) $M^2>2m^2$. (i) $W^2<4M^2-(M^4/m^2)$: C_1, C_2, and C_3 contain a common point for all R.

(ii) $4M^2-(M^4/m^2)<W^2<4m^2$: C_2 and C_3 overlap for all R, but C_1, C_2, and C_3 only contain a common point for $R^2>M^4/(4M^2-W^2)$.

(iii) $4m^2<W^2<2M^2$: C_2 and C_3 overlap for $R>\frac{1}{2}W$ but C_1, C_2, C_3 only contain a common point for $R^2>M^4/(4M^2-W^2)$.

(iv) $W^2>2M^2$: either $R<\frac{1}{2}W$ and C_2 and C_3 do not overlap, or $R>\frac{1}{2}W$ and C_1, C_2, and C_3 contain a common point.

When C_1, C_2, and C_3 contain a common point, it is possible to shift the origin to that point. The contours in the complex x and y planes can then be rotated to the imaginary axes, with no contributions from the poles. When C_2 and C_3 do not overlap, on the other hand, the situation is analogous to that of displaced poles,[1] and there is a contribution to the integral corresponding to the coincidence of the poles on these two circles. It is when C_2 and C_3 overlap but C_1, C_2, and C_3 do not contain a common point that a new ("abnormal") situation arises. This can occur only for $M^2>2m^2$ and along a finite segment of the W axis:

$$4M^2-M^4/m^2<W^2<2M^2.$$

These abnormal singularities were discovered by Karplus, Sommerfield, and Wichmann.[2]

When $W=2M$ the centers of the three circles become collinear, and the function joins on smoothly to its value when the p_i are real Lorentz vectors.

The transition between a normal situation and an abnormal one evidently occurs when the circles are in the configuration of Fig. 2(a). Since the lines joining the centers of the circles represent the vectors p', p_1', and p_2', this situation can happen only if the diagram in Fig. 2(b) can be drawn. Here, the sides of the triangle represent the Euclidian vectors p_i' in length and direction, and the lengths of the internal lines represent the masses m. In order to correspond to a configuration like Fig. 2(a) the common point of the three broken lines must lie inside the continuous line triangle. With this condition, the construction in Fig. 2(b) determines the critical value of W uniquely (provided the value of M is such that the diagram can be drawn at all).

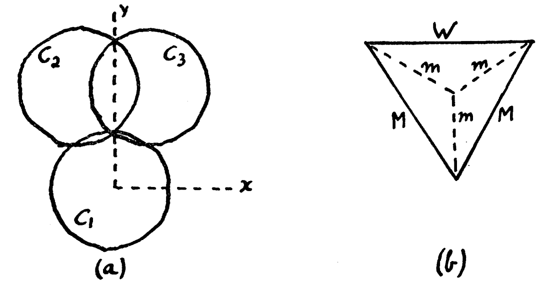

FIG. 2. (a) Critical configuration of circles on which the zeros of the Feynman denominators lie. (b) Dual diagram expressing the relationships between the momenta for the situation in (a).

[4] S. Mandelstum, Phys. Rev. **112**, 1344 (1958).
[5] Y. Nambu, Nuovo cimento **9**, 610 (1958).
[6] K. Symanzik, Progr. Theoret. Phys. **20**, 690 (1958).

The construction in Fig. 2(b) was given in the Appendix of reference 3. Figure 2(a) explains its significance.

3. THE EUCLIDIAN CASE TO ALL ORDERS

Once its significance is understood, the construction of the last section can be generalized to apply to a graph of any order. The result of Nambu[5] and Symanzik[6]—roughly that the region of analyticity is determined by a few lowest order graphs—is then an almost immediate consequence.

The singularity corresponding to Fig. 2(b) arises when the momentum on each internal line is equal to the mass on that line. In higher order graphs, there *may* be a singularity of this type whenever the momenta on a certain subset of internal lines can be equal to the corresponding masses. We therefore first draw a reduced graph consisting only of this subset of internal lines, the remainder being omitted and vertices being coalesced where necessary. For vertex parts to which the external momenta form a Euclidian set, it is sufficient to take *all* the momenta to be two-dimensional Euclidian vectors. This is because the singularities determined by an integral of the form (2) have the same *position* whatever the dimensionality of the integration. The conditions on the momenta for a singularity to occur can then be represented by a plane Euclidian diagram, which is in an obvious sense the dual of the reduced graph. Thus Fig. 3(b) is the dual of the graph in Fig. 3(a) and the external momenta and internal masses in (a) determine the lengths of the lines in (b).

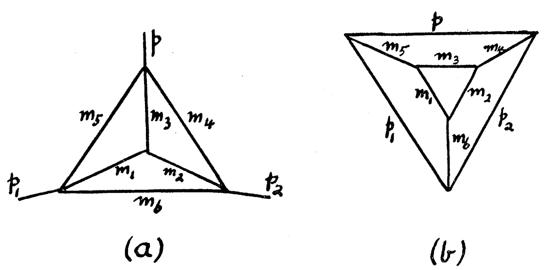

FIG. 3. (a) An example of a Feynman graph, and (b) its dual diagram.

The conditions for a reduced graph to correspond to a singularity of the type under consideration are then: (I) that a dual diagram exists, (II) that just three lines meet at each internal vertex of the dual figure, and (III) that none of the angles at an internal vertex exceeds π. The two last conditions allow that at each internal vertex three[7] circles can intersect in the relative configuration of Fig. 2(a). The graph in Fig. 3(a) clearly fulfills all three conditions. Examples of graphs breaking conditions I and II are shown in Figs. 4(a) and (b), respectively.

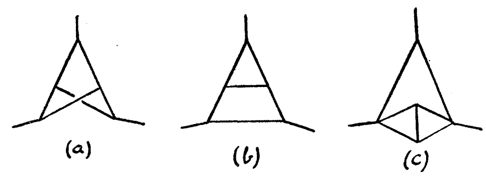

FIG. 4. Examples of Feynman graphs with no admissible dual diagrams.

If a dual diagram satisfies conditions II and III, and if the lengths of, say, p_1 and p_2 are fixed, then the length of p is just determined. Also, the lines of the external triangle are "under tension" and the internal lines are "under compression," in the sense that if any of the internal masses is decreased the critical value of $|p|$ decreases. Thus, if it is possible, without breaking selection rules, to reduce the graph further by omitting more internal lines, then the region of analyticity will not be increased. This proves the assertion made at the beginning of this section.

It has been implicitly assumed that all the external and internal particles are stable ($W<2M$ in Fig. 2, $m_6<m_1+m_2$ in Fig. 3, etc.). If they are not, the above considerations must be supplemented by including "normal" threshold conditions.[8] These can be included by generalizing condition II to allow vertices at which only two lines meet. Condition III then demands that these two lines be parallel. A similar example occurs in Fig. 5(e) and (f). Figure 4(c) appears to be an example where the normal thresholds of the diamond-shaped subgraph would be relevant, but the dual

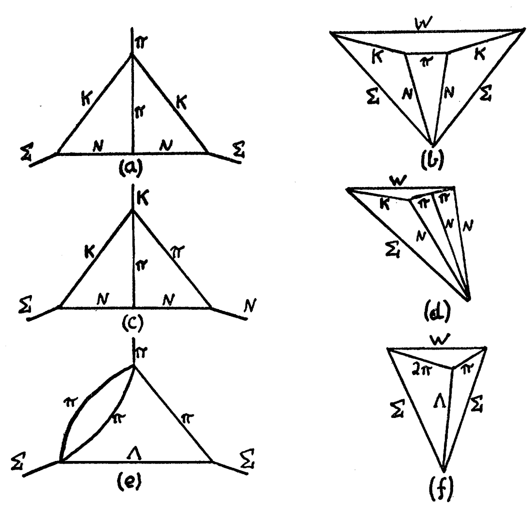

FIG. 5. (a), (c), and (e) are examples of Feynman graphs, and (b), (d), and (f) are the corresponding dual diagrams. (d) is inadmissible because of an internal reflex angle.

[7] If four circles intersected in a point and one of them was slightly displaced outwards, then that circle and some pair of the others would be in an abnormal configuration. In other words, the remaining circle would be irrelevant. The expression of this is that the dual diagram for such a configuration would be overdetermined.

[8] These are in fact just a linear version of the stated two-dimensional conditions.

diagram violates condition III and it is correct not to include Fig. 4(c) as a reduced graph.

The only cases in which one must go beyond the third order to find the limit of analyticity of vertex parts are when selection rules limit the reductions that are permissible. Some examples are given in Fig. 5. Diagrams (b), (d), and (f) are the duals of graphs (a), (c), and (e), respectively. The particle symbols denote that the lines in the dual diagrams have lengths determined by the appropriate masses. All external baryons are supposed to be free particles, whereas the external mesons have energy W, to be determined. (The dual diagrams are not drawn to scale.)

In graph (a), the critical value of W comes out to be about the Λ mass, and therefore this abnormal threshold is well above the normal 3π threshold. In the dual diagram (d), the condition III is not satisfied, and so there is no corresponding abnormal threshold. Graph (e) is an example where a normal singularity has to be taken into account, in the way indicated in (f). The critical value of W turns out to be about 2.98 π-masses, that is just below the normal 3π threshold.

The main idea of the above constructions may be generalized to apply to Green functions with four external lines. Then a three-dimensional Euclidian space is required, and the circles become spherical surfaces. The generalization of Fig. 2(b) for example, is a tetrahedron with each of its vertices joined to an internal point—a construction also given in an Appendix to reference 3. Four of the sides of the tetrahedron represent the external momenta, p_1, q_1, p_2, q_2; and the other two sides represent two out of the three combinations

$$W^2=(p_1+q_1)^2,\quad \bar{W}^2=(p_1-q_2)^2,\quad \Delta^2=-\tfrac{1}{4}(p_1-p_2)^2.$$

Unfortunately, however, the cases of physical interest are scattering Green's functions, for which the momenta can form a Euclidian set only if the momentum transfer, Δ, is zero. But if this happens, the tetrahedron degenerates into a plane figure, and it is sufficient to study the two triangles (p_1,q_1,W) and $(p_1,q_2,\bar{W})$ by the method described for vertex parts. This fact will be exploited in Sec. 5.

4. SCATTERING WITH NONZERO MOMENTUM TRANSFER

In this section our method is applied to a fourth-order perturbation theory amplitude for two-particle scattering. For this case, L' is not in general Euclidian, and it is of interest to enquire what will be the result corresponding to that of Sec. 2. It turns out that singularities are to be expected on a certain curve, in a manner which has already been incorporated in Mandelstam's[4] double dispersion-relation. Thus, we have a new way of looking at this particular feature of Mandelstam's representation.

Let all particles have equal mass, M, to simplify the formulas. Consider the "self-crossed" graph in Fig. 6, with the notation defined at the end of Sec. 3. For $W^2>4(\Delta^2+M^2)$, the four momenta are real Lorentz vectors, and the Dyson-Eden theory applies. For $W^2<4(\Delta^2+M^2)$, one requires an L' with signature $(+,+;-,-)$. Then, in a Breit frame, write

$$p_1'=(\lambda,0;\Delta,0),\qquad q_1'=(\omega,\eta;-\Delta,0),$$
$$p_2'=(\lambda,0;-\Delta,0),\quad q_2'=(\omega,\eta;\Delta,0),$$

where

$$\lambda=(M^2+\Delta^2)^{\frac{1}{2}}=(\omega^2+\eta^2)^{\frac{1}{2}},$$
$$W^2=2\lambda(\lambda+\omega),\quad \bar{W}^2=2\lambda(\lambda-\omega).$$

With a convenient choice of the integration momentum $(x,y;z,k)$, the zeros of the "Feynman" denominators lie on the following four circles in the x-y plane.

$$C_1:\quad \left(x+\frac{\lambda-\omega}{2}\right)^2+(y-\tfrac{1}{2}\eta)^2=\left(\frac{\lambda\omega-\Delta^2}{2\Delta}-z\right)^2+R^2,$$

$$C_2:\quad \left(x-\frac{\lambda+\omega}{2}\right)^2+(y-\tfrac{1}{2}\eta)^2=\left(\frac{\lambda\omega+\Delta^2}{2\Delta}-z\right)^2+R^2,$$

$$C_3:\quad \left(x-\frac{\lambda-\omega}{2}\right)^2+(y+\tfrac{1}{2}\eta)^2=\left(\frac{\lambda\omega-\Delta^2}{2\Delta}-z\right)^2+R^2,$$

$$C_4:\quad \left(x+\frac{\lambda+\omega}{2}\right)^2+(y+\tfrac{1}{2}\eta)^2=\left(\frac{\lambda\omega+\Delta^2}{2\Delta}-z\right)^2+R^2,$$

where

$$R=(M^2+k^2)^{\frac{1}{2}}>M.$$

The pairs of circles C_1 and C_2, C_2 and C_3, C_3 and C_4, and C_4 and C_1 are each overlapping pairs for all R and z. This corresponds to the fact that each of the external particles is stable. The four centers are at the vertices of a rhombus whose center is at the origin, and the figure is symmetrical about either diagonal.

If C_1 and C_3 overlap for all z and R, which is so for $\bar{W}<2M$, then it can be verified that it is also the case that C_1, C_2, and C_3 contain a common point and C_1, C_3, and C_4 have a point in common.[9] This is therefore a normal situation, with only, at worst, C_2 and C_4 mutually displaced. Similarly, for $W<2M$, C_2 and C_4 overlap for all z and R, and there is a normal situation.

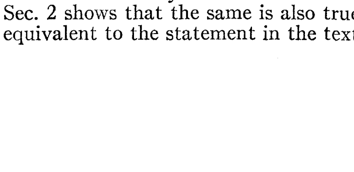

FIG. 6. A Feynman graph for equal-mass particle scattering.

[9] That this must be so can be seen as follows. If one is concerned with only three of the circles, one need consider a set of only three of the external vectors, say p_1, q_1, and W. These must form either a real Lorentz or a real Euclidian set. In the former case there are certainly no abnormal thresholds, and the result of Sec. 2 shows that the same is also true in the latter case. This is equivalent to the statement in the text.

If $\Delta > M$, neither of the above conditions can be fulfilled for $|\omega| < (\Delta^2 - M^2)/\lambda$. However, there is still not *necessarily* an abnormal situation.

There is an abnormal situation only if there is some *one* value of z and R for which *neither* C_1 and C_3 *nor* C_2 and C_4 are overlapping pairs. The condition for this is

$$(\lambda\omega)^2 < \Delta^2(\Delta^2 - 2M^2).$$

This is just the region in which the spectral function of Mandelstam's[4] representation does not vanish. The argument of this section, therefore, affords a new interpretation of this region.

5. DISPERSION RELATIONS

Since it is not clear at present how to generalize the preceding section to higher orders, we follow Symanzik[6] and replace the external momenta by a Euclidian set which, he proves, does not remove any singularities; thus placing rather crude bounds on the positions of the singularities. Having taken this initial step, however, we are able to take advantage of the remark at the end of Sec. 3. The rest of Symanzik's results then follow very quickly, and can be improved upon.

Consider first nucleon-nucleon scattering. Using Symanzik's Euclidian momenta [reference 6, Eq. (7b)], for which the momentum transfer is zero, we can, by the remark at the end of Sec. 3, reduce the problem to the consideration of two-dimensional Euclidian sets, in the manner of Sec. 3. The argument that the internal lines in the dual diagrams are under pressure and can be removed (if selection rules allow) immediately shows that we need only consider Symanzik's graphs in Fig. 6 of reference 6. But it is not possible to draw any dual diagram for the graphs (b), (c), and (d) of that Fig. 6; and so one is left with Fig. 6(a), and not with Fig. 7. The limit on momentum transfer then comes out to be $\Delta^2 = \mu^2$.

Consider next pion-nucleon scattering, first of all allowing 3π vertices to exist. Then Symanzik's choice of external momenta and our constructional technique lead us to study only fourth-order graphs, like that in Fig. 7(a), where $q^2 = \mu^2$ and $p^2 = M^2 + 2\Delta^2$. For $\Delta^2 < \frac{1}{2}(M\mu + \mu^2)$ there are only normal singularities, with cuts only for $|\omega| > (M\mu - \mu^2)/2\lambda$ and a gap in between. At $\Delta^2 = \frac{1}{2}(M\mu + \mu^2)$ abnormal singularities begin, corresponding to the dual diagram in Fig. 7(b), but a finite gap may still remain. The gap disappears, and our method consequently fails, when there is an abnormal singularity at $\omega = 0$ ($W = \bar{W}$). Then the dual diagram

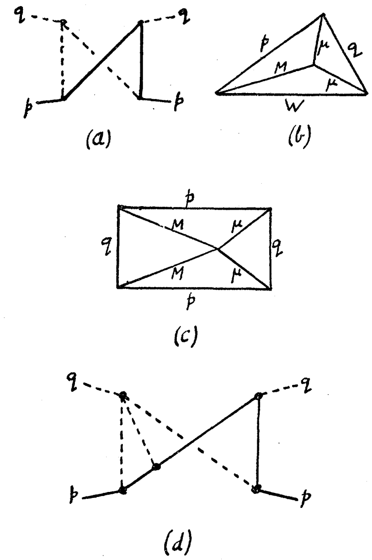

FIG. 7. (a) A reduced graph for pion-nucleon scattering, neglecting the fact that the pion is pseudoscalar. (b) A dual diagram connected with (a). (c) A critical configuration of two dual diagrams connected with (a). (d) A typical reduced graph for pion-nucleon scattering, observing the fact that the pion is pseudoscalar.

can be drawn in the rectangular form of Fig. 7(c), and trigonometry shows that

$$\Delta^2 = \frac{\sqrt{3}}{2} M\mu \left(1 + \frac{\mu^2}{4M^2}\right)^{\frac{1}{2}} + \tfrac{1}{4}\mu^2.$$

Now one can see what happens if the condition that the pion is pseudoscalar be imposed. Then graphs like Fig. 7(d) must be considered. For this graph, no dual diagram, corresponding to Fig. 7(b), can be drawn; for the 4π vertex would appear as a rhombus, which cannot fit into a triangle with which it shares an edge. Thus the method will not fail until the *normal* singularities meet at $\omega = 0$, which occurs when $\Delta^2 = M\mu$. This is, therefore, a new lower bound on the maximum permissible value of Δ^2.

ACKNOWLEDGMENTS

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Note added in proof.—Similar work to this paper has been done by Landau (Report to 1959 International Conference on Physics of High Energy Particles, Kiev).