support. This bound state may be removed and re-expressed as in II, and the analytic information of interest may be deduced from the beginning of the continuum of mass values.

When the external particle (the projectile) is strongly interacting, the restriction to the mass shell makes $\Lambda$ have a finite segment lying inside $\mathcal{S}$ for *all* nonvanishing momentum transfers, since $n \leqslant \kappa + m$. Thus it is not possible to derive dispersion relations for nonforward scattering of physical particles from this representation. When the momentum transfer tends to zero, $\Lambda$ moves over to the boundary of $\mathcal{S}$ and in the limit of the forward-scattering amplitude we obtain the dispersion relations derived in II.

Here one needs to point out that it is not essential to take both the projectile states to be on the mass shell. One may consider those specializations of the Green's function which correspond to off-the-mass-shell functions in a completely analogous manner and study the analyticity properties of these amplitudes for virtual processes. But since the consideration of these various restrictions of the Green's function adds nothing essentially new in principle we shall not devote any further attention to them. Similar considerations apply to the three-momentum and multi-momentum amplitudes considered in Sec. 5.

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# Some Aspects of the Covariant Two-Body Problem. I. The Bound-State Problem*

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A study has been made of the bound states of the Bethe-Salpeter equation for the nucleon-antinucleon system, including the ladder and pair-annihilation diagrams. For simplicity, nucleons and mesons were taken to be scalar, the latter having zero rest mass. Pair effects enter only in $S$-states with the bound states corresponding to the poles of the meson propagator $D_F'$. The Bethe-Salpeter equation is closely related to the integral equation for the generalized vertex operator $\Gamma$; this has been solved by using an integral-transform method similar to that of Wick and Cutkosky, under the assumption that the nucleon mass is large compared to the binding energy. After performing a self-energy subtraction, the energy eigenvalues are found as a function of the coupling constant. These have the form given by the usual Bohr formula plus corrections. Finally, some comments are made with respect to the extension of the formalism to mesons with nonzero mass and spinor nucleons.

## I. INTRODUCTION

A FEW years ago, several authors[1,2] showed how to write down a relativistic wave equation for two-body systems. Subsequently, there has been a good deal of interest in the study of methods of solving this equation.

Although it is the bound-state problem for the case of two spinor particles that is the most interesting from both a theoretical and practical standpoint, the corresponding covariant or Bethe-Salpeter (B-S) equation is very difficult to handle. Indeed, for this case, several authors[3] at one time maintained that no solution exists at all with vector coupling because of the singular nature of the kernel appearing in the integral equation. However, Goldstein[4] has found a solution for a very special situation, i.e., when the total relativistic energy of the system equals zero.

The solution in this case is very peculiar in that one is led to a continuous rather than a discrete bound-state energy-level spectrum. We will return to a discussion of this solution in the last part of this paper (Sec. V) at which time it will become clear that the results obtained for the bound states of two spinor particles will depend materially on the nature of the interaction with the meson field; in particular, the $\gamma_5$ and $\gamma_\mu$ interactions behave very differently from one another.

In contrast to the spinor case, the B-S equation can be

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1 E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

2 J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 435 (1951).

3 C. Hayashi and Y. Munakata, Progr. Theoret. Phys. (Kyoto) **7**, 481 (1952).

4 J. S. Goldstein, Phys. Rev. **91**, 1516 (1953).

solved exactly for the bound states of two scalar particles, and here one finds a discrete level spectrum. The proofs were given by Wick[5] and Cutkosky[6] who worked in the ladder approximation and who assumed zero-mass mesons (scalar photons).

The problem which is posed in this paper can be considered to be an extension of that of Wick and Cutkosky in that we add a nucleon-antinucleon pair-annihilation diagram to the usual ladder diagram, the nucleon once again being taken to be a scalar particle (boson). Our problem then is to investigate the bound states of the nucleon-antinucleon system.

Of course, since we are concerned with two spinless particles bound by a scalar photon field, we are dealing with a fictitious situation. Nevertheless, the general formulation of the problem as given here holds good for the case of spinor nucleons, and it is to be hoped that one may ultimately be able to apply similar methods in investigating the more realistic problem which is considerably more complicated.

As will be seen, it turns out that the B-S equation is related to the integral equation for the generalized vertex operator $\Gamma$ in field theory; the bound states themselves correspond to the poles of the meson propagator $D_F'$. We can then employ usual methods to subtract out the divergences in the latter; in this way, we obtain ultimately the energy eigenvalues of the system.

To get an explicit form for $D_F'$, we must solve the integral equation for the modified vertex operator $\Gamma$, which we do by using a Stieltjes-transform method that is essentially a generalization of Wick's method.[5] The integral equation is then cast in the form of a second-order partial differential equation involving two parameters and subject to certain boundary conditions. The whole problem is reduced to that of solving this differential equation. From the solution, we can obtain the poles of $D_F'$, and thus the determination of the bound states is effected.

The general formulation of the problem, the integral-transform method, and the computation of the energy eigenvalues are discussed in Secs. II, III, and IV, respectively. In Sec. V, we shall conclude this paper by considering the extension to the cases of mesons of finite mass and spinor nucleons.

## II. FORMULATION OF THE PROBLEM

For the interaction Hamiltonian at the space-time point $x$, we take

$$H_1 = g[\phi(x)]^2 A(x), \tag{1}$$

where $\phi(x)$ is a neutral scalar nucleon field, and $A(x)$, a neutral scalar meson field with zero rest mass (scalar photon field); later, we will consider briefly the case of a meson field of finite mass. The strength of the meson-nucleon interaction is measured by the coupling constant $g$.

[5] G. C. Wick, Phys. Rev. **96**, 1124 (1954).
[6] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

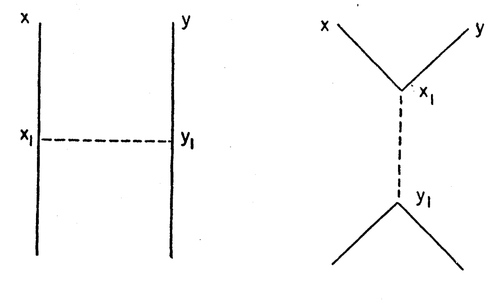


FIG. 1. (a) The ladder diagram; (b) the pair-annihilation diagram.

According to Gell-Mann and Low,[7] the B-S wave function for the two-body problem is given by

$$\chi(x,y) = (\Psi_0, P[\phi(x)\phi(y)]\Psi), \tag{2}$$

where $\Psi_0$ and $\Psi$ are the state vectors for the true vacuum and the bound states of the two-body system, respectively; $P$ is the chronological ordering operator of Dyson.[8] Notice that, since we are dealing with a neutral scalar nucleon field, we may refer to $\chi$, equivalently, as the two-nucleon or nucleon-antinucleon wave function.

If we assume that the two-body interaction arises in virtue of the exchange of single mesons (ladder interaction) and of the annihilation of a nucleon pair, we find that the bound-state B-S wave function $\chi$ satisfies the following homogeneous integral equation:

$$\begin{aligned}\chi(x,y) = &-4g^2 \int d^4x_1 d^4y_1\, \Delta_F(x-x_1) \\ &\times \Delta_F(y-y_1) D_F(x_1-y_1)\chi(x_1,y_1) \\ &-2g^2 \int d^4x_1 d^4y_1\, \Delta_F(x-x_1) \\ &\times \Delta_F(y-x_1) D_F(x_1-y_1)\chi(y_1,y_1);\end{aligned} \tag{3}$$

the first term on the right-hand side of (3) corresponds to the ladder diagram [Fig. 1(a)], the second term to the pair-annihilation diagram [Fig. 1(b)]. For the derivation of Eq. (3) as well as the definition of the free-particle propagation functions $\Delta_F$ and $D_F$, the reader is referred to the original paper of Gell-Mann and Low.[7]

The wave function $\chi$ can now be expressed as the product of two terms, describing the center-of-mass motion and the relative motion, respectively, of the two particles, *viz.*,[9]

$$\chi(x,y) = e^{i(E/2)(x+y)} f(x-y). \tag{4}$$

Here, $E$ denotes the energy-momentum four-vector for the center-of-mass motion, and $f$ is the wave function for the internal motion.

[7] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
[8] F. J. Dyson, Phys. Rev. **75**, 486 (1949).
[9] We use natural units with $\hbar = c = 1$ throughout this paper.

Upon introducing the wave function in momentum space $f(p)$ by means of the Fourier transformation

$$f(x-y)=\int d^4p\, f(p)e^{ip(x-y)}, \tag{5}$$

we find that Eq. (3) may be rewritten in the form

$$f(p)=-\frac{4ig^2}{(2\pi)^4}\frac{1}{(\frac{1}{2}E+p)^2+m^2}\times\frac{1}{(\frac{1}{2}E-p)^2+m^2}\int d^4k\, f(k)\left[\frac{1}{(p-k)^2}+\frac{1}{2E^2}\right]; \tag{6}$$

it is here to be understood that all denominators include a vanishingly small negative imaginary part $-i\delta$ in view of the causal nature of the functions $\Delta_F$ and $D_F$. Furthermore, if we set

$$f(p)=\left[-\frac{2ig^2}{(2\pi)^4}\frac{1}{E^2}\int d^4k\, f(k)\right]\psi(p), \tag{7}$$

Eq. (6) becomes

$$\psi(p)=\frac{1}{(\frac{1}{2}E+p)^2+m^2}\frac{1}{(\frac{1}{2}E-p)^2+m^2}\times\left[1-\frac{4ig^2}{(2\pi)^4}\int d^4k\frac{\psi(k)}{(p-k)^2}\right]. \tag{8}$$

It is evident that the substitution expressed by Eq. (7) is permissible only if

$$\int d^4k\, f(k)\neq 0.$$

This implies that, in the center-of-mass system, we are dealing only with $S$ waves. For all other angular-momentum states, we have

$$\int d^4k\, f(k)=0,$$

in which case Eq. (6) reduces to

$$f(p)=-\frac{4ig^2}{(2\pi)^4}\frac{1}{(\frac{1}{2}E+p)^2+m^2}\times\frac{1}{(\frac{1}{2}E-p)^2+m^2}\int d^4k\frac{f(k)}{(p-k)^2}. \tag{9}$$

It is thus clear that the pair-annihilation term will not affect states other than $S$ states.

Equation (9) will be recognized as identical to that solved by Wick[5] and Cutkosky.[6] The essential problem with which we are concerned in this paper is to study the more general situation given by Eq. (6); equivalently, we want to solve the inhomogeneous integral equation (8) for $\psi(p)$. Thus, in what follows, we confine our attention to the $S$-wave problem only.

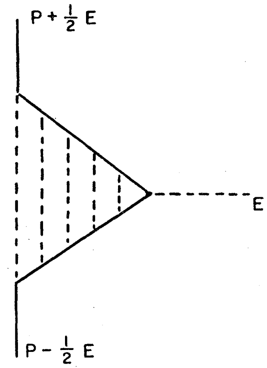


FIG. 2. Diagram for the generalized vertex operator $\Gamma$ in the ladder approximation.

It is of interest to notice that there exists a close relation between the wave function $\psi(p)$ and the generalized vertex operator[10] $\Gamma$. If we make the ladder approximation which is diagrammed in Fig. 2, then $\Gamma$ satisfies the inhomogeneous integral equation

$$\Gamma(p+\tfrac{1}{2}E,\, p-\tfrac{1}{2}E)=1-\frac{4ig^2}{(2\pi)^4}\int d^4k\frac{1}{(p-k)^2}\frac{1}{(\frac{1}{2}E+k)^2+m^2}\times\frac{1}{(\frac{1}{2}E-k)^2+m^2}\Gamma(k+\tfrac{1}{2}E,\, k-\tfrac{1}{2}E). \tag{10}$$

On comparing this with (8), we see that

$$\psi(p)=\frac{1}{(\frac{1}{2}E+p)^2+m^2}\times\frac{1}{(\frac{1}{2}E-p)^2+m^2}\Gamma(p+\tfrac{1}{2}E,\, p-\tfrac{1}{2}E). \tag{11}$$

While we have derived Eq. (11) for the case of the ladder approximation, the relation seems to hold true generally provided we replace the free-particle propagators $[(\frac{1}{2}E+p)^2+m^2]^{-1}$, $[(\frac{1}{2}E-p)^2+m^2]^{-1}$ by the modified propagators.

Returning to our problem, we now consider how to obtain the bound-state eigenvalues $E^2$. Upon integrating (7) over all $p$ and dividing both sides of this equation by the common factor $\int f(p)d^4p$, we obtain

$$1=-\frac{2ig^2}{(2\pi)^4}\frac{1}{E^2}\int\psi(p)d^4p. \tag{12}$$

From (11) it is clear that the quantity $\int\psi(p)d^4p$ corresponds to the closed-loop diagram given in Fig. 3.

[10] F. J. Dyson, Phys. Rev. 75, 1736 (1949).

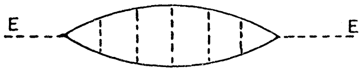

FIG. 3. Diagram for the quantity $\int\psi(p)d^4p$.

Indeed, if we set

$$\Pi^*(E)=\frac{2g^2}{(2\pi)^4}\int\psi(p)d^4p, \tag{13}$$

we find that this is exactly the same quantity as defined originally by Dyson.[10] Hence, Eq. (12) may be rewritten in the form

$$1=D_F(E)\Pi^*(E), \tag{14}$$

where

$$D_F(E)=-i/E^2.$$

On the other hand, the modified propagator $D_F'(E)$ is given by

$$D_F'(E)=D_F(E)[1-D_F(E)\Pi^*(E)]^{-1}.$$

It is therefore evident that, in solving (14), we are determining the poles of $D_F'(E)$. It is well-known that the poles of $D_F'(E)$ will give rise to a spectrum of bound states; it is now clear that they correspond to the bound states of the nucleon-antinucleon system.

In point of fact, $\Pi^*(E)$ is a divergent quantity, but the divergence can be removed by a self-energy subtraction. Thus, Eq. (12) should be replaced by

$$E^2=-\frac{2ig^2}{(2\pi)^4}\left\{\int\psi(p)d^4p-\left[\int\psi(p)d^4p\right]_{E^2=0}\right\}; \tag{15}$$

this insures that $E^2=0$, corresponding to the free-meson state with mass zero, is a solution. It is not necessary in our problem to carry out a charge renormalization, since the self-energy diagram is the only primitive divergent graph we encounter.

### III. INTEGRAL-TRANSFORM METHOD

Our basic problem is to develop a procedure for solving the integral equation (8). By way of generalization of methods used by several authors,[5,6,11] we express $\psi(p)$ in the following form:

$$\psi(p)=\int_0^\infty dx\int_0^\infty dy\frac{2H(x,y)}{[x(p^2+\frac{1}{4}E^2+m^2+pE)+y(p^2+\frac{1}{4}E^2+m^2-pE)+1]^3}. \tag{16}$$

Notice that the representation (16) as a function of $p$ is quite similar to the general representation of Green's functions given by Nambu[12] and Schwinger.[13]

On using the integral identity[11]

$$\int d^4k\frac{1}{(p-k)^2}\frac{1}{(k^2+\Lambda)^3}=\frac{i\pi^2}{2}\frac{1}{\Lambda(p^2+\Lambda)}, \tag{17}$$

we find

$$\int d^4k\frac{\psi(k)}{(p-k)^2}=\frac{i\pi^2}{2}\int_0^\infty dx\int_0^\infty dy\frac{h(x,y)}{[x(p^2+\frac{1}{4}E^2+m^2+pE)+y(p^2+\frac{1}{4}E^2+m^2-pE)+1]}, \tag{18a}$$

where, for simplicity, we have introduced the notation

$$h(x,y)=\frac{2H(x,y)}{(\frac{1}{4}E^2+m^2)(x+y)^2-\frac{1}{4}E^2(x-y)^2+(x+y)}. \tag{18b}$$

The integral formula (17) is essential in this work as well as in that of Wick. These methods are therefore useful only in the case of mesons of zero mass. We will return to this point in Sec. V.

Using next the identity

$$\frac{1}{abc}=2\int_0^\infty du\int_0^\infty dv\frac{1}{[au+bv+c]^3},$$

we have

$$I\equiv\frac{1}{(\frac{1}{2}E+p)^2+m^2}\frac{1}{(\frac{1}{2}E-p)^2+m^2}\int d^4k\frac{\psi(k)}{(p-k)^2}=i\pi^2\int_0^\infty du\int_0^\infty dv\int_0^\infty dx\int_0^\infty dy$$
$$\times\frac{h(x,y)}{[(u+x)(p^2+\frac{1}{4}E^2+m^2+pE)+(v+y)(p^2+\frac{1}{4}E^2+m^2-pE)+1]^3}.$$

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[11] S. Okubo, Progr. Theoret. Phys. (Kyoto) **10**, 692 (1953); **11**, 80 (1954).
[12] Y. Nambu, Nuovo cimento **6**, 1064 (1957).
[13] J. Schwinger, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957).

Upon changing the variables of integration $u$ and $v$ into $u-x$ and $v-y$, and noticing that

$$\int_0^\infty dx\int_x^\infty du=\int_0^\infty du\int_0^u dx,\ \text{etc.},$$

we obtain

$$I=i\pi^2\int_0^\infty du\int_0^\infty dv\int_0^u dx\int_0^v dy\frac{h(x,y)}{[u(p^2+\frac{1}{4}E^2+m^2+pE)+v(p^2+\frac{1}{4}E^2+m^2-pE)+1]^3}.$$

Equation (8) then becomes

$$\psi(p)=\int_0^\infty du\int_0^\infty dv\frac{1}{[u(p^2+\frac{1}{4}E^2+m^2+pE)+v(p^2+\frac{1}{4}E^2+m^2-pE)+1]^3}\left\{2+\frac{g^2}{4\pi^2}\int_0^u dx\int_0^v dy\, h(x,y)\right\}.$$

On comparing this with (16) and making use of (18b), we obtain finally

$$H(u,v)=1+\frac{g^2}{4\pi^2}\int_0^u dx\int_0^v dy\frac{H(x,y)}{(\frac{1}{4}E^2+m^2)(x+y)^2-\frac{1}{4}E^2(x-y)^2+(x+y)}.\tag{19}$$

We have reduced the four-dimensional integral equation (8) to a two-dimensional integral form.

It is readily seen that (19) is equivalent to the differential equation

$$\frac{\partial^2 H(u,v)}{\partial u\partial v}=\frac{g^2}{4\pi^2}\frac{H(u,v)}{(\frac{1}{4}E^2+m^2)(u+v)^2-\frac{1}{4}E^2(u-v)^2+(u+v)},\tag{19a}$$

with the boundary conditions

$$H(u=0,\,v)\equiv H(u,\,v=0)\equiv 1.\tag{19b}$$

It is, in fact, much more convenient to introduce new variables $\xi$ and $\eta$ by

$$u+v=\xi,\quad u-v=\eta;\tag{20}$$

Eqs. (19a,b) then go over into the form

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)G(\xi,\eta)=\frac{g^2}{\pi^2}\frac{G(\xi,\eta)}{\epsilon^2\eta^2+4(\omega^2\xi^2+\xi)},\tag{21a}$$

with

$$G(\xi,|\eta|=\xi)\equiv 1;\tag{21b}$$

we have here set

$$H(u,v)\equiv G(\xi,\eta),\tag{22}$$

and have also introduced the notations

$$\begin{aligned}&E^2=-\epsilon^2<0,\\&\omega^2=\tfrac{1}{4}E^2+m^2=m^2-\tfrac{1}{4}\epsilon^2>0.\end{aligned}\tag{23}$$

It is evident that $\epsilon$ is the rest mass of the bound system and is therefore also the total energy in the center-of-mass system. For bound states, we must have $\epsilon<2m$, whence $\omega^2>0$ and we can take $\omega>0$. The relation between $\omega$ and the binding energy $W$ is given by

$$\omega^2=W(m-\tfrac{1}{4}W).\tag{24}$$

It is quite difficult to solve Eqs. (21a,b) exactly. Hence, we will consider the case that the binding energy is quite small compared to the nucleon mass; this amounts to a form of nonrelativistic approximation. However, we must bear in mind that this is not equivalent to the usual nonrelativistic treatment in three-dimensional space, since our formalism is still covariant in four-dimensional space.

Thus, we assume

$$\epsilon\gg\omega,\tag{25}$$

or, equivalently,

$$\epsilon\cong 2m.$$

We now notice that, in the limit $\epsilon\to\infty$, we may make the replacement

$$\frac{1}{\epsilon^2\eta^2+4(\omega^2\xi^2+\xi)}\to\frac{\pi}{2\epsilon}\frac{1}{(\omega^2\xi^2+\xi)^{\frac{1}{2}}}\delta(\eta),\tag{26}$$

as shown originally by Wick.[5] With this approximation, Eq. (21a) can be solved subject to the boundary condition (21b); the solution has the form

$$G(\xi,\eta)=F(\xi-|\eta|),\tag{27a}$$

where

$$F(x)=\left[\frac{(\omega^2x^2+x)^{\frac{1}{2}}+\omega x}{(\omega^2x^2+x)^{\frac{1}{2}}-\omega x}\right]^\theta,\tag{27b}$$

and

$$\theta=g^2/4\pi\epsilon\omega.\tag{28}$$

It is now evident from (27b) that, for large values of the argument,

$$F(x)\sim x^\theta,\quad (x\to\infty)$$

and so, from Eqs. (27a), (22), and (20), we see that (16) diverges if $\theta\geqslant 1$. This fact is connected with the existence of a bound-state solution of the homogeneous

equation (9) at $\theta=1$. Actually, we can show that $\psi(p)$ as given by (16) has simple poles at

$$\theta=n, \quad (n=1,2,3,\cdots)$$

when we consider $\psi(p)$ as an analytic function of $\theta$ (see the Appendix). The residues are the corresponding solutions of the bound-state problem when we neglect the pair-annihilation interaction. We will see later that, when we apply Eq. (15) to determine the eigenvalues of the nucleon-antinucleon system, we will find

$$\theta=n+O(\omega/\epsilon).$$

We pause briefly to examine the relation between the results just obtained for the homogeneous equation and the Bohr formula for hydrogen-like atoms. From the nonrelativistic reduction of (9), using the method of Lévy,[14] one can establish the following relation between $g^2$ and the effective coupling $e^2Z$ which is involved in the ordinary Coulomb interaction:

$$g^2/4\pi m^2=e^2Z. \tag{29}$$

Then, from (28), (29), and the relation

$$\theta=n, \quad (n=1,2,3,\cdots)$$

we obtain the binding energies for our system (always with the neglect of the pair-annihilation term)

$$W=2m-\epsilon=(me^4Z^2/4n^2)[1+O(e^4Z^2)], \tag{30}$$

i.e., the Bohr formula, apart from higher-order corrections.

Evidently, our assumption (25) is equivalent to

$$4W/m\cong(4\omega/\epsilon)^2\cong(g^2/4\pi m^2)^2\ll 1, \tag{31}$$

i.e., the weak-coupling approximation. This is consistent with our neglect of graphs other than the ladder type.

Later, we will calculate higher-order corrections to the Bohr formula for the case of the nucleon-antinucleon bound-state problem; these come from relativistic and nucleon-antinucleon pair effects.

### IV. CALCULATION OF ENERGY EIGENVALUES

We now return to our original problem. According to Eq. (15), we must calculate $\int d^4p\,\psi(p)$. Using (27), (22), (20), and (16), we obtain

$$\int d^4p\,\psi(p)=i\pi^2\int_0^\infty d\xi\int_0^\xi d\eta\,\frac{F(\xi-|\eta|)}{\xi(\omega^2\xi^2+\xi+\frac{1}{4}\epsilon^2\eta^2)}\times[1+O(\omega/\epsilon)]$$

where we have made use of the integral formula

$$\int d^4p\,\frac{1}{(p^2+\Lambda)^3}=\frac{i\pi^2}{2}\frac{1}{\Lambda};$$

$\psi(p)$ is in error by an amount of order $\omega/\epsilon$ in view of the approximation (26). Upon changing $\eta$ into $\xi-\eta$ and then interchanging the order of the $\xi$ and $\eta$ integrations, we have

$$\int d^4p\,\psi(p)=i\pi^2\int_0^\infty d\eta\,F(\eta)I(\eta)[1+O(\omega/\epsilon)], \tag{32}$$

where

$$I(\eta)\equiv\int_\eta^\infty d\xi\,\frac{1}{\xi[\omega^2\xi^2+\xi+\frac{1}{4}\epsilon^2(\xi-\eta)^2]}. \tag{33}$$

The integral $I(\eta)$ may be evaluated by an elementary calculation; the result is as follows:

$$I(\eta)=\frac{2}{\epsilon^2\eta^2}\ln\left[\frac{4(\omega^2\eta+1)}{(4\omega^2+\epsilon^2)\eta}\right]-\frac{2}{\epsilon^2\eta^2}\times\frac{2-\epsilon^2\eta}{(\epsilon^2\omega^2\eta^2+\epsilon^2\eta-1)^{\frac{1}{2}}}\tan^{-1}\left[\frac{(\epsilon^2\omega^2\eta^2+\epsilon^2\eta-1)^{\frac{1}{2}}}{2\omega^2\eta+1}\right],$$
$$(\epsilon^2\omega^2\eta^2+\epsilon^2\eta-1>0), \tag{34a}$$

and

$$I(\eta)=\frac{2}{\epsilon^2\eta^2}\ln\left[\frac{4(\omega^2\eta+1)}{(4\omega^2+\epsilon^2)\eta}\right]+\frac{1}{\epsilon^2\eta^2}\frac{2-\epsilon^2\eta}{(1-\epsilon^2\eta-\epsilon^2\omega^2\eta^2)^{\frac{1}{2}}}\times\ln\frac{2\omega^2\eta+1-(1-\epsilon^2\eta-\epsilon^2\omega^2\eta^2)^{\frac{1}{2}}}{2\omega^2\eta+1+(1-\epsilon^2\eta-\epsilon^2\omega^2\eta^2)^{\frac{1}{2}}},$$
$$(\epsilon^2\omega^2\eta^2+\epsilon^2\eta-1<0). \tag{34b}$$

Since we are concerned with large $\epsilon/\omega$, the condition $\epsilon^2\omega^2\eta^2+\epsilon^2\eta-1>(<)0$ is essentially equivalent to $\eta>(<)1/\epsilon^2$; this leads to a neglect of terms of order $(\omega/\epsilon)^2$ in (32).

The integral (32) diverges for $\eta=0$, but this divergence will be canceled when we subtract $[\int d^4p\,\psi(p)]_{E^2=0}$. We cannot, of course, use (32) for the evaluation of $[\int d^4p\,\psi(p)]_{E^2=0}$, since we have assumed there that $-E^2$ is quite large. This term must therefore be calculated separately. From covariance, $\int d^4p\,\psi(p)$ is a function of $E^2$ only, and so

$$\left[\int d^4p\,\psi(p)\right]_{E^2=0}=\int d^4p\,\psi(p,E=0).$$

The function $\psi(p,E=0)$ has, however, already been evaluated[11] and is given by

$$\psi(p,E=0)=2\int_0^\infty d\eta\,\frac{\eta F(\alpha,\beta,2;-m^2\eta)}{[\eta(p^2+m^2)+1]^3}, \tag{35}$$

where $F$ represents the hypergeometric function, and $\alpha$, $\beta$ are given by

$$\begin{aligned}\alpha&=\tfrac{1}{2}\{1+[1+(g^2/\pi^2m^2)]^{\frac{1}{2}}\},\\ \beta&=\tfrac{1}{2}\{1-[1+(g^2/\pi^2m^2)]^{\frac{1}{2}}\}.\end{aligned} \tag{36}$$

[14] M. M. Lévy, Phys. Rev. 88, 725 (1952).

It is evident that (35) has essentially the same form as (16) for $E=0$.

Proceeding as before, we have

$$\int d^4p\,\psi(p,E=0)=i\pi^2\int_0^\infty d\eta\,\frac{F(\alpha,\beta,2;-m^2\eta)}{\eta(m^2\eta+1)},$$

and so

$$\int d^4p\,\psi(p)-\left[\int d^4p\,\psi(p)\right]_{E^2=0}$$
$$=i\pi^2\lim_{a\to+0}\left\{\int_{1/\epsilon^2}^{\infty}I(\eta)F(\eta)d\eta+\int_a^{1/\epsilon^2}I(\eta)F(\eta)d\eta-\int_a^\infty\frac{F(\alpha,\beta,2;-m^2\eta)}{\eta(m^2\eta+1)}d\eta\right\}[1+O(\omega/\epsilon)]. \quad (37)$$

We proceed to calculate the second term on the right-hand side of Eq. (37). Now

$$\int_a^{1/\epsilon^2}I(\eta)F(\eta)d\eta=\frac{1}{\epsilon^2}\int_{a\epsilon^2}^1 I(x/\epsilon^2)F(x/\epsilon^2)dx.$$

Neglecting terms of order $\omega/\epsilon$ compared to unity, we can set

$$F(x/\epsilon^2)\cong 1$$

and

$$I(x/\epsilon^2)\cong\epsilon^2\left\{\frac{2}{x^2}\ln\left(\frac{4}{x}\right)+\frac{1}{x^2}\frac{2-x}{(1-x)^{\frac{1}{2}}}\ln\frac{1-(1-x)^{\frac{1}{2}}}{1+(1-x)^{\frac{1}{2}}}\right\}.$$

In view of the indefinite integral

$$\int dx\left\{\frac{2}{x^2}\ln\left(\frac{4}{x}\right)+\frac{1}{x^2}\frac{2-x}{(1-x)^{\frac{1}{2}}}\ln\frac{1-(1-x)^{\frac{1}{2}}}{1+(1-x)^{\frac{1}{2}}}\right\}=\frac{2}{x}\ln\left(\frac{x}{4}\right)-\frac{2(1-x)^{\frac{1}{2}}}{x}\ln\frac{1-(1-x)^{\frac{1}{2}}}{1+(1-x)^{\frac{1}{2}}},$$

we have, in the limit as $a\epsilon^2\to+0$,

$$\int_a^{1/\epsilon^2}I(\eta)F(\eta)d\eta\cong 1-\ln(4a\epsilon^2). \quad (38)$$

Next we consider the third integral on the right-hand side of (37). One can show, after a somewhat lengthy calculation, that, in the limit as $am^2\to+0$,

$$\int_a^\infty d\eta\,\frac{F(\alpha,\beta,2;-m^2\eta)}{\eta(m^2\eta+1)}\cong-\ln(am^2)-[\Psi(\alpha)+\Psi(\beta)-\Psi(2)-\Psi(1)+1/\alpha\beta], \quad (39)$$

where

$$\Psi(x)=\frac{d}{dx}\ln\Gamma(x).$$

Actually, in the derivation of (39), we use only the relation $\alpha+\beta=1$. However, if we note the explicit form of $\alpha$, $\beta$ given by (36), and expand in a power series in $g^2/m^2$, then it may be easily seen that

$$\int_a^\infty d\eta\,\frac{F(\alpha,\beta,2;-m^2\eta)}{\eta(m^2\eta+1)}=-\ln(am^2)+O(g^2/m^2),\quad (am^2\to+0). \quad (39')$$

This result is equivalent to the replacement of $F(\alpha,\beta,2;-m^2\eta)$ in (39′) by unity, involving an error of the order of $g^2/m^2$. We will see shortly [this is already indicated by (31)] that

$$g^2/m^2=O(\omega/\epsilon),$$

whence this replacement is justified.

From (38) and (39′) we now have

$$\lim_{a\to+0}\left[\int_a^{1/\epsilon^2}I(\eta)F(\eta)d\eta-\int_a^\infty\frac{F(\alpha,\beta,2;-m^2\eta)}{\eta(m^2\eta+1)}d\eta\right]=1-\ln(4\epsilon^2/m^2)+O(\omega/\epsilon),$$

whence (37) becomes

$$\int d^4p\,\psi(p)-\left[\int d^4p\,\psi(p)\right]_{E^2=0}=i\pi^2\left[\int_{1/\epsilon^2}^\infty I(\eta)F(\eta)d\eta+1-\ln(4\epsilon^2/m^2)\right]+O(\omega/\epsilon). \quad (40)$$

Notice that the divergence in the limit as $a\to+0$ has been eliminated.

Finally, let us calculate

$$I\equiv\int_{1/\epsilon^2}^\infty I(\eta)F(\eta)d\eta.$$

We introduce a number $\lambda$ that satisfies the inequality

$$1/\omega^2\gg\lambda\gg1/\epsilon^2.$$

For example, we can take

$$\lambda=\epsilon^{-2}(\epsilon/\omega)^\Delta,\quad(2>\Delta>0).$$

Then, dividing the region of integration into two parts, we have

$$I=\left(\int_\lambda^\infty+\int_{1/\epsilon^2}^\lambda\right)I(\eta)F(\eta)d\eta.$$

Now, if we choose $1>\Delta>0$, then in the second region of integration where $\lambda>\eta>1/\epsilon^2$, we may replace $F(\eta)$ by unity, and we may write

$$I_1\equiv\int_{1/\epsilon^2}^\lambda I(\eta)F(\eta)d\eta\cong\int_{1/\epsilon^2}^\lambda I(\eta)d\eta$$
$$\cong\int_1^{\lambda\epsilon^2}dx\left[\frac{2}{x^2}\ln\left(\frac{4}{x}\right)-\frac{2}{x^2}\frac{2-x}{(x-1)^{\frac{1}{2}}}\tan^{-1}(x-1)^{\frac{1}{2}}\right]$$
$$=2\ln4-2\pi(\epsilon^2\lambda)^{-\frac{1}{2}}+O(\omega/\epsilon).$$

In the first region of integration where $\infty > \eta > \lambda$, we can approximate (34a) by

$$I(\eta) = (\pi/\epsilon\eta)(\omega^2\eta^2+\eta)^{-\frac{1}{2}}[1+O(\omega/\epsilon)],$$

so that

$$I_2 \equiv \int_\lambda^\infty I(\eta)F(\eta)d\eta = \frac{\pi}{\epsilon}\int_\lambda^\infty F(\eta)\frac{1}{\eta(\omega^2\eta^2+\eta)^{\frac{1}{2}}}d\eta + O(\omega/\epsilon).$$

Furthermore, it is evident that

$$\frac{\pi}{\epsilon}\int_{1/\epsilon^2}^{\lambda}\frac{d\eta}{\eta(\omega^2\eta^2+\eta)^{\frac{1}{2}}} = 2\pi - \frac{2\pi}{(\lambda\epsilon^2)^{\frac{1}{2}}} + O(\omega/\epsilon),$$

whence we have

$$\begin{aligned} I &= I_1 + I_2 \\ &= 2\ln 4 - 2\pi + \frac{\pi}{\epsilon}\int_{1/\epsilon^2}^{\infty}\frac{F(\eta)}{\eta(\omega^2\eta^2+\eta)^{\frac{1}{2}}}d\eta + O(\omega/\epsilon). \end{aligned} \quad (41)$$

Notice that the arbitrary number $\lambda$ has disappeared.

We now need to evaluate the integral

$$J \equiv \int_{1/\epsilon^2}^{\infty} d\eta \frac{F(\eta)}{\eta(\omega^2\eta^2+\eta)^{\frac{1}{2}}}.$$

Upon changing the integration variable from $\eta$ to $z$ by setting

$$z = \frac{(\omega^2\eta^2+\eta)^{\frac{1}{2}} - \omega\eta}{(\omega^2\eta^2+\eta)^{\frac{1}{2}} + \omega\eta},$$

we obtain

$$J = 4\omega\int_0^s (1-z)^{-2}z^{-\theta}dz,$$

where

$$s = 1 - 2(\omega/\epsilon) + O((\omega/\epsilon)^2).$$

To evaluate $J$, we consider the analytic function of $x$

$$J(x) = 4\omega\int_0^s (1-z)^x z^{-\theta}dz.$$

Assuming $\mathrm{Re}x > -1$, we have

$$\begin{aligned} J(x) &= 4\omega\int_0^1 (1-z)^x z^{-\theta}dz - 4\omega\int_s^1 (1-z)^x z^{-\theta}dz \\ &= 4\omega\frac{\Gamma(1+x)\Gamma(1-\theta)}{\Gamma(2+x-\theta)} - 4\omega\int_s^1 (1-z)^x z^{-\theta}dz. \end{aligned}$$

Since $s$ is close to unity, the second integral can be evaluated by expanding $z^{-\theta}$ in powers of $1-z$. This gives

$$J(x) = 4\omega\left[\frac{\Gamma(1+x)\Gamma(1-\theta)}{\Gamma(2+x-\theta)} - \frac{1}{1+x}(1-s)^{1+x} - \frac{\theta}{2+x}(1-s)^{2+x} + \cdots\right],$$

or, in the limit as $x \to -2$,

$$J = 4\omega\{\theta[\Psi(1) - \Psi(-\theta) + 1] - \ln(2\omega/\epsilon) + (\epsilon/2\omega) + O(1)\}.$$

Inserting this into (41), we have finally

$$\begin{aligned} I &\equiv \int_{1/\epsilon^2}^{\infty} I(\eta)F(\eta)d\eta \\ &= 2\ln 4 + 4\pi(\omega/\epsilon)\{\theta[\Psi(1) - \Psi(-\theta) + 1] \\ &\qquad - \ln(2\omega/\epsilon) + \tfrac{1}{2}\} + O(\omega/\epsilon). \end{aligned}$$

It will be clear later that

$$\theta = n + O(\omega/\epsilon),$$

and so $\Psi(-\theta)$ is of the order $\epsilon/\omega$. Therefore,

$$I = 2\ln 4 - 4\pi(\omega/\epsilon)\theta\Psi(-\theta) + O(\omega/\epsilon).$$

From this and (40), we have

$$\int d^4p\,\psi(p) - \left[\int d^4p\,\psi(p)\right]_{E^2=0} = i\pi^2\{-4\pi(\omega/\epsilon)\theta\Psi(-\theta) + 1 + O(\omega/\epsilon)\};$$

we have here discarded a term in $\ln(\epsilon^2/4m^2)$ which is approximately $4\omega^2/\epsilon^2$. Returning to our fundamental Eq. (15), we find that

$$\begin{aligned} 1 = &-(32\pi^2)^{-1}(g^2/m^2) \\ &\times[-4\pi(\omega/\epsilon)\theta\Psi(-\theta) + 1 + O(\omega/\epsilon)]. \end{aligned} \quad (42)$$

We can now show that it is consistent to set

$$\theta = n + O(\omega/\epsilon), \quad (n = 1, 2, 3, \cdots).$$

According to the definition of $\theta$ given by (28), this statement also means that

$$g^2/8\pi m^2 \cong (\omega/m)\theta = O(\omega/\epsilon).$$

It is evident from (42) that the right-hand side will be of the order $\omega/\epsilon$ unless $\Psi(-\theta)$ is of the order $(\epsilon/\omega)^2$. But this would lead to a contradiction since the left-hand side of (42) is unity. If we notice that

$$\Psi(-\theta) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{n-\theta} - \frac{1}{n+1}\right),$$

then the only possibility that remains is for $\theta$ to be near a pole of $\Psi(-\theta)$, *viz.*, $\theta \cong n$ $(n = 1, 2, \cdots)$; we exclude $n = 0$ because it has little physical meaning.

We may therefore set

$$\Psi(-\theta) = (\theta - n)^{-1} + O(1),$$

whence we can deduce from (42) and (28) that

$$\omega = (g^2/8\pi m)n^{-1}[1 + O(\omega/\epsilon)],$$

and also

$$\begin{aligned} \theta - n = &-\tfrac{1}{8}(g^2/4\pi m^2)^2 \\ &\times[1 - (8\pi)^{-1}(g^2/4\pi m^2) + O((g^2/m^2)^2)]^{-1}. \end{aligned} \quad (43)$$

From Eq. (28) we can now calculate $\omega$ and therefore $\epsilon$ in terms of $g^2/m^2$; our final result is

$$\epsilon/m=[1+D^{-1}(g^2/4\pi m^2)]^{\frac{1}{2}}+[1-D^{-1}(g^2/4\pi m^2)]^{\frac{1}{2}}, \quad (44)$$

where

$$D=n-\tfrac{1}{8}(g^2/4\pi m^2)^2 \times\{1-(8\pi)^{-1}(g^2/4\pi m^2)+O[(g^2/4\pi m^2)^2]\}^{-1}, \quad (45)$$

or, expressed as a series in $g^2/m^2$, we have

$$\frac{\epsilon}{m}=2-\frac{1}{4n^2}\left(\frac{g^2}{4\pi m^2}\right)^2-\left(\frac{5}{64}\frac{1}{n^4}+\frac{1}{16}\frac{1}{n^3}\right)\left(\frac{g^2}{4\pi m^2}\right)^4 -\frac{1}{128\pi}\frac{1}{n^3}\left(\frac{g^2}{4\pi m^2}\right)^5+O[(g^2/4\pi m^2)^6]. \quad (46)$$

From (29) we see that the bound-state energy eigenvalues, as given by Eq. (46), have the form of the Bohr formula for hydrogen-like atoms plus corrections. The $(g^2/4\pi m^2)^6$ term cannot be calculated within our approximation.

## V. REMARKS ON THE CASE OF MESONS OF NONZERO MASS AND THE FERMION-ANTIFERMION PROBLEM

We take up first the extension to mesons of nonzero mass. The only difference from the zero-mass case consists in the fact that the meson propagator $D_F$ is now replaced by $\Delta_F$. Therefore, instead of (8), we have the equation

$$\psi(p)=\frac{1}{(\frac{1}{2}E+p)^2+m^2}\frac{1}{(\frac{1}{2}E-p)^2+m^2} \times\left[1-\frac{4ig^2}{(2\pi)^4}\int d^4k\frac{\psi(k)}{(k-p)^2+\mu^2}\right], \quad (47)$$

where $\mu$ is the meson mass.

In this case, it is useless to express $\psi(p)$ in the generalized Stieltjes form (16), since, unfortunately, we cannot now use the convenient integral identity (17) to eliminate the momentum variables.

However, we can show that the momentum variables can be separated out by expressing $\psi(p)$ in the form of a generalized Gaussian transform. It will be noted that a similar transform was used recently by Nambu[12] in the problem of the general representation of Green's functions.

Let us write

$$\psi(p)=i\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\, z^2F(x,y,z)\exp\{-iz \times[x(p^2+\omega^2+pE)+y(p^2+\omega^2-pE)+1]\}, \quad (48)$$

where $\omega^2$ is given by (23). We can now derive an integral equation for $F$ from (47), *viz.*,

$$F(u,v,t)=1+\frac{ig^2}{4\pi^2}\int_0^u dx\int_0^v dy\int_t^\infty dz\frac{F(x,y,z)}{x+y} \times\exp\left[-i(x+y)(z-t)\omega^2+\frac{iE^2}{4}\frac{(x-y)^2}{x+y} \times(z-t)-i(z-t)-i\frac{zt}{z-t}(x+y)\mu^2\right]. \quad (49)$$

For the derivation of this equation, the reader is referred to the sequel to this paper.[15] In the case $\mu=0$, $F$ is independent of the third variable $z$. If we set

$$F(x,y,z)\equiv H(x,y),$$

then it can be seen that Eq. (48) reduces to (16) and Eq. (49) to (19). Evidently, the Gaussian representation (48) is the natural generalization of the Stieltjes form (16) which was used before.

Although it may appear that, in the case of mesons of nonzero mass, we have had to introduce a three-parameter representation, essentially only two of the parameters are independent. Thus, let us rewrite (48) in the form

$$\psi(p)=\int_0^\infty dx\int_0^\infty dy\,K(x,y) \times\exp[-ix(p^2+\omega^2+pE)-iy(p^2+\omega^2-pE)], \quad (50a)$$

where

$$K(x,y)=i\int_0^\infty dz\,e^{-iz}F(x/z,y/z,z). \quad (50b)$$

Then, from (49), we can derive the following integral equation:

$$K(u,v)=1-\frac{g^2}{4\pi^2}\int_0^1 dt\int_0^{u/t} dx\int_0^{v/t} dy\frac{K(x,y)}{x+y} \times\exp\left[-i(x+y)(1-t)\omega^2+\frac{iE^2}{4}\frac{(x-y)^2}{x+y} \times(1-t)-i\frac{t}{1-t}(x+y)\mu^2\right]. \quad (51)$$

Unfortunately, it is very difficult to solve (51) analytically, and we shall have nothing further to say about this problem.

We turn next to the fermion-antifermion problem. In this case, the general method given in Sec. II may be taken over with some modifications. Thus $\psi(p)$ will now

[15] S. Okubo and D. Feldman, following paper [Phys. Rev. **117**, 292 (1960)].

be a 4×4 matrix. Proceeding by analogy with the development which led to Eq. (8), we now obtain

$$\psi(p)=\frac{i\gamma(p+\frac{1}{2}E)-m}{(p+\frac{1}{2}E)^2+m^2}\left[\gamma_5+\frac{ig^2}{(2\pi)^4}\gamma_5\int d^4k\,\frac{\psi(k)}{(p-k)^2}\gamma_5\right]\times\frac{i\gamma(p-\frac{1}{2}E)-m}{(p-\frac{1}{2}E)^2+m^2};\quad(52)$$

instead of (11), the relation between $\psi$ and the generalized vertex operator $\Gamma_5$ is given by

$$\psi(p)=\frac{i\gamma(p+\frac{1}{2}E)-m}{(p+\frac{1}{2}E)^2+m^2}\Gamma_5(p+\tfrac{1}{2}E,\,p-\tfrac{1}{2}E)\times\frac{i\gamma(p-\frac{1}{2}E)-m}{(p-\frac{1}{2}E)^2+m^2};\quad(53)$$

also, instead of (12), we now have

$$E^2=\frac{-ig^2}{(2\pi)^4}\int d^4p\ \mathrm{tr}[\gamma_5\psi(p)].$$

Thus, as before, we can show that, in solving this equation for $E^2$, we are, in effect, determining the poles of the meson propagator $D_F'(E)$.

Notice that we have here assumed that the nucleon spinors interact via pseudoscalar coupling with a massless neutral pseudoscalar meson field. To extend the formalism to quantum electrodynamics, we have only to replace $\gamma_5$ by $\gamma_\mu$.

In principle, therefore, when we are dealing with Dirac particles, the situation is not much different than when we have scalar nucleons. However, Eq. (52) is quite difficult to solve because of the $\gamma$ matrices. We may write

$$\psi(p)=\gamma_5\psi_1+\gamma_5(\gamma E)\psi_2+\gamma_5(\gamma p)\psi_3+\gamma_5[(\gamma p)(\gamma E)-(\gamma E)(\gamma p)]\psi_4,$$

and assume that all the $\psi_i$ have the form (48); we can then separate out the $\gamma$ matrices and the momentum variables, obtaining a set of simultaneous integral equations for a set of functions $F_i$ which correspond to $F$ in (48). These equations are quite complicated so that we do not discuss the general case any further here.

There is still another difficulty in the fermion case, *viz.*, Eq. (52) has no finite solution because of the well-known divergence of the generalized vertex operator. However, this divergence can be eliminated by carrying out a charge renormalization. In view of Eq. (53), we may write

$$\psi(p)=Z_1^{-1}\psi_c(p),\quad(54)$$

where $Z_1$ is the renormalization constant defined by Dyson[10] and $\psi_c$ is finite. In place of (52), we now have the equation

$$\psi_c(p)=\frac{i\gamma(p+\frac{1}{2}E)-m}{(p+\frac{1}{2}E)^2+m^2}\left[\gamma_5Z_1+\frac{ig^2}{(2\pi)^4}\gamma_5\int d^4k\,\frac{\psi_c(k)}{(p-k)^2}\gamma_5\right]\times\frac{i\gamma(p-\frac{1}{2}E)-m}{(p-\frac{1}{2}E)^2+m^2}.\quad(55)$$

Of course, the general discussion of (55) is quite difficult. Here, we will examine the special case $E=0$ and compare with the results given previously in the scalar-nucleon case.

With $E=0$, the solution has the simple form

$$\psi_c(p)=\gamma_5F(p^2),\quad(56)$$

where $F$ is independent of the $\gamma$ matrices. Then, Eq. (55) reduces to

$$F(p^2)=\frac{Z_1}{p^2+m^2}+\frac{ig^2}{(2\pi)^4}\frac{1}{p^2+m^2}\int d^4k\,\frac{F(k^2)}{(p-k)^2}.\quad(57)$$

In the derivation of Eq. (57), we have assumed that the meson field is neutral. In the symmetrical theory, we find an additional sign change for the second term on the right-hand side of (57) due to the fact that the $\tau$-spin operators anticommute; in this case, instead of (57), we find

$$F_s(p^2)=\frac{Z_1}{p^2+m^2}-\frac{ig^2}{(2\pi)^4}\frac{1}{p^2+m^2}\int d^4k\,\frac{F_s(k^2)}{(p-k)^2}.\quad(58)$$

For the corresponding problem in quantum electrodynamics, one obtains, after having made some approximations,

$$F'(p^2)=\frac{Z_1}{p^2+m^2}-\frac{2ie^2}{(2\pi)^4}\frac{1}{p^2+m^2}\int d^4k\,\frac{F'(k^2)}{(p-k)^2}.\quad(58')$$

Equation (58), or its equivalent (58′), was first investigated by Edwards.[16] The minus sign in the second term on the right-hand side of (58′) comes from the fact that, in their commutation properties, the matrices $\gamma_5$ and $\gamma_\mu$ are not alike.

As we will see shortly, this difference of sign has important consequences for the solutions of Eqs. (57) and (58) [or (58′)]. In the latter case, Edwards showed that, in fact, no inhomogeneous solutions exist at all, i.e., $Z_1=0$; this result corresponds essentially to the finding by Goldstein[4] of a continuum of solutions in the bound-state fermion-fermion problem. The reason for this is the following: although we are investigating the nucleon-antinucleon problem or, equivalently, the integral equation for the generalized vertex operator $\Gamma_\mu$, the homogeneous equation corresponding to (58) or (58′) is actually the same as that obtained in the ladder approximation for the bound-state problem of the fermion-fermion system with fermions of opposite charge.

[16] S. F. Edwards, Phys. Rev. 90, 284 (1953).

In the case of the neutral $\gamma_5$-interaction (57), on the other hand, we will see that $Z_1=\infty$, so that, for this case, no solution of the homogeneous bound-state integral equation exists at all. Both of these results are completely different from the scalar-nucleon case, which we treated earlier.

We now discuss the solution of Eqs. (57) and (58). By analogy with the scalar-nucleon case, we express $F(p^2)$ as a Stieltjes transform, i.e.,

$$F(p^2)=\int_0^\infty dt\frac{G(t)}{(p^2+m^2+t)^3}. \tag{59}$$

Then, using (17) and noticing that

$$\frac{1}{p^2+m^2}\int_0^\infty dt\frac{G(t)}{m^2+t}\frac{1}{p^2+m^2+t}$$
$$=\int_0^\infty dt\frac{2G(t)}{t(m^2+t)}\int_0^t dx\int_x^\infty dy\frac{1}{(p^2+m^2+y)^3}$$
$$=\int_0^\infty dy\frac{1}{(p^2+m^2+y)^3}\int_0^y dx\int_x^\infty dt\frac{2G(t)}{t(m^2+t)},$$

we find that (57) will be satisfied provided

$$G(t)=2tZ_1-\frac{g^2}{16\pi^2}\int_0^t dx\int_x^\infty dy\frac{G(y)}{y(m^2+y)}. \tag{60}$$

This is equivalent to the differential equation

$$t(m^2+t)\frac{d^2}{dt^2}G(t)-\frac{g^2}{16\pi^2}G(t)=0 \tag{61}$$

subject to the boundary conditions

$$G(t=0)=0, \tag{62a}$$

$$(dG/dt)_{t=\infty}=2Z_1. \tag{62b}$$

Equation (61) has the two independent solutions

$$G=(x^2-1)^{\frac{1}{2}}P_\nu^1(x),\quad (x^2-1)^{\frac{1}{2}}Q_\nu^1(x),$$

where

$$x=(2t/m^2)+1, \tag{63}$$

and $\nu$ is given by

$$\nu(\nu+1)=g^2/16\pi^2; \tag{64}$$

the functions $P_\nu^1$ and $Q_\nu^1$ are the associated Legendre functions of the first and second kind, respectively. Since the solution $Q_\nu^1$ does not satisfy the boundary condition (62a), our solution must be of the form

$$G=C(x^2-1)^{\frac{1}{2}}P_\nu^1(x),$$

where $C$ is a constant to be determined.

Without loss of generality, we can take $\nu>0$, since the other solution of (64) is given by $-(\nu+1)$ and

$$P_{-(\nu+1)}^1(x)\equiv P_\nu^1(x).$$

Then, from

$$\frac{dG}{dt}=\frac{2}{m^2}C\nu(\nu+1)P_\nu(x),$$

and the asymptotic relation for the Legendre functions

$$P_\nu(x)=\frac{\Gamma(1+2\nu)}{\Gamma^2(1+\nu)}\left(\frac{t}{m^2}\right)^\nu[1+O(m^2/t)],\quad (\nu>0)$$

for $t/m^2\to\infty$, we find, from (62b), that

$$C=\frac{m^2}{\nu(\nu+1)}\frac{\Gamma^2(1+\nu)}{\Gamma(1+2\nu)}L^{-\nu}Z_1,$$

where $L$ is the value of $t/m^2$ at infinity. We have finally

$$G=\frac{m^2}{\nu(\nu+1)}\frac{\Gamma^2(1+\nu)}{\Gamma(1+2\nu)}L^{-\nu}Z_1(x^2-1)^{\frac{1}{2}}P_\nu^1(x). \tag{65}$$

The renormalization constant $Z_1$ can be determined in the standard way.[10] From (53), (54), and (56), we have

$$\Gamma_{5c}(p,p)=(p^2+m^2)F(p^2)\gamma_5\to\gamma_5,\quad (p^2+m^2\to 0).$$

From (57), this is equivalent to

$$[(p^2+m^2)F(p^2)]_{p^2+m^2=0}$$
$$=1=Z_1+\frac{ig^2}{(2\pi)^4}\left[\int d^4k\frac{F(k^2)}{(p-k)^2}\right]_{p^2+m^2=0}$$
$$=Z_1-\frac{g^2}{32\pi^2}\int_0^\infty dt\frac{G(t)}{t(m^2+t)}.$$

On comparing this with Eq. (60) (differentiated once with respect to $t$), we find

$$(dG/dt)_{t=0}=2.$$

Hence, from (65), we must have

$$Z_1=\frac{\Gamma(1+2\nu)}{\Gamma^2(1+\nu)}L^\nu, \tag{66}$$

and so

$$G(t)=\frac{m^2}{\nu(\nu+1)}(x^2-1)^{\frac{1}{2}}P_\nu^1(x). \tag{67}$$

Notice that $Z_1=\infty$ since $\nu>0$ and $L\to+\infty$.

The function $F(p^2)$ may be calculated next from (59); after carrying out an integration by parts, we have

$$F(p^2)=\int_0^\infty dt\frac{P_\nu[1+(2/m^2)t]}{(p^2+m^2+t)^2} \tag{68a}$$

$$=\frac{1}{m^2}\frac{\pi\nu(\nu+1)}{\sin\nu\pi}F(2+\nu,1-\nu,2;-p^2/m^2). \tag{68b}$$

In the symmetrical meson theory or in quantum electrodynamics, where Eq. (58) or (58′) is our starting point, we can proceed in a similar way. An essential difference appears when we come to the analogue of Eq. (64) which will now read

$$\nu(\nu+1)=-g^2/16\pi^2. \tag{69}$$

The two solutions of this equation are both negative with $0>\nu>-1$. Once again, without loss of generality, we can take $\nu$ to be the larger solution and so

$$0>\nu>-\tfrac{1}{2}.$$

The solution will again have the form (67), with the renormalization constant $Z_1$ given by (66).

The only difference is that $\nu$ is now negative, and so, from (66), we have

$$Z_1=0,$$

which is Edwards' result.[16] Then Eq. (58) is essentially a homogeneous equation whose solution is given by (68b), where $\nu$ is defined by (69). This result agrees exactly with the solution given by Goldstein[4] who treated the homogeneous equation corresponding to (58) which, as we saw earlier, is equivalent to the bound-state integral equation of the nucleon-antinucleon system without the annihilation diagram, or to that of the two-nucleon system with opposite charge in the ladder approximation. There is, in this case, no solution of the inhomogeneous equation (58), and we accordingly have the continuous level spectrum of Goldstein.

In the case of the neutral $\gamma_5$ coupling, we see, on the other hand, that there is no solution of the homogeneous equation of (57); hence, there are no bound states for the nucleon-antinucleon system without the annihilation diagram, or, equivalently, for the two-nucleon system in the simple ladder approximation.

Thus, we see that there is a considerable difference in our results depending on whether the nucleon is a scalar or a spinor particle. The latter case is peculiar, at least when $E=0$; it is not at all clear whether a reasonable solution exists for the general situation $E\neq 0$.


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### APPENDIX. INVESTIGATION OF THE ANALYTIC PROPERTIES OF THE WAVE FUNCTION $\psi(p)$

Here, we prove that $\psi(p)$, regarded as a function of $\theta$, has poles at $\theta=n$ ($n=1, 2, 3, \cdots$) as was stated in the argument following Eq. (28).

By using Eqs. (20), (22), and (27a), we can rewrite (16) as follows:

$$\psi(p)=\int_0^\infty d\xi\int_{-\xi}^{\xi} d\eta\,\frac{F(\xi-|\eta|)}{[\xi(p^2+\omega^2)+\eta pE+1]^3}$$
$$=\int_0^\infty d\xi\int_0^{\xi} d\eta\,F(\eta)\left(\frac{1}{[\xi(p^2+\omega^2)+(\xi-\eta)pE+1]^3}+\frac{1}{[\xi(p^2+\omega^2)-(\xi-\eta)pE+1]^3}\right).$$

On interchanging the order of the $\xi$ and $\eta$ integrations and then carrying out the $\xi$ integration, we find

$$\psi(p)=\tfrac{1}{2}\left(\frac{1}{p^2+\omega^2+pE}+\frac{1}{p^2+\omega^2-pE}\right)\times\int_0^\infty d\eta\,\frac{F(\eta)}{[\eta(p^2+\omega^2)+1]^2},$$

where $F(\eta)$ is given by (27b).

Upon changing the integration variable from $\eta$ to $z$ according to the transformation

$$z=\frac{(\omega^2\eta^2+\eta)^{\frac{1}{2}}+\omega\eta}{(\omega^2\eta^2+\eta)^{\frac{1}{2}}-\omega\eta},$$

we obtain

$$\psi(p)=\left(\frac{1}{p^2+\omega^2+pE}+\frac{1}{p^2+\omega^2-pE}\right)\times\frac{1}{\omega^2}\left(\frac{\omega^2}{p^2+\omega^2}\right)^2 I(\theta), \tag{A1}$$

where

$$I(\theta)=\int_1^\infty dz\,\frac{z^\theta(z^2-1)}{[(1-z)^2+2\lambda z]^2}, \tag{A2}$$

and

$$\lambda=2\omega^2/(p^2+\omega^2). \tag{A3}$$

Taking a number $N$ which is arbitrary but large ($N\gg 1$), we divide the region of integration in (A2) into two parts, *viz.*,

$$I(\theta)=I_1(\theta)+I_2(\theta)=\left(\int_1^N+\int_N^\infty\right)dz\,\frac{z^\theta(z^2-1)}{[(1-z)^2+2\lambda z]^2}.$$

Now $I_1(\theta)$ and all its derivatives with respect to $\theta$ are finite for all $\theta$, so that $I_1(\theta)$ is a regular function in the complex $\theta$ plane except for $\theta=\infty$; it therefore has no poles. In $I_2(\theta)$, on the other hand, we can expand the denominator in a power series in $1/z$ since $z\geqslant N\gg 1$.

Then

$$I_2(\theta)=\int_N^\infty dz\frac{z^{\theta-2}-z^{\theta-4}}{[1-2(1-\lambda)(1/z)+(1/z^2)]^2}$$
$$=\sum_{n=0}^{\infty}\int_N^\infty dz\,[z^{\theta-2}-z^{\theta-4}]C_n^2(1-\lambda)\left(\frac{1}{z}\right)^n;$$

the $C_n^2$ are the Gegenbauer polynomials. Upon carrying out the $z$ integration, we have

$$I_2(\theta)=\sum_{n=0}^{\infty}C_n^2(1-\lambda)$$
$$\times\left[\frac{1}{n+1-\theta}\left(\frac{1}{N}\right)^{n+1-\theta}-\frac{1}{n+3-\theta}\left(\frac{1}{N}\right)^{n+3-\theta}\right].$$

Evidently, $I_2(\theta)$ has poles at

$$\theta=1,2,3,\cdots,$$

and its residue at $\theta=n$ is given by

$$-C_{n-1}^2(1-\lambda)+C_{n-3}^2(1-\lambda)$$
$$=-C_{n-1}^2\left(\frac{p^2-\omega^2}{p^2+\omega^2}\right)+C_{n-3}^2\left(\frac{p^2-\omega^2}{p^2+\omega^2}\right),$$

where it is understood that $C_m^2\equiv0$ for $m<0$.

The residues of $\psi(p)$ are evidently solutions of the homogeneous equation (9), and thus are the bound-state wave functions in the absence of the pair-annihilation interaction; the corresponding energy eigenvalues are determined by the relation $\theta=n=1,2,3,\cdots$. From (A1) and (A2), the solutions of (9) are given by

$$\phi_n(p)=\text{const}\times\left(\frac{1}{p^2+\omega^2+pE}+\frac{1}{p^2+\omega^2-pE}\right)\left(\frac{\omega}{p^2+\omega^2}\right)^2$$
$$\times\left[C_{n-1}^2\left(\frac{p^2-\omega^2}{p^2+\omega^2}\right)-C_{n-3}^2\left(\frac{p^2-\omega^2}{p^2+\omega^2}\right)\right].\quad\text{(A4)}$$

Of course, these wave functions are valid only for large $\epsilon$ since we have used the approximation (26).

When $E=0$, on the other hand, we can get the exact solutions for the bound states of (9). Equation (35) can be integrated,[17] yielding

$$\psi(p,E=0)=\frac{\pi(\alpha\beta)}{\sin(\pi\beta)}\frac{m^2}{(p^2+m^2)^3}$$
$$\times F\left(\alpha+1,\beta+1,2;\frac{p^2}{p^2+m^2}\right),\quad\text{(A5)}$$

where we have used $\alpha+\beta=1$. Evidently, $\psi(p)$, regarded as an analytic function of $\alpha$ or $\beta$, has simple poles. In particular, it has such poles when

$$\beta=-n,\quad(n=1,2,3,\cdots),$$

which, in view of (36), lead to the eigenvalues

$$g^2/4\pi^2m^2=n(n+1).\quad\text{(A6)}$$

The corresponding residues of $\psi(p,E=0)$ at $\beta=-n$ are given by

$$\phi_n(p)=\text{const}\times\frac{1}{(p^2+m^2)^3}F\left(n+2,1-n,2;\frac{p^2}{p^2+m^2}\right)$$
$$=\text{const}\times\frac{1}{(p^2+m^2)^3}(x^2-1)^{-\frac{1}{2}}P_n^1(x),$$
$$(n=1,2,\cdots),\quad\text{(A7)}$$

where $x=-(p^2-m^2)/(p^2+m^2)$ and $P_n^1(x)$ is the associated Legendre function. The eigenvalues (A6) and eigenfunctions (A7) agree with those given by Cutkosky[6] who investigated the bound-state problem in the absence of the pair-annihilation term, using a generalization of Fock's stereographic-projection method.[18] Of course, in (A7), we have obtained the solution for $S$ states.

[17] A. Erdélyi *et al.*, *Tables of Integral Transforms* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, p. 400, Eq. (10).
[18] V. Fock, Z. Physik **98**, 145 (1935).