Some Aspects of the Covariant Two-Body Problem. II. The Scattering Problem*

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By way of extension of a previous study of the bound-state problem within the framework of the covariant Bethe-Salpeter formalism, the scattering of nucleons by nucleons has been investigated in the ladder approximation, assuming scalar nucleons and zero-mass scalar mesons. The solution of the scattering problem can be effected by using an integral-transform method which is similar to that developed for the bound-state problem, subject to the assumption that the nucleon mass is large compared to the kinetic energy in the center-of-mass system. An essential complication which is now encountered is the appearance of an infrared divergence. Two methods of circumventing this difficulty are discussed. The cross section can be calculated from the Green's function by a limiting process; the usual S-matrix formalism leads to incorrect results in this case, and an amplitude renormalization is required. In this connection, it is instructive to re-examine, in detail, the Coulomb-scattering problem in momentum space, since this is very closely related to the Bethe-Salpeter scattering problem. The Green's-function method turns out to be unsuitable for the calculation of higher-order corrections to the nucleon-nucleon scattering cross section, so that here a cutoff procedure must be used.

I. INTRODUCTION

IN a previous paper,[1] we developed a procedure for handling the bound states of the nucleon-antinucleon system for the case of scalar nucleons. Here, we extend the method to study the nucleon-nucleon scattering problem in the ladder approximation in terms of the covariant Bethe-Salpeter (B-S) formalism. Once again, it is quite difficult to treat the spinor-nucleon case, and so we restrict ourselves to the same model as was used previously, *viz.*, scalar nucleons and zero-mass scalar mesons. One can hope that an extension of the method may eventually be helpful in treating the problem of the scattering of spinor nucleons by nucleons.

In our discussion of the scattering problem, we shall find it very helpful to use the integral-transform method developed previously for the bound-state problem. A similar approach has been presented independently by Nishijima,[2] but his results appear to be in error since he did not treat properly the infrared divergence which appears in the theory. The appearance of an infrared divergence in the scattering problem represents an important difference from the bound-state problem and leads to complications. These are connected with the existence of a distorted incoming wave in the Coulomb-scattering problem, as will be seen later.

As is well-known, the infrared divergences which appear in quantum electrodynamics have their origin in the emission of virtual soft photons and should be canceled by terms which come from the emission of real soft photons.[3] This is true to any order of perturbation theory and is also true in our scalar-photon case. The difficulty in our problem is that our solution of the B-S equation does not correspond to simple perturbation theory, and so we must take into account real soft-photon emission in a non-perturbation-theoretical way. To do this properly seems to be quite difficult; hence, we content ourselves in this paper with solving the B-S equation without considering soft-photon emission. We introduce, instead, a low-frequency cutoff for the virtual-photon energy.

Our treatment of the scattering problem is based on a covariant form of nonrelativistic approximation which is similar to that used in I. It turns out that, in the lowest-order approximation, all infrared divergences are contained in a phase factor and so do not lead to any difficulties at all. In fact, the cross section is then given exactly by the classical Rutherford formula for the scattering of a particle by a Coulomb potential. However, in the calculation of higher-order corrections to the Rutherford formula, the infrared divergences appear in a nontrivial way.

In the extreme low-energy limit, the cross section may be obtained from the Green's function by means of a limiting process. Since we are dealing with zero-mass mesons, however, the usual S-matrix formalism does not give the correct answer, and an amplitude renormalization for the wave function is required. This situation is clarified by considering the classical Rutherford scattering problem in momentum space.

For reasons that will be given later, the Green's-function method is not suitable for the calculation of higher-order corrections, so that here we must use a cutoff procedure. In this case, an amplitude renormalization is not necessary.

In Sec. II, we set up the general formulation of our scattering problem; in Sec. III, we consider the application of the integral-transform method; in Sec. IV,

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[1] S. Okubo and D. Feldman, preceding paper [Phys. Rev. **117**, 279 (1960)], henceforth referred to as I.

[2] K. Nishijima, Progr. Theoret. Phys. (Kyoto) **14**, 203 (1955).

[3] See, for example, J. Jauch and F. Rohrlich, Helv. Phys. Acta **27**, 613 (1954).

we investigate, in detail, the Coulomb-scattering problem in momentum space; in Sec. V, we return to consider the solution of the B-S equation.

II. FORMULATION OF THE PROBLEM

For simplicity, we consider a charged scalar nucleon field ϕ in interaction with a neutral scalar field A with zero rest mass, the interaction Hamiltonian being given by

$$H_1 = g\phi^*\phi A. \tag{1}$$

Assuming, for definiteness, that both particles have positive charge, the B-S wave function $\chi(x,y)$ which characterizes the two-nucleon scattering state satisfies, in the ladder approximation, the following inhomogeneous integral equation:

$$\chi(x,y) = \chi_{\text{in}}(x,y) - g^2 \int\int d^4x_1 d^4y_1\, \Delta_F(x-x_1) \times \Delta_F(y-y_1) D_F(x_1-y_1)\chi(x_1,y_1), \tag{2}$$

where χ_{in} represents the wave function of the incident free particles. The functions Δ_F and D_F are the same as were used in I; notice also that g^2 in (2) corresponds to $4g^2$ of I. Equation (2) may be derived by the method of Gell-Mann and Low.[4]

We now separate out the center-of-mass motion from the internal motion by writing

$$\chi(x,y) = e^{i(E/2)(x+y)} f(x-y), \tag{3}$$

where E is the energy-momentum four-vector of the total system (we use natural units with $\hbar = c = 1$). Denoting by p_1 and p_2 the four-momenta of the two incident nucleons, we may set

$$p_1 = \tfrac{1}{2}E + k, \quad p_2 = \tfrac{1}{2}E - k;$$

since

$$p_1^2 + m^2 = p_2^2 + m^2 = 0,$$

we have also

$$k^2 = -(\tfrac{1}{4}E^2 + m^2), \tag{4a}$$

$$kE = 0. \tag{4b}$$

The function χ_{in} may now be written as follows:

$$\chi_{\text{in}}(x,y) = \text{const} \times e^{i(E/2)(x+y)} e^{ik(x-y)}.$$

Since the two nucleons are identical and obey Bose statistics, we should, strictly speaking, symmetrize χ_{in} with respect to an interchange of the two particles. However, this symmetrization can be effected at the end by symmetrizing the total wave function with respect to the replacement of k by $-k$.

On defining the Fourier transform of (3) by

$$\chi(x,y) = e^{i(E/2)(x+y)} \int d^4p\, e^{ip(x-y)} \psi(p),$$

we find that $\psi(p)$ satisfies the following equation:

$$\psi(p) = \delta(p-k) - \frac{ig^2}{(2\pi)^4} \frac{1}{p^2 - k^2 + pE} \frac{1}{p^2 - k^2 - pE} \times \int d^4p' \frac{\psi(p')}{(p-p')^2}. \tag{5}$$

A vanishingly small negative imaginary part is understood to be contained in all denominators; also, for convenience, the constant in χ_{in} has been taken to be unity.

As we will see later, Eq. (5) has no finite solution because of the appearance of an infrared divergence. One way to avoid this difficulty is to consider that the meson field has a small but finite rest mass μ; another way is to work with the Green's function instead of the B-S wave function. For the sake of reference, we write down the equation for the Green's function in the general case of a meson field of finite mass:

$$G(p,q,E) = \delta(p-q) - \frac{ig^2}{(2\pi)^4} \frac{1}{p^2 - k^2 + pE} \frac{1}{p^2 - k^2 - pE} \times \int d^4p' \frac{G(p',q,E)}{(p-p')^2 + \mu^2}; \tag{6}$$

here, q is an arbitrary four-vector, and k^2 must be interpreted from (4a) as a function of E^2. Equation (6) represents the transcription in terms of momentum variables of the inhomogeneous integral equation satisfied by the two-nucleon Green's function $G(x,y;x',y')$ in the ladder approximation.[4]

With $q = k$, $G(p,q,E)$ is essentially the wave function $\psi(p)$ of the scattering problem with a finite meson mass, where $\psi(p)$ satisfies the equation

$$\psi(p) = \delta(p-k) - \frac{ig^2}{(2\pi)^4} \frac{1}{p^2 - k^2 + pE} \frac{1}{p^2 - k^2 - pE} \times \int d^4p' \frac{\psi(p')}{(p-p')^2 + \mu^2}. \tag{5'}$$

On setting $\mu = 0$, we of course regain Eq. (5).

Finally, we consider the physical meaning of the vector k. In the center-of-mass system,

$$E = (0,0,0,i\epsilon), \tag{7}$$

where ϵ is the total energy of the system. From (4b), it follows that k must have the form

$$k = (\mathbf{k}, 0), \tag{8}$$

where $\mathbf{k}$ is a three-dimensional vector. From (4a), we then have

$$\epsilon = 2(\mathbf{k}^2 + m^2)^{\frac{1}{2}}. \tag{9}$$

Evidently, $\mathbf{k}$ and $-\mathbf{k}$ are the wave numbers of the two nucleons in the center-of-mass system.

[4] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

III. THE INTEGRAL-TRANSFORM METHOD

We now turn to a consideration of Eq. (6). Let

$$G(p,q,E)=\delta(p-q)+F(p,q,E); \tag{10}$$

we then find that

$$F(p,q,E)=-\frac{ig^2}{(2\pi)^4}\frac{1}{p^2-k^2+pE}\frac{1}{p^2-k^2-pE}\times\left[\frac{1}{(p-q)^2+\mu^2}+\int d^4p'\frac{F(p',q,E)}{(p-p')^2+\mu^2}\right]. \tag{11}$$

Next, we express $F(p,q,E)$ in the form of a Gaussian transform, *viz.*,

$$F(p,q,E)=\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\, z^2H(x,y,z)\times\exp\{-iz[(p^2-k^2+pE)x+(p^2-k^2-pE)y+(p-q)^2+\mu^2]\}. \tag{12}$$

It is worth while noticing that Nambu,[5] in his investigation of the general representation of Green's functions, found a form which is almost the same as (12). Since we are interested in the p dependence of F (q and E are fixed), $H(x,y,z)$ is independent of p but may depend upon q and E. In (12), p^2 is understood to include vanishingly small imaginary parts.

Now, using

$$\frac{1}{(p-p')^2+\mu^2}=i\int_0^\infty du\,\exp\{-iu[(p-p')^2+\mu^2]\},$$

and (12), we find, upon replacing the integration variable u by uz,

$$I\equiv\int d^4p'\frac{F(p',q,E)}{(p-p')^2+\mu^2}$$
$$=i\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\int_0^\infty du\, z^3H(x,y,z)\times\int d^4p'\exp(-iz\{(p'^2-k^2+p'E)x+(p'^2-k^2-p'E)y+(p'-q)^2+\mu^2+[(p'-p)^2+\mu^2]u\}).$$

The integration over p' may be carried out with aid of the integral formula

$$\int d^4p'\exp(iap'^2)=i\frac{\pi^2}{a|a|}. \tag{13}$$

Upon carrying out, subsequently, a change of the integration variable from u to t by setting

$$t=uz/(x+y+u+1),$$

and then interchanging the order of the z and t integrations, we find that

$$I=\pi^2\int_0^\infty dx\int_0^\infty dy\int_0^\infty dt\int_t^\infty dz\frac{H(x,y,z)}{x+y+1}K(x,y,z,t)\times\exp\{-it[x(p^2-k^2+pE)+y(p^2-k^2-pE)+(p-q)^2+\mu^2]\},$$

where we have introduced the abbreviation

$$K(x,y,z,t)=\exp\left\{ik^2(x+y)(z-t)-iq^2\frac{x+y}{x+y+1}(z-t)+iqE\frac{y-x}{x+y+1}(z-t)+i\frac{E^2}{4}\frac{(x-y)^2}{x+y+1}(z-t)-i\mu^2\left[\frac{zt}{z-t}(x+y+1)+z-t\right]\right\}. \tag{14}$$

Next, we observe that

$$\frac{1}{p^2-k^2+pE}\frac{1}{p^2-k^2-pE}=-\int_0^\infty du\int_0^\infty dv\times\exp[-iu(p^2-k^2+pE)-iv(p^2-k^2-pE)].$$

Multiplying this by I and replacing the variables u and v by ut and vt, respectively, we obtain

$$J\equiv\frac{1}{p^2-k^2+pE}\frac{1}{p^2-k^2-pE}\int d^4p'\frac{F(p',q,E)}{(p-p')^2+\mu^2}$$
$$=-\pi^2\int_0^\infty du\int_0^\infty dv\int_0^\infty dx\int_0^\infty dy\int_0^\infty dt\int_t^\infty dz\times t^2\frac{H(x,y,z)}{x+y+1}K(x,y,z,t)\exp\{-it\times[(x+u)(p^2-k^2+pE)+(y+v)(p^2-k^2-pE)+(p-q)^2+\mu^2]\}.$$

With the substitution of $u-x$, $v-y$ for the variables u, v, respectively, followed by an interchange of the order of the x and u integrations, etc., we obtain finally

$$J=-\pi^2\int_0^\infty du\int_0^\infty dv\int_0^\infty dt\, t^2\exp\{-it[u(p^2-k^2+pE)+v(p^2-k^2-pE)+(p-q)^2+\mu^2]\}\times\int_0^u dx\int_0^v dy\int_t^\infty dz\frac{H(x,y,z)}{x+y+1}K(x,y,z,t). \tag{15}$$

Notice that this has exactly the same form as (12).

[5] Y. Nambu, Nuovo cimento **6**, 1064 (1957).

For the inhomogeneous term of (11), we write

$$\frac{1}{p^2-k^2+pE}\,\frac{1}{p^2-k^2-pE}\,\frac{1}{(p-q)^2+\mu^2} =-i\int_0^\infty du\int_0^\infty dv\int_0^\infty dt\, t^2 \exp\{-it[u(p^2-k^2+pE) +v(p^2-k^2-pE)+(p-q)^2+\mu^2]\}. \quad (16)$$

Upon inserting (12), (15), and (16) into (11), we infer that H must satisfy the integral equation

$$H(u,v,t)=-\frac{g^2}{(2\pi)^4}+\frac{ig^2}{16\pi^2} \times\int_0^u dx\int_0^v dy\int_t^\infty dz\frac{H(x,y,z)}{x+y+1}K(x,y,z,t), \quad (17)$$

where K is given by (14).

Now, in view of Eqs. (10) and (12), it is clear that (17) is an integral equation for the Green's function for fixed q and E. Since we are interested in the real scattering problem, we have to set $q=k$. Making use of (14) and (4b), we find

$$H(u,v,t)=-\frac{g^2}{(2\pi)^4}+\frac{ig^2}{16\pi^2}\int^u dx\int_0^v dy\int_t^\infty dz\frac{H(x,y,z)}{x+y+1} \times\exp\left\{ik^2\frac{(x+y)^2}{x+y+1}(z-t)-i\frac{\epsilon^2}{4}\frac{(x-y)^2}{x+y+1}(z-t) -i\mu^2\left[\frac{zt}{z-t}(x+y+1)+z-t\right]\right\}, \quad (q=k), \quad (18)$$

where we have replaced E^2 by $-\epsilon^2$, according to (7).

We investigate next the relation between $H(x,y,z)$ and the scattering cross section. The wave function in coordinate space is defined by

$$\psi(x)=\int d^4p\,\psi(p)e^{ipx}.$$

Recalling that

$$\psi(p)=G(p,\,q=k,\,E),$$

and making use of Eqs. (10) and (12), we obtain

$$\psi(x)=e^{ikx}+\int_0^\infty du\int_0^\infty dv\int_0^\infty dz\, z^2H(u,v,z) \times\int d^4p\,\exp\{ipx-iz[u(p^2-k^2+pE) +v(p^2-k^2-pE)+(p-k)^2+\mu^2]\},$$

where, to avoid confusion, we have written u and v instead of x and y in Eq. (12). Upon carrying out the p integration with the help of (13), we derive the following equation:

$$\psi(x)=e^{ikx}-i\pi^2\int_0^\infty du\int_0^\infty dv\int_0^\infty dz\frac{H(u,v,z)}{(u+v+1)^2} \times\exp\left(\frac{ix^2}{4}\frac{1}{z(u+v+1)}+ikx\frac{1}{u+v+1}-i\frac{Ex}{2}\frac{u-v}{u+v+1} +\frac{iE^2}{4}\frac{z(u-v)^2}{u+v+1}+ik^2\frac{z(u+v)^2}{u+v+1}-i\mu^2z\right). \quad (19)$$

Actually, the physical interpretation of the B-S wave function $\psi(x)$ is not well established, particularly since it involves x_4, the relative time coordinate of the two nucleons. Since, in scattering experiments, we measure the scattered waves of the two nucleons at the same time, it seems reasonable to take $x_4=0$. Indeed, in this case, the B-S wave function corresponds to a Tamm-Dancoff amplitude,[6] whose physical meaning is clear.

Then, going over to the center-of-mass system for which (7) is applicable, so that $Ex=0$, and introducing the notations

$$x^2=\mathbf{x}^2=r^2,\quad kx=\mathbf{k}\cdot\mathbf{x}=kr\cos\theta,$$

we obtain from Eq. (19)

$$\psi(\mathbf{x})\equiv\psi(\mathbf{x},\,x_4=0) =e^{i\mathbf{k}\cdot\mathbf{x}}-i\pi^2\int_0^\infty du\int_0^\infty dv\int_0^\infty dz\frac{H(u,v,z)}{(u+v+1)^2} \times\exp\left(i\frac{r^2}{4}\frac{1}{z(u+v+1)}+ikr\cos\theta\frac{1}{u+v+1} -i\frac{\epsilon^2}{4}\frac{z(u-v)^2}{u+v+1}+ik^2\frac{z(u+v)^2}{u+v+1}-i\mu^2z\right).$$

Now, we are interested in the asymptotic form of $\psi(\mathbf{x})$ as $r\to\infty$. Since the exponential factor will oscillate quite rapidly in this limit, we can use the so-called method of stationary phase.[7] We then find

$$\psi(\mathbf{x})\sim e^{i\mathbf{k}\cdot\mathbf{x}}-\frac{2\pi^3i}{\epsilon}\frac{e^{ikr}}{r}\int_0^\infty dz\,H\left(\frac{r}{4kz},\frac{r}{4kz},z\right) \times\exp\{-iz[2k^2(1-\cos\theta)+\mu^2]\},\quad (r\to\infty). \quad (20)$$

Defining a four-vector k' by $k'=(\mathbf{k}',0)$, where $\mathbf{k}'^2=\mathbf{k}^2$ and $\mathbf{k}'\|\mathbf{x}$, we have finally

$$\psi(\mathbf{x})\sim e^{i\mathbf{k}\cdot\mathbf{x}}-\frac{2\pi^3i}{\epsilon}\frac{e^{ikr}}{r}\int_0^\infty dz\,H(u=\infty,\,v=\infty,\,z) \times\exp\{-iz[(k-k')^2+\mu^2]\} \quad (21)$$

[6] M. M. Lévy, Phys. Rev. **88**, 725 (1952).

[7] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1945), second edition, pp. 225, 229.

provided $H(u=\infty, v=\infty, z)$ exists. The scattering cross section from the initial relative momentum k to the final relative momentum k' is therefore given by

$$\frac{d\sigma}{d\Omega}=\frac{4\pi^6}{\epsilon^2}\left|\int_0^\infty dz\, H(u=\infty, v=\infty, z)\times\exp\{-iz[(k-k')^2+\mu^2]\}\right|^2. \quad (22)$$

Strictly speaking, Eq. (22) is true only for the scattering of two nonidentical particles, since we have not symmetrized the scattering amplitude with respect to the replacement of k by $-k$. For the sake of simplicity, however, we will not indicate this symmetrization explicitly.

Our expression for the cross section [Eq. (22)] must agree with the result derived from the usual S-matrix formalism. Apart from a multiplicative factor, the T matrix is given by[8]

$$T_{p,k}=\frac{ig^2}{(2\pi)^4}\int d^4p'\frac{\psi(p')}{(p-p')^2+\mu^2}, \quad (23)$$

where $p=(\mathbf{p},0)$, $k=(\mathbf{k},0)$, and $p^2=k^2$, i.e., we are concerned here only with that part of the T matrix which is on the energy shell; the function $\psi(p')$ in (23) satisfies (5′). The T matrix given here is related to the S matrix in the usual way, *viz.*,

$$S_{p,k}=-2\pi i\delta(E_p-E_k)T_{p,k}.$$

On evaluating (23) with the aid of (10), (12), and the relation $\psi(p')=G(p',k,E)$, one finds

$$T_{p,k}=\int_0^\infty dz\left\{-\frac{g^2}{(2\pi)^4}+\frac{ig^2}{16\pi^2}\int_0^\infty dx\int_0^\infty dy\int_z^\infty dt\times\frac{H(x,y,t)}{x+y+1}\exp\left[ik^2\frac{(x+y)^2}{x+y+1}(t-z)-i\frac{\epsilon^2}{4}\frac{(x-y)^2}{x+y+1}(t-z)-i\mu^2\left(\frac{tz}{t-z}(x+y+1)+t-z\right)\right]\right\}\times\exp\{-iz[(p-k)^2+\mu^2]\}.$$

In view of the integral equation (18), this can be rewritten in the form

$$T_{p,k}=\int_0^\infty dz\, H(u=\infty, v=\infty, z)\times\exp\{-iz[(p-k)^2+\mu^2]\}, \quad (24)$$

so that we are once again led to the formula (22) with the replacement of p by k'.

In both derivations, we have assumed the existence of $H(u=\infty, v=\infty, z)$; indeed, when this is the case, the two determinations of the cross section agree with one another. But, if $H(u=\infty, v=\infty, z)$ does not exist, then we will see that the usual S-matrix formalism is useless. On the other hand, our first approach which was based on the evaluation of the asymptotic form of the wave function needs to be re-examined, since it now appears that the wave function does not have the usual asymptotic form, *viz.*, that of an incoming plane wave plus an outgoing spherical wave, but, in fact, these waves are distorted, and indeed, at the same time, the normalization of the incident wave is altered. This has the consequence that the derivations of the cross section based on the analysis of the coordinate-space wave function and on the S-matrix formalism will not agree. This peculiar result is obtained when $\mu=0$ and is essentially due to the infrared divergences which are present in the theory.

In this paper, we will treat, in detail, only the case $\mu=0$, the more general situation which obtains when $\mu\neq 0$ being too difficult to handle. When $\mu=0$, it follows from (17) and (14) that $H(u,v,t)$ is independent of t,

$$H(u,v,t)\equiv H(u,v),$$

and satisfies the equation

$$H(u,v)=-\frac{g^2}{(2\pi)^4}+\frac{g^2}{16\pi^2}\int_0^u dx\int_0^v dy\, H(x,y)\times\frac{1}{\frac{1}{4}\epsilon^2(x-y)^2-k^2(x+y)(x+y+1)+q^2(x+y)+qE(x-y)-i\delta}, \quad (25)$$

where $-i\delta$ is the vanishingly small negative imaginary term which was implicitly included in the p^2 of all denominators and exponential factors. Equation (12) becomes correspondingly

$$F(p,q,E)=2i\int_0^\infty dx\int_0^\infty dy\times\frac{H(x,y)}{[x(p^2-k^2+pE)+y(p^2-k^2-pE)+(p-q)^2]^3}. \quad (26)$$

Evidently, the Gaussian representation (12) reduces to the Stieltjes form (26) in the case $\mu=0$.

In point of fact, Eq. (25), which is valid when $\mu=0$, may be obtained more easily by expressing F in the form (26) and then following a procedure similar to that used in Sec. III of I. An equation corresponding to (25) has, in fact, been derived independently by Nishijima[2] who used such a method.

When we deal with the real scattering problem, we have only to set $q=k$ in Eq. (25); in view of (4b), we obtain

$$H(u,v)=-\frac{g^2}{(2\pi)^4}+\frac{g^2}{16\pi^2}\int_0^u dx\int_0^v dy\times\frac{H(x,y)}{\frac{1}{4}\epsilon^2(x-y)^2-k^2(x+y)^2-i\delta}, \quad (27)$$

[8] K. Nishijima, Progr. Theoret. Phys. (Kyoto) **10**, 549 (1953); **12**, 279 (1954); **13**, 305 (1955).

which, in turn, is equivalent to the differential equation

$$\frac{\partial^2}{\partial u \partial v} H(u,v) = \frac{g^2}{16\pi^2} \frac{H(u,v)}{\frac{1}{4}\epsilon^2(u-v)^2 - k^2(u+v)^2 - i\delta}, \quad (28)$$

subject to the boundary conditions

$$H(u=0,\, v) = H(u,\, v=0) = -g^2/(2\pi)^4. \quad (29)$$

Nishijima[2] inferred from Eqs. (28) and (29) that the solution must be of the form

$$H(u,v) = f((u-v)/(u+v)), \quad f(\pm 1) = -g^2/(2\pi)^4.$$

But then $H(u,v)$ is a function of u/v only, whence it follows from (27) that the integral on the right-hand side is divergent for $x=0$ or $y=0$. This means that (27) has, in fact, no solution at all.

One way out of this difficulty is to try to solve, not (27), but rather the more general equation (25) for the Green's function, and then let $q \to k$ at the very end. In this method, the infrared divergence appears only at the final stage of the calculation, i.e., when the limiting process $q \to k$ is performed, but not in the solution of (25) itself. Physically, this procedure corresponds to calculating the T matrix, first, off the energy shell and then going, in the limit, on to the energy shell.

Another way to proceed is to modify Eq. (27) by introducing a small cutoff near $x=0$ and $y=0$, so that

$$H(u,v) = -\frac{g^2}{(2\pi)^4} + \frac{g^2}{16\pi^2} \int_s^u dx \int_s^v dy \times \frac{H(x,y)}{\frac{1}{4}\epsilon^2(x-y)^2 - k^2(x+y)^2 - i\delta}, \quad (30)$$

and then taking the limit $s \to +0$ at the end of the calculation. Under these circumstances, Eq. (28) still holds good, but the boundary conditions now read

$$H(u=s,\, v) = H(u,\, v=s) = -g^2/(2\pi)^4. \quad (31)$$

For later use, we write down, at this point, the differential-equation formulation of (25), *viz.*,

$$\frac{\partial^2}{\partial u \partial v} H(u,v) = \frac{g^2}{16\pi^2} \times \frac{H(u,v)}{\frac{1}{4}\epsilon^2(u-v)^2 - k^2(u+v)(u+v+1) + q^2(u+v) - i\delta}, \quad (32)$$

with the boundary conditions

$$H(u=0,\, v) = H(u,\, v=0) = -g^2/(2\pi)^4. \quad (33)$$

In the derivation of (32), we have restricted ourselves to those q that satisfy the relation $qE=0$, i.e., q has a form similar to (8), but we do not assume that $q^2=k^2$. This restriction is adequate for our needs, since, at the end, we will take the limit $q \to k$.

IV. INVESTIGATION OF THE COULOMB-SCATTERING PROBLEM IN MOMENTUM SPACE

Before we proceed to solve (30) or (32), we will study the Coulomb-scattering problem in momentum space. As we will see later, this will help in our understanding of the various difficulties involved in the solution of the B-S equation.

The equation for the relative motion of two particles which interact via the Coulomb potential $V = V_0/r$ is

$$(\nabla^2 + k^2 - mV_0/r)\psi(\mathbf{x}) = 0; \quad (34)$$

in momentum space, this equation assumes the form

$$(\mathbf{p}^2 - k^2)\psi(\mathbf{p}) = -\frac{mV_0}{2\pi^2} \int d^3p' \frac{\psi(\mathbf{p}')}{(\mathbf{p}'-\mathbf{p})^2}. \quad (34')$$

The usual discussion[9] of the scattering solution of (34′) leads to the integral equation

$$\psi(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{k}) - \frac{mV_0}{2\pi^2} \frac{1}{\mathbf{p}^2 - k^2 - i\delta} \int d^3p' \frac{\psi(\mathbf{p}')}{(\mathbf{p}-\mathbf{p}')^2}. \quad (35)$$

On the other hand, the complete nonrelativistic reduction[6] of the B-S equation (5) yields an equation similar to (35) with

$$g^2 = -16\pi m^2 V_0. \quad (36)$$

The customary method of handling the Coulomb-scattering problem is to solve Eq. (34) directly in coordinate space. However, it is well known that, while Eq. (35) gives the exact Rutherford formula for the scattering cross section in first Born approximation, it leads to a divergent result in the second Born approximation. Actually, this divergence has the character of an infrared catastrophe, as was already evident in our discussion of the B-S equation.

Basically, Eq. (35) has no solution, because of the fact that the first term on the right-hand side represents a pure incoming plane wave and the second term an outgoing spherical wave; but we know from the solution in coordinate space that, because of the long range of the Coulomb potential, both the incoming plane wave and the outgoing spherical wave are distorted. We must therefore be careful in applying the usual S-matrix formalism in this case.

We proceed to modify (35) in the manner described in the preceding section, and calculate the scattering cross section according to both the Green's-function method and the cutoff procedure.

Instead of the wave function $\psi(\mathbf{p})$, we introduce the Green's function $G(\mathbf{p},\mathbf{q})$ which satisfies the equation

$$G(\mathbf{p},\mathbf{q}) = \delta(\mathbf{p}-\mathbf{q}) - \frac{mV_0}{2\pi^2} \frac{1}{\mathbf{p}^2 - k^2 - i\delta} \int d^3p' \frac{G(\mathbf{p}',\mathbf{q})}{(\mathbf{p}-\mathbf{p}')^2}, \quad (37)$$

where $\mathbf{q}$ is an arbitrary vector which does not necessarily

[9] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

satisfy $\mathbf{q}^2=k^2$. Unlike (35), Eq. (37) has a finite solution. Evidently, Eq. (37) is the analog of (6), and, indeed, may be obtained from (6) on carrying out the complete nonrelativistic reduction.[6] To solve (37), we set

$$G(\mathbf{p},\mathbf{q})=\delta(\mathbf{p}-\mathbf{q})+\int_0^\infty d\xi\frac{H(\xi)}{[\xi(\mathbf{p}^2-k^2)+(\mathbf{p}-\mathbf{q})^2-i\delta]^2},\tag{38}$$

which is essentially the nonrelativistic analog of (10) and (26), and use the integral identity

$$\int d^3p'\frac{1}{(\mathbf{p}-\mathbf{p}')^2}\frac{1}{(\mathbf{p}'^2-\lambda^2-i\delta)^2}=\frac{\pi^2 i}{\lambda}\frac{1}{\mathbf{p}^2-\lambda^2-i\delta},\quad (\mathrm{Re}\lambda>0),\tag{39}$$

which, in turn, corresponds to the relativistic formula

$$\int d^4p'\frac{1}{(p-p')^2-i\delta}\frac{1}{(p'^2+\Lambda-i\delta)^3}=\frac{i\pi^2}{2}\frac{1}{\Lambda}\frac{1}{p^2+\Lambda-i\delta}.$$

We can then deduce the integral equation

$$H(\xi)=-\frac{mV_0}{2\pi^2}-\frac{imV_0}{2}\int_0^\xi dz\frac{H(z)}{[z(z+1)k^2-z\mathbf{q}^2-i\delta]^{\frac{1}{2}}};\tag{40}$$

the method of derivation of Eq. (40) is very similar to that used in Sec. III of I.

Equation (40) is, in turn, equivalent to the differential equation

$$\frac{d}{d\xi}H(\xi)=-\frac{imV_0}{2}\frac{H(\xi)}{[\xi(\xi+1)k^2-\xi\mathbf{q}^2]^{\frac{1}{2}}},\tag{41a}$$

with

$$H(0)=-mV_0/2\pi^2,\tag{41b}$$

which has the solution

$$H(\xi)=-\frac{mV_0}{2\pi^2}\left(\frac{[\xi(\xi+1)k^2-\xi\mathbf{q}^2]^{\frac{1}{2}}+k\xi}{[\xi(\xi+1)k^2-\xi\mathbf{q}^2]^{\frac{1}{2}}-k\xi}\right)^{i\alpha},\tag{42}$$

where

$$\alpha=-mV_0/2k.\tag{43}$$

Equation (42) is similar to (I, 27b); accordingly, the Green's function, regarded as an analytic function of α, has simple poles at $i\alpha=n$ ($n=1, 2, 3, \cdots$). This implies the existence of bound states for the Coulomb interaction, provided $V_0<0$. With $k=i(-mE)^{\frac{1}{2}}$, we are led once again to the Bohr formula for hydrogen-like atoms. Correspondingly, the residues of the Green's function will yield the bound-state eigenfunctions (see the Appendix of I).

Now, in the real scattering problem, we should set $\mathbf{q}=\mathbf{k}$. Under these circumstances, however, it is evident that (42) will diverge, and hence Eq. (35) has no solution. We will therefore refrain from taking this limit until the very end of the calculation.

Next, we calculate the T matrix, using the result just obtained for the Green's function. According to the usual S-matrix formalism,[9] the T matrix is given by

$$T_{\mathbf{p},\mathbf{q}}=\frac{V_0}{2\pi^2}\int d^3p'\frac{G(\mathbf{p}',\mathbf{q})}{(\mathbf{p}-\mathbf{p}')^2}.\tag{44}$$

Using Eqs. (38) and (39), we have

$$T_{\mathbf{p},\mathbf{q}}=\frac{V_0}{2\pi^2}\left(\frac{1}{(\mathbf{p}-\mathbf{q})^2}+i\pi^2\int_0^\infty d\xi\frac{H(\xi)}{\xi(\mathbf{p}^2-k^2)+(\mathbf{p}-\mathbf{q})^2}\times\frac{1}{[\xi(\xi+1)k^2-\xi\mathbf{q}^2]^{\frac{1}{2}}}\right).\tag{44'}$$

For the real scattering problem, we have, of course,

$$\mathbf{p}^2\to k^2,\quad \mathbf{q}^2\to k^2,\tag{45}$$

whence we see that divergences occur in the ξ integration at $\xi=0$ and $\xi=\infty$; we therefore continue to defer the taking of this limit.

Using Eqs. (41a, b) and carrying out an integration by parts, we find

$$T_{\mathbf{p},\mathbf{q}}=-\frac{1}{m}\int_0^\infty d\xi\frac{(\mathbf{p}^2-k^2)}{[\xi(\mathbf{p}^2-k^2)+(\mathbf{p}-\mathbf{q})^2]^2}H(\xi).\tag{46}$$

Now, assuming for the sake of definiteness, that

$$\mathbf{p}^2>k^2>\mathbf{q}^2,\tag{47}$$

and carrying out the change of variable

$$\xi\to(\mathbf{p}-\mathbf{q})^2(\mathbf{p}^2-k^2)^{-1}\xi,$$

we obtain

$$T_{\mathbf{p},\mathbf{q}}=\frac{V_0}{2\pi^2}\frac{1}{(\mathbf{p}-\mathbf{q})^2}\int_0^\infty d\xi\frac{1}{(\xi+1)^2}\left(\frac{(\xi^2+\beta'\xi)^{\frac{1}{2}}+\xi}{(\xi^2+\beta'\xi)^{\frac{1}{2}}-\xi}\right)^{i\alpha},$$

where

$$\beta'=\frac{\mathbf{p}^2-k^2}{(\mathbf{p}-\mathbf{q})^2}\beta,\tag{48a}$$

and

$$\beta=1-(\mathbf{q}^2/k^2);\tag{48b}$$

from (47), we have

$$\beta'>0.$$

In view of (45), we must now take the limit

$$\beta'\to+0.$$

Our expression for $T_{\mathbf{p},\mathbf{q}}$ then becomes

$$T_{\mathbf{p},\mathbf{q}}=\frac{V_0}{2\pi^2}\left(\frac{4}{\beta'}\right)^{i\alpha}\frac{1}{(\mathbf{p}-\mathbf{q})^2}\int_0^\infty d\xi\frac{\xi^{i\alpha}}{(\xi+1)^2},\quad(\beta'\to+0),$$

or finally

$$T_{\mathbf{p},\mathbf{k}}=\frac{V_0}{2\pi^2}\left(\frac{4}{\beta'}\right)^{i\alpha}\frac{\pi\alpha}{\sinh\pi\alpha}\frac{1}{(\mathbf{p}-\mathbf{k})^2},\quad(\beta'\to+0).\tag{49}$$

We notice that all the divergences appear in the phase factor

$$(4/\beta')^{i\alpha} = \exp[i\alpha \ln(4/\beta')],$$

and so will lead to no difficulties in the calculation of the cross section.

From (49), the differential cross section for scattering through the angle θ is given by

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \frac{m^2}{4} |T_{\mathbf{p},\mathbf{k}}|^2 = \left(\frac{\pi\alpha}{\sinh\pi\alpha}\right)^2 \left(\frac{d\sigma}{d\Omega}\right)_R, \tag{50}$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{\alpha^2}{4k^2} \frac{1}{\sin^4(\frac{1}{2}\theta)} \tag{51}$$

is the classical Rutherford formula for the Coulomb-scattering cross section. We see, therefore, that we have obtained a result which differs from the usual Rutherford formula. Since the latter should be correct, this result is quite peculiar.

In deriving Eqs. (49) and (50), we assumed the inequality (47). Since this assumption has little meaning, we carry out the limiting process (45) in a somewhat different way. Thus, suppose

$$k^2 > \mathbf{q}^2, \quad k^2 > \mathbf{p}^2. \tag{47'}$$

Then, proceeding as before, but now carrying out the change of variable

$$\xi \to -(\mathbf{p}-\mathbf{q})^2(\mathbf{p}^2-k^2)^{-1}\xi,$$

we find that (46) goes over into

$$T_{\mathbf{p},\mathbf{q}} = -\frac{V_0}{2\pi^2} \frac{1}{(\mathbf{p}-\mathbf{q})^2} \int_0^\infty d\xi \frac{1}{(\xi-1+i\delta)^2} \times \left[\frac{(\xi^2+|\beta'|\xi)^{\frac{1}{2}}+\xi}{(\xi^2+|\beta'|\xi)^{\frac{1}{2}}-\xi}\right]^{i\alpha}.$$

On taking the limit $|\beta'| \to 0$, and rotating the ξ axis through 180° in a positive direction, we obtain

$$T_{\mathbf{p},\mathbf{q}} = \frac{V_0}{2\pi^2} \left(\frac{4}{|\beta'|}\right)^{i\alpha} \frac{\pi\alpha}{\sinh\pi\alpha} e^{-\pi\alpha} \frac{1}{(\mathbf{p}-\mathbf{q})^2}, \quad (|\beta'| \to 0). \tag{49'}$$

It is evident, on comparison with Eq. (49), that we have here an extra factor of $e^{-\pi\alpha}$. The cross section will therefore be given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{\pi\alpha}{\sinh\pi\alpha} e^{-\pi\alpha}\right)^2 \left(\frac{d\sigma}{d\Omega}\right)_R. \tag{50'}$$

Of course, both (50) and (50′) lead to the Rutherford formula in first Born approximation.

It is now clear that, not only does the Green's-function method lead to an incorrect formula for the scattering cross section, but the formula depends, in fact, on the nature of the limiting process that is used. We proceed, in the following, to explain this peculiar result, and, in so doing, we will see how to interpret the cross section properly. The essential point is that one has to take note of the change of amplitude of the incoming wave in the case of scattering by a Coulomb field.

To see how this comes about, we calculate the wave function in coordinate space; this is given by

$$\psi(\mathbf{x}) = \int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}} G(\mathbf{p},\mathbf{q}), \quad (\mathbf{q} \to \mathbf{k}).$$

From (38), we find

$$\psi(\mathbf{x}) = e^{i\mathbf{q}\cdot\mathbf{x}} + i\pi^2 \int_0^\infty d\xi \frac{H(\xi)}{(\xi+1)[\xi(\xi+1)k^2-\xi\mathbf{q}^2]^{\frac{1}{2}}} \times \exp\left(\frac{i}{\xi+1}\{\mathbf{q}\cdot\mathbf{x}+r[\xi(\xi+1)k^2-\xi\mathbf{q}^2]^{\frac{1}{2}}\}\right), \quad (\mathbf{q} \to \mathbf{k})$$

where $r = |\mathbf{x}|$. In what follows, we assume, for the sake of simplicity, that

$$k^2 > \mathbf{q}^2,$$

and

$$\mathbf{q} = (0,0,q).$$

Making use once again of Eqs. (41a, b), we obtain, after a partial integration,

$$\psi(\mathbf{x}) = \frac{2\pi^2}{mV_0} \int_0^\infty d\xi\, H(\xi) \frac{d}{d\xi} \times \left[\frac{1}{\xi+1} \exp\left(\frac{i}{\xi+1}[qz+kr(\xi^2+\beta\xi)^{\frac{1}{2}}]\right)\right], \quad (\mathbf{q} \to \mathbf{k})$$

where β is defined by (48b) and satisfies

$$\beta > 0.$$

We are now in a position to carry out the limiting process $q \to k$, i.e., $\beta \to 0$. We then have

$$\begin{aligned}
\psi(\mathbf{x}) &= -\left(\frac{4}{\beta}\right)^{i\alpha} \int_0^\infty d\xi\, \xi^{i\alpha} \frac{d}{d\xi}\left[\frac{1}{\xi+1} \exp\left(\frac{i}{\xi+1}(kz+kr\xi)\right)\right] \\
&= -\left(\frac{4}{\beta}\right)^{i\alpha} \lim_{\theta \to i\alpha+0} \int_0^\infty d\xi\, \xi^\theta \frac{d}{d\xi} \times \left[\frac{1}{\xi+1} \exp\left(\frac{i}{\xi+1}(kz+kr\xi)\right)\right] \\
&= \left(\frac{4}{\beta}\right)^{i\alpha} \lim_{\theta \to i\alpha+0} \theta \int_0^\infty d\xi \frac{\xi^{\theta-1}}{\xi+1} \exp\left(\frac{i}{\xi+1}(kz+kr\xi)\right).
\end{aligned}$$

If we set $u=\xi/(\xi+1)$, we find[10]

$$\psi(\mathbf{x})=i\alpha\left(\frac{4}{\beta}\right)^{i\alpha}\lim_{\theta\to i\alpha+0}e^{ikz}\int_0^1 du\, u^{\theta-1}(1-u)^{-\theta}e^{iuk(r-z)}$$

$$=\frac{\pi\alpha}{\sinh\pi\alpha}\left(\frac{4}{\beta}\right)^{i\alpha}e^{ikz}\,{}_1F_1(i\alpha,1;ik(r-z)),\tag{52}$$

where ${}_1F_1$ is the confluent hypergeometric function, and all divergences again appear in the phase factor.

Equation (52) agrees, except for a multiplicative factor, with the usual scattering solution. As we will see shortly, the essential point here consists in the appearance of this additional constant factor.

The form of the wave function $\psi(\mathbf{x})$ at infinity may be determined from the asymptotic expansion of the confluent hypergeometric function; this gives

$$\psi(\mathbf{x})\sim\frac{\pi\alpha}{\sinh\pi\alpha}\left(\frac{4}{\beta}\right)^{i\alpha}\frac{e^{-\pi\alpha/2}}{\Gamma(1-i\alpha)}\left(e^{ikz-i\alpha\ln(kr-kz)}+\alpha\frac{\Gamma(1-i\alpha)}{\Gamma(1+i\alpha)}\frac{e^{ikr+i\alpha\ln(kr-kz)}}{k(r-z)}\right).\tag{53}$$

Thus we see that the amplitude of the incoming wave is now given by

$$Z_2^{\frac{1}{2}}=\frac{\pi\alpha}{\sinh\pi\alpha}\left(\frac{4}{\beta}\right)^{i\alpha}\frac{e^{-\pi\alpha/2}}{\Gamma(1-i\alpha)},\tag{54}$$

even though it was originally normalized to unity [see Eq. (35)].

The physical meaning of $Z_2^{\frac{1}{2}}$ is essentially the same as in the case of quantum field theory; it plays the role of the renormalization constant of the amplitude of an external line.[11] This factor has its origin in the long-range character of the Coulomb potential. As in quantum field theory, we must therefore divide the cross section by $|Z_2|\,|Z_2'|$, where Z_2 and Z_2' are the constants appropriate to the initial and final states, respectively. For $\beta>0$, it follows from (54) that

$$|Z_2|=(\pi\alpha/\sinh\pi\alpha)e^{-\pi\alpha},\quad(\beta>0).\tag{55}$$

On the other hand, when $\beta<0$, we have

$$|Z_2|=(\pi\alpha/\sinh\pi\alpha)e^{\pi\alpha},\quad(\beta<0),\tag{56}$$

since

$$(4/\beta)^{i\alpha}=(4/|\beta|)^{i\alpha}e^{\pi\alpha}.$$

In applying these results to Eq. (50), we notice that, since it is based on the inequality (47), we must use Eqs. (55) and (56) for $Z_2(q)$ and $Z_2(p)$, respectively. The result is to give

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{renormalized}}=\frac{1}{|Z_2(p)|\,|Z_2(q)|}\left(\frac{d\sigma}{d\Omega}\right)=\left(\frac{d\sigma}{d\Omega}\right)_R.\tag{57}$$

In the same way, when we consider (50′), we must use (55) for both $Z_2(q)$ and $Z_2(p)$ in view of Eq. (47′); in consequence, we obtain (57) once again. Evidently, after we have carried out an amplitude renormalization, we can indeed obtain the correct cross section, no matter which form of limiting process is used.

It is interesting to notice that, of the two possible expressions for $|Z_2|$ [Eqs. (55) and (56)], one will always be greater than unity, the other will be less than unity, depending on the sign of α. This is unlike the case of field theory[12] for which, formally at least, one has $|Z_2|<1$.

Next, we investigate the cutoff procedure proposed in the previous section; this will turn out to be much more suitable than the Green's-function method for the discussion of the scattering problem within the B-S formalism. When we apply a cutoff, we take $\mathbf{q}=\mathbf{k}$ from the very beginning; then, Eq. (40) becomes

$$H(\xi)=-\frac{mV_0}{2\pi^2}-\frac{imV_0}{2k}\int_0^\xi dz\frac{H(z)}{z}.$$

But, as we have already noted earlier, this equation has no solution [see Eq. (42)]. We therefore introduce a small cutoff s and consider the following integral equation:

$$H(\xi)=-\frac{mV_0}{2\pi^2}+i\alpha\int_s^\xi dz\frac{H(z)}{z},\tag{58}$$

where α is defined by (43), and, at the conclusion of our calculation, we must take the limit $s\to+0$.

The solution of (58) is given by

$$H(\xi)=-(mV_0/2\pi^2)(\xi/s)^{i\alpha}.\tag{59}$$

From (44′), we find that the T matrix is equal to

$$T_{\mathbf{p},\mathbf{k}}=\frac{V_0}{2\pi^2}\left(\frac{1}{(\mathbf{p}-\mathbf{k})^2}+\frac{i\pi^2}{k}\int_s^\infty d\xi\frac{1}{\xi(\mathbf{p}^2-k^2)+(\mathbf{p}-\mathbf{k})^2}\frac{H(\xi)}{\xi}\right).$$

It is consistent to set $\mathbf{p}^2=k^2$ here, too, whence we have

$$T_{\mathbf{p},\mathbf{k}}=\frac{V_0}{2\pi^2}\frac{1}{(\mathbf{p}-\mathbf{k})^2}\left[1+i\alpha\int_s^\infty d\xi\frac{1}{\xi}\left(\frac{\xi}{s}\right)^{i\alpha}\right].$$

There is still a divergence at $\xi=\infty$; we therefore introduce an upper bound L for the ξ variable, so that

$$T_{\mathbf{p},\mathbf{k}}=\frac{V_0}{2\pi^2}\frac{1}{(\mathbf{p}-\mathbf{k})^2}\left[1+i\alpha\int_s^L d\xi\frac{1}{\xi}\left(\frac{\xi}{s}\right)^{i\alpha}\right].$$

[10] A. Erdélyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 255.

[11] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[12] G. Källén, Helv. Phys. Acta **25**, 417 (1952); H. Lehmann, Nuovo cimento **11**, 342 (1954); but see also T. D. Lee, Phys. Rev. **95**, 1329 (1954).

On carrying out the integration, we have

$$T_{\mathbf{p},\mathbf{k}}=\frac{V_0}{2\pi^2}\frac{1}{(\mathbf{p}-\mathbf{k})^2}\left(\frac{L}{s}\right)^{i\alpha}, \tag{60}$$

where $s\rightarrow +0$ and $L\rightarrow +\infty$.

Once again, it is evident that all divergences appear in a phase factor, so that the scattering cross section is given by

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_R.$$

We have, by use of the cutoff procedure, obtained the Rutherford scattering formula, without encountering extraneous multiplicative factors. If we calculate the wave function in coordinate space, we find that, in this case, the wave function no longer contains distorted waves, but consists, asymptotically, of the usual plane plus outgoing spherical waves. With the cutoff, the second term on the right-hand side of (35) represents only an outgoing wave, and so, now, $Z_2=1$.

These results are connected with the existence of $H(\xi=\infty)$. In our discussion of the formalism of the nucleon-nucleon scattering problem in Sec. III, we saw that, if $H(x=\infty, y=\infty, z)$ exists, then there are essentially no difficulties. Our cutoff procedure evidently assures the existence of this quantity, i.e., it corresponds to a cutoff of the long-range Coulomb potential at large distances.

It is also interesting to note that, if we take, first, the limit $L\rightarrow\infty$, we obtain the usual wave function (52) with distorted waves; then, we must take into account the amplitude renormalization factor $Z_2^{\frac{1}{2}}$ just as we did in the Green's-function case. On the other hand, if we let $r\rightarrow\infty$ with L fixed, the wave function at infinity is nondistorted, and we get the Rutherford formula. Thus, the two limiting processes $L\rightarrow\infty$ and $r\rightarrow\infty$ do not commute. In both cases, nonzero s simply serves to assure the nondivergence of our integrals at $\xi=0$.

We have had to introduce two cutoff parameters L and s so as to obtain convergence at $\xi=\infty$ and $\xi=0$. In the final result, both parameters always appear in the combination L/s [see Eq. (60) and Sec. V].

V. SOLUTION OF THE B-S EQUATION

We will first apply the cutoff procedure, i.e., we must solve Eq. (30) or, equivalently, Eqs. (28) and (31). Having done so, we will have, from (24) and the identity

$$H(u,v,z)\equiv H(u,v),$$

which obtains when $\mu=0$, the relation

$$T_{p,k}=[-i/(p-k)^2]H(u=L, v=L); \tag{61}$$

we have here also introduced a cutoff at $u=L$, $v=L$ according to our prescription. In view of Eq. (22), the cross section will then be given by

$$\frac{d\sigma}{d\Omega}=\frac{4\pi^6}{\epsilon^2}|T_{p,k}|^2=\frac{4\pi^6}{\epsilon^2}\left(\frac{1}{(p-k)^2}\right)^2|H(u=L, v=L)|^2.$$

Since k and p both have the form (8), we find

$$\frac{d\sigma}{d\Omega}=\frac{\pi^6}{4\epsilon^2k^4}|H(u=L, v=L)|^2\frac{1}{\sin^4(\frac{1}{2}\theta)}, \tag{62}$$

where θ is the scattering angle. The angular dependence is evidently the same as given by the Rutherford formula, and does not depend on a perturbation calculation.

Our remaining problem is to obtain H. Instead of using the variables u and v in Eqs. (28) and (31), it is convenient to introduce ξ and η which are defined by

$$u+v=\xi,\quad u-v=\eta. \tag{63}$$

Equations (28) and (31) then become

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)G(\xi,\eta)=\frac{g^2}{16\pi^2}\frac{1}{\frac{1}{4}\epsilon^2\eta^2-k^2\xi^2-i\delta}G(\xi,\eta), \tag{64}$$

with

$$G(\xi, |\eta|=\xi-2s)=-g^2/(2\pi)^4, \tag{65}$$

where we have written

$$H(u,v)\equiv G(\xi,\eta); \tag{66}$$

we have also the inequalities

$$\xi\geqslant 2s,\quad |\eta|\leqslant\xi-2s.$$

Because of the boundary condition (65), Eq. (64) no longer has a solution which is a function only of η/ξ, unlike the case of no cutoff [see the argument following Eq. (29)].

Now, it is quite difficult to solve Eqs. (64, 65) exactly. Hence, we will use an approximation similar to that employed in I, i.e., we will assume that $\epsilon/k\gg 1$ which, again, amounts to a form of nonrelativistic approximation.

In the lowest-order approximation, we can make use of the relation[13]

$$\frac{1}{\frac{1}{4}\epsilon^2\eta^2-k^2\xi^2-i\delta}\rightarrow\frac{\pi i}{k\epsilon}\frac{2}{\xi}\delta(\eta),\quad(\epsilon/k\rightarrow\infty). \tag{67}$$

The solution of Eqs. (64, 65) is then given by

$$G(\xi,\eta)=-\frac{g^2}{(2\pi)^4}\left(\frac{\xi-|\eta|}{2s}\right)^{i\alpha}, \tag{68}$$

where

$$\alpha=g^2/16\pi k\epsilon. \tag{69}$$

Using (36) and the relation $\epsilon\cong 2m$, we find that α as given by (69) is exactly the same as that given by (43) for the Coulomb-scattering problem.

[13] G. C. Wick, Phys. Rev. **96**, 1124 (1954); see also (I,26).

Putting together Eqs. (63), (66), (68), and (61), we obtain

$$T_{p,k}=\frac{ig^2}{(2\pi)^4}\frac{1}{(p-k)^2}\left(\frac{L}{s}\right)^{i\alpha}, \tag{70}$$

which, in turn, corresponds to Eq. (60); the difference in the multiplicative factors in the two cases arises simply from the difference in the definition of T for the nonrelativistic and covariant situations. In view of Eqs. (62) and (69), we have finally

$$d\sigma/d\Omega=(d\sigma/d\Omega)_R.$$

The approximation (67) thus yields nothing other than the results already obtained in Sec. IV.

We proceed to calculate the corrections of next higher order to the Rutherford scattering. Toward this end, we note the formula

$$\frac{1}{\frac{1}{4}\epsilon^2\eta^2-k^2\xi^2-i\delta}=\frac{1}{\frac{1}{4}\epsilon^2\eta^2-k^2\xi^2}+\frac{\pi i}{\epsilon k}\frac{1}{\xi}\left[\delta\left(\eta+\frac{2k}{\epsilon}\xi\right)+\delta\left(\eta-\frac{2k}{\epsilon}\xi\right)\right]. \tag{71}$$

In the limit as $k/\epsilon\to 0$, the second term on the right-hand side of (71) gives (67). Evidently, the first term, which is to be interpreted as the Cauchy principal value, is of higher order than the second with respect to the parameter k/ϵ. Expanding G in (64) in the form of a power series

$$G=G_0+G_1+\cdots, \tag{72}$$

and retaining only the first two terms, we have

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)G_0=\frac{i\alpha}{\xi}\left[\delta\left(\eta+\frac{2k}{\epsilon}\xi\right)+\delta\left(\eta-\frac{2k}{\epsilon}\xi\right)\right]G_0, \tag{73}$$

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)G_1=\frac{i\alpha}{\xi}\left[\delta\left(\eta+\frac{2k}{\epsilon}\xi\right)+\delta\left(\eta-\frac{2k}{\epsilon}\xi\right)\right]G_1+\frac{g^2}{16\pi^2}\frac{1}{\frac{1}{4}\epsilon^2\eta^2-k^2\xi^2}G_0, \tag{74}$$

where we have used (69).

To solve (73), we notice that this equation, together with the boundary condition (65), implies that

$$G_0(\xi,\eta)=F_1(\xi-|\eta|), \qquad [|\eta|>(2k/\epsilon)\xi],$$
$$G_0(\xi,\eta)=F_2(\xi+\eta)+F_2(\xi-\eta), \qquad [|\eta|<(2k/\epsilon)\xi],$$

where the functions F_1 and F_2 are to be determined from the requirements that $G_0(\xi,\eta)$ be continuous, and also be a solution of (73), at $\eta=\pm(2k/\epsilon)\xi$. We then find that

$$F_1\left(\xi-\frac{2k}{\epsilon}\xi\right)=F_2\left(\xi-\frac{2k}{\epsilon}\xi\right)+F_2\left(\xi+\frac{2k}{\epsilon}\xi\right),$$

$$2\frac{d}{d\xi}F_2\left(\xi+\frac{2k}{\epsilon}\xi\right)=\frac{i\alpha}{\xi}F_1\left(\xi-\frac{2k}{\epsilon}\xi\right).$$

Since it is still difficult to solve these equations exactly, we proceed by expanding F_1 and F_2 in the form of a power series in k/ϵ and consistently neglect terms of order $(k/\epsilon)^2$ with respect to the lowest-order approximation. The final result for the solution of (73), subject to the boundary condition (65) and correct to first order in k/ϵ, is as follows:

$$G_0(\xi,\eta)=F(\xi-|\eta|)\left[1+\frac{2k}{\epsilon}\alpha^2\ln\left(\frac{\xi-|\eta|}{2s}\right)\right], \qquad \left(|\eta|>\frac{2k}{\epsilon}\xi\right), \tag{75a}$$

$$G_0(\xi,\eta)=\tfrac{1}{2}F(\xi+\eta)\left[1-\frac{2k}{\epsilon}i\alpha+\frac{2k}{\epsilon}\alpha^2\ln\left(\frac{\xi+\eta}{2s}\right)\right]+\tfrac{1}{2}F(\xi-\eta)\left[1-\frac{2k}{\epsilon}i\alpha+\frac{2k}{\epsilon}\alpha^2\ln\left(\frac{\xi-\eta}{2s}\right)\right], \qquad \left(|\eta|<\frac{2k}{\epsilon}\xi\right), \tag{75b}$$

where

$$F(x)=-\frac{g^2}{(2\pi)^4}\left(\frac{x}{2s}\right)^{i\alpha}. \tag{76}$$

It is to be emphasized that these formulas are not the results of a weak-coupling perturbation theory. The parameter of smallness in the calculation is k/ϵ, and we need not assume that α is either small or large.

Now that we have solved Eq. (73), we consider Eq. (74). Since we are interested only in corrections of orders k/ϵ with respect to the limiting case $k/\epsilon\to 0$, we can simplify (74) as follows:

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)G_1(\xi,\eta)\cong\frac{2i\alpha}{\xi}\delta(\eta)G_1(\xi,\eta)+\frac{g^2}{16\pi^2}\frac{F(\xi-|\eta|)}{\frac{1}{4}\epsilon^2\eta^2-k^2\xi^2}. \tag{74'}$$

The general solution of (74′) may be expressed in the form

$$G_1(\xi,\eta)=H(\xi-|\eta|)+\frac{g^2}{16\pi^2}\int_{2s}^{\xi-|\eta|}dx\,F(x)[I(x,\xi+|\eta|)+I(x,\xi-|\eta|)-2I(x,x)], \tag{77}$$

where

$$I(x,y)=\int^{y}dy\frac{1}{\frac{1}{4}\epsilon^2(x-y)^2-k^2(x+y)^2}, \tag{78}$$

and $H(x)$ is a function to be determined.

The evaluation of $H(x)$ may be performed by integrating (74′) with respect to η from $\eta=-i\delta$ to $\eta=+i\delta$

and then taking the limit $\delta \to +0$. We then find that

$$\frac{d}{d\xi}H(\xi)=\frac{i\alpha}{\xi}H(\xi)+\frac{g^2}{8\pi^2}\frac{i\alpha}{\xi}\int_{2s}^{\xi}dx\,F(x)[I(x,\xi)-I(x,x)].$$

With aid of the boundary condition

$$H(\xi-|\eta|=2s)=0,$$

which follows from (65), (72), and (75a), we may integrate the previous equation to give

$$H(\xi)=\frac{g^2}{8\pi^2}(i\alpha)F(\xi)\int_{2s}^{\xi}dy\frac{1}{F(y)}\frac{1}{y}\int_{2s}^{y}dx\,F(x)\times[I(x,y)-I(x,x)].$$

If we now take note of the relation

$$\frac{i\alpha}{y}\frac{1}{F(y)}=-\frac{d}{dy}\frac{1}{F(y)},$$

and integrate by parts with respect to y, we can write

$$H(\xi)=-\frac{g^2}{8\pi^2}\int_{2s}^{\xi}dx\,F(x)[I(x,\xi)-I(x,x)]+\frac{g^2}{8\pi^2}F(\xi)\times\int_{2s}^{\xi}dy\frac{1}{F(y)}\int_{2s}^{y}dx\frac{F(x)}{\frac{1}{4}\epsilon^2(x-y)^2-k^2(x+y)^2},\quad(79)$$

where we have made use of Eq. (78). This result, together with (77), yields

$$G_1(\xi,\eta)=\frac{g^2}{16\pi^2}\int_{2s}^{\xi-|\eta|}dx\,F(x)[I(x,\xi+|\eta|)-I(x,\xi-|\eta|)]+\frac{g^2}{8\pi^2}F(\xi-|\eta|)\int_{2s}^{\xi-|\eta|}dy\frac{1}{F(y)}\times\int_{2s}^{y}dx\frac{F(x)}{\frac{1}{4}\epsilon^2(x-y)^2-k^2(x+y)^2}.\quad(80)$$

Next, we calculate $H(u=L,\ v=L)$ and the scattering cross section. From Eqs. (66), (72), (75b), (76), and (80), we find

$$\begin{aligned}H(u=L,\ v=L)&\cong G_0(\xi=2L,\ \eta=0)+G_1(\xi=2L,\ \eta=0)\\&\cong-\left(\frac{L}{s}\right)^{i\alpha}\frac{g^2}{(2\pi)^4}\left[1-\frac{2k}{\epsilon}(i\alpha)+\frac{2k}{\epsilon}\alpha^2\ln(L/s)+\frac{g^2}{8\pi^2}\int_{2s}^{2L}dy\,y^{-i\alpha}\int_{2s}^{y}dx\frac{x^{i\alpha}}{\frac{1}{4}\epsilon^2(x-y)^2-k^2(x+y)^2}\right].\end{aligned}$$

The evaluation of the integral in the preceding equation is quite complicated. We shall here simply quote the final result. One finds

$$H(u=L,\ v=L)\cong-\frac{g^2}{(2\pi)^4}\left(\frac{L}{s}\right)^{i\alpha}\Big(1-\frac{2k}{\epsilon}(i\alpha)+\frac{2k}{\epsilon}\alpha^2\ln(L/s)+\frac{8\alpha}{\pi}\frac{k}{\epsilon}\{-\tfrac{1}{2}+(i\alpha)[\Psi(1+i\alpha)-\Psi(1)-1-\ln(\epsilon/4k)]\}\ln(L/s)+\frac{8\alpha}{\pi}\frac{k}{\epsilon}[\Psi(1+i\alpha)-\Psi(1)-1-\ln(\epsilon/4k)+(i\alpha)\Psi'(1+i\alpha)]\Big),\quad(81)$$

where $\Psi(x)=(d/dx)\ln\Gamma(x)$ and $\Psi'(x)$ is its derivative; we have in (81) omitted terms of order higher than k/ϵ and also terms which go to zero for vanishing s/L.

It is interesting to notice that s and L appear together in the combination L/s, so that, from a practical standpoint, we have effectively introduced only one parameter L/s, which tends to infinity. It will also be observed that, this time, the infrared divergence does not occur wholly in a phase factor.

With aid of the identity

$$\Psi(1+i\alpha)-\Psi(1-i\alpha)=(1/i\alpha)(1-\pi\alpha\coth\pi\alpha),$$

we see finally from (62) and (81) that

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_R\Big[1+\frac{4k}{\epsilon}\alpha^2(1-2\coth\pi\alpha)\ln(L/s)+\frac{16}{\pi}\frac{k}{\epsilon}\alpha\{\tfrac{1}{2}[\Psi(1+i\alpha)+\Psi(1-i\alpha)]+\tfrac{1}{2}i\alpha[\Psi'(1+i\alpha)-\Psi'(1-i\alpha)]-\Psi(1)-1-\ln(\epsilon/4k)\}\Big],\quad(82)$$

so that we have obtained corrections of order k/ϵ with respect to the Rutherford formula. However, unlike the simple Coulomb-scattering problem, we still have an infrared divergence ($d\sigma/d\Omega\to\infty$ as $L/s\to\infty$), which should be canceled by taking into account the cross section for the emission of soft photons. That one encounters an infrared divergence in the first-order corrections to the Rutherford formula, but not in the Rutherford formula itself, is explained roughly by the fact that higher-order corrections to the nonrelativistic Coulomb potential contain terms such as $r^{-1}\ln r$; these are of too long range to allow the existence of the usual scattering solutions.

Next, we consider the Green's-function method. As we saw in Sec. IV, it will be necessary in this case to carry out an amplitude renormalization. It will turn out that this is feasible in the lowest-order approximation in k/ϵ, but difficulties appear when we calculate higher-order corrections.

We must solve Eq. (32) subject to the boundary conditions (33). Introducing the variables ξ, η and the function G, defined by Eqs. (63) and (66), respectively,

we find that Eqs. (32) and (33) assume the following form:

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)G(\xi,\eta)=\frac{g^2}{16\pi^2}\frac{G(\xi,\eta)}{\frac{1}{4}\epsilon^2\eta^2-k^2\xi^2-(k^2-q^2)\xi-i\delta}, \quad (83)$$

$$G(\xi,\eta=\pm\xi)=-g^2/(2\pi)^4. \quad (84)$$

Of course, if we set $q^2=k^2$, Eq. (83) goes over into (64).

In the lowest-order approximation, i.e., for $k/\epsilon\to 0$, we can use the relation

$$\frac{1}{\frac{1}{4}\epsilon^2\eta^2-k^2\xi^2-(k^2-q^2)\xi-i\delta}\to\frac{\pi i}{k\epsilon}\frac{2}{(\xi^2+\beta\xi)^{\frac{1}{2}}}\delta(\eta), \quad (85)$$

where β is defined by (48b); it is clear that Eq. (85) is an extension of (67). The solution of Eqs. (84, 85) is then given by

$$G(\xi,\eta)=F(\xi-|\eta|),$$
$$F(x)=-\frac{g^2}{(2\pi)^4}\left[\frac{(x^2+\beta x)^{\frac{1}{2}}+x}{(x^2+\beta x)^{\frac{1}{2}}-x}\right]^{i\alpha}, \quad (86)$$

where α is defined by (69). This result corresponds to Eq. (42) of Sec. IV.

We can next calculate the cross section, and we obtain Eqs. (50) or (50′). To determine the amplitude renormalization constants, we compute the wave function in coordinate space in the manner discussed in Sec. III; we find that the wave function is once again given by Eq. (52), and so the renormalization constants Z_2 are the same as (55) and (56). Thus, in the lowest-order approximation, everything is essentially the same as in the nonrelativistic approximation treated in Sec. IV, except for the fact that the formalism that we are using here is covariant. Actually, it can be shown that our Green's function goes over into the nonrelativistic Green's function for the Coulomb-scattering problem in the nonrelativistic limit.

We go on to consider corrections of order k/ϵ with respect to the nonrelativistic result. By using a decomposition similar to (71) and (72), we obtain a set of equations which are the analogs of (73) and (74), *viz.*,

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)G_0=\frac{i\alpha}{(\xi^2+\beta\xi)^{\frac{1}{2}}}\left\{\delta\left[\eta+\frac{2k}{\epsilon}(\xi^2+\beta\xi)^{\frac{1}{2}}\right]+\delta\left[\eta-\frac{2k}{\epsilon}(\xi^2+\beta\xi)^{\frac{1}{2}}\right]\right\}G_0, \quad (87)$$

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)G_1=\frac{i\alpha}{(\xi^2+\beta\xi)^{\frac{1}{2}}}\left\{\delta\left[\eta+\frac{2k}{\epsilon}(\xi^2+\beta\xi)^{\frac{1}{2}}\right]+\delta\left[\eta-\frac{2k}{\epsilon}(\xi^2+\beta\xi)^{\frac{1}{2}}\right]\right\}G_1+\frac{g^2}{16\pi^2}\frac{1}{\frac{1}{4}\epsilon^2\eta^2-k^2(\xi^2+\beta\xi)}G_0. \quad (88)$$

The same method which was used in solving (73) and (74) can be applied here; one finds that

$$G_0(\xi,\eta)\cong F(\xi-|\eta|)\left[1-\frac{2k}{\epsilon}(i\alpha)\ln\frac{F(\xi-|\eta|)}{F(0)}\right],$$
$$\left[|\eta|>\frac{2k}{\epsilon}(\xi^2+\beta\xi)^{\frac{1}{2}}\right], \quad (89a)$$

$$G_0(\xi,\eta)\cong\tfrac{1}{2}F(\xi+\eta)\left[1-\frac{2k}{\epsilon}(i\alpha)-\frac{2k}{\epsilon}(i\alpha)\ln\frac{F(\xi+\eta)}{F(0)}\right]+\tfrac{1}{2}F(\xi-\eta)\left[1-\frac{2k}{\epsilon}(i\alpha)-\frac{2k}{\epsilon}(i\alpha)\ln\frac{F(\xi-\eta)}{F(0)}\right],$$
$$\left[|\eta|<\frac{2k}{\epsilon}(\xi^2+\beta\xi)^{\frac{1}{2}}\right], \quad (89b)$$

where $F(x)$ is defined by (86), and where we have omitted terms of order $(k/\epsilon)^2$ and higher.

The solution of (88) parallels that of (74); we obtain

$$G_1(\xi,\eta)=\frac{g^2}{16\pi^2}\int_0^{\xi-|\eta|}dx\,F(x)[I(x,\xi+|\eta|)-I(x,\xi-|\eta|)]+\frac{g^2}{8\pi^2}F(\xi-|\eta|)\int_0^{\xi-|\eta|}dy\frac{1}{F(y)}\times\int_0^y dx\frac{F(x)}{\frac{1}{4}\epsilon^2(x-y)^2-k^2(x+y)^2-2k^2\beta(x+y)}, \quad (90)$$

where $I(x,y)$ is now given by

$$I(x,y)=\int^y dy\frac{1}{\frac{1}{4}\epsilon^2(x-y)^2-k^2(x+y)^2-2k^2\beta(x+y)}. \quad (91)$$

Unlike the cutoff procedure, we cannot now use the formula (22) to calculate the cross section, because $H(u=\infty, v=\infty)$ does not exist. Instead, we calculate the T matrix from (23), replacing ψ by $G(p,q,E)$ which is defined by (6) with $\mu=0$; using Eqs. (10), (26), (63), and (66), we find

$$T_{p,q}=\frac{ig^2}{(2\pi)^4}\left(\frac{1}{(p-q)^2}-\frac{\pi^2}{2}\int_0^\infty d\xi\times\int_{-\xi}^{\xi}d\eta\frac{1}{\xi(p^2-k^2)+\eta pE+(p-q)^2}\times\frac{G(\xi,\eta)}{\frac{1}{4}\epsilon^2\eta^2+\eta qE-k^2\xi^2-(k^2-q^2)\xi-i\delta}\right). \quad (92)$$

Once again, we must let p^2, $q^2\to k^2$. The T matrix will depend on the nature of the limiting process, as was the case in our discussion of Coulomb scattering. Assuming the inequality (47), for definiteness [note that the fourth components of p and q can be taken to be zero in view of (8)], we can evaluate (92) by using (89a, b)

and (90). Neglecting terms of order $(k/\epsilon)^2$, we find, in the limit p^2, $q^2 \to k^2$, the following result:

$$T_{p,q} \cong \frac{ig^2}{(2\pi)^4} \frac{1}{(p-q)^2} \frac{\pi\alpha}{\sinh\pi\alpha} \left(\frac{4}{\beta'}\right)^{i\alpha} [A + B\ln(4/\beta')], \quad (93)$$

where

$$A = 1 + \frac{g^2}{4\pi^2\epsilon^2}\left(-\frac{\pi i}{2}(1-\pi\alpha\coth\pi\alpha) + 2[\Psi(1+i\alpha) - \Psi(1) - 1 - \ln(\epsilon/4k) + i\alpha\Psi'(1+i\alpha)] - \frac{i}{\pi\alpha}(1-\pi\alpha\coth\pi\alpha)\{-1+2i\alpha[\Psi(1+i\alpha) - \Psi(1) - 1 - \ln(\epsilon/4k)]\}\right), \quad (94a)$$

$$B = \frac{g^2}{4\pi^2\epsilon^2}\{(\pi\alpha/2) - 1 + 2i\alpha[\Psi(1+i\alpha) - \Psi(1) - 1 - \ln(\epsilon/4k)]\}, \quad (94b)$$

where β' and α are defined by (48) and (69), respectively. We do not give the details of the complicated calculation leading to (93) and (94).

In the lowest order in k/ϵ, Eqs. (93) and (94) are essentially equivalent to (49) except for a nonessential constant factor which has its origin in the difference in normalization of the relativistic and nonrelativistic T matrices.

In the lowest-order approximation, the cross section is given by (50), and the amplitude renormalization constant by (55). We can, of course, calculate the cross section with relativistic corrections from (93) and (94); the result is given by

$$\frac{d\sigma}{d\Omega} \cong \left(\frac{\pi\alpha}{\sinh\pi\alpha}\right)^2 \left(\frac{d\sigma}{d\Omega}\right)_R \left[1 + \frac{g^2}{\pi^2\epsilon^2}\left(\tfrac{1}{2}[\Psi(1+i\alpha) + \Psi(1-i\alpha)] - \Psi(1) - 1 - \ln(\epsilon/4k) + \frac{i\alpha}{2}[\Psi'(1+i\alpha) - \Psi'(1-i\alpha)] + \frac{1}{\pi}(1-\pi\alpha\coth\pi\alpha)\{\tfrac{1}{2}[\Psi(1+i\alpha) + \Psi(1-i\alpha)] - \Psi(1) - 1 - \ln(\epsilon/4k)\}\right) + \frac{g^2}{\pi^2\epsilon^2}\left(\frac{\pi\alpha}{4} - \frac{1}{2} + \frac{i\alpha}{2}[\Psi(1+i\alpha) - \Psi(1-i\alpha)]\right)\ln\left(\frac{4}{\beta'}\right)\right]. \quad (95)$$

Notice that the last term of (95) depends on $\ln(4/\beta')$, and so our expression for the cross section is divergent.

To obtain the correct cross section, we must carry out an amplitude renormalization; to find the renormalization constant Z_2, we need to calculate the wave function in coordinate space, *viz.*,

$$\psi(x) = \lim_{q\to k} \int d^4p\, e^{ipx}[\delta(p-q) + F(p,q,E)]$$
$$= e^{ikx} + \lim_{q\to k} i \int_0^\infty d\xi \int_{-\xi}^{\xi} d\eta\, G(\xi,\eta) \times \int d^4p \frac{e^{ipx}}{[\xi(p^2-k^2) + \eta pE + (p-q)^2]^3}. \quad (96)$$

Assuming

$$k^2 > q^2,$$

and setting $x_4 = 0$, we can calculate $\psi(\mathbf{x})$ up to and including terms of order k/ϵ. This task is quite complicated, and, again, we do not reproduce the details here. If we put, for simplicity,

$$\mathbf{k} = (0,0,k),$$

and make use of (89) and (90), the result is to find, in the limit as $\beta \to 0$,

$$\psi(\mathbf{x}) \cong \left(\frac{4}{\beta}\right)^{i\alpha} \frac{\pi\alpha}{\sinh\pi\alpha} e^{ikz}\, {}_1F_1(i\alpha, 1; ik(r-z)) \times \left\{1 - \frac{ig^2}{4\pi\epsilon^2} + \frac{\alpha g^2}{8\pi\epsilon^2}\ln\left(\frac{4}{\beta}\right) + \frac{g^2}{8\pi^2}\int_0^1 dt \times \frac{t^{i\alpha}}{\frac{1}{4}\epsilon^2(1-t)^2 - k^2(1+t)^2}\left[\ln\left(\frac{4}{\beta}t\right) - \frac{1-t}{2}\right] - \frac{g^2}{4\pi^2\epsilon^2}\right\}$$
$$+ \left(\frac{4}{\beta}\right)^{i\alpha} \frac{\pi\alpha}{\sinh\pi\alpha} e^{ikz}\left((i\pi)\coth\pi\alpha\, {}_1F_1(i\alpha, 1; ik(r-z)) + \frac{d}{d(i\alpha)}\, {}_1F_1(i\alpha, 1; ik(r-z))\right)\left(\frac{\alpha g^2}{8\pi\epsilon^2} + \frac{g^2}{8\pi^2}\int_0^1 dt \times \frac{t^{i\alpha}}{\frac{1}{4}\epsilon^2(1-t)^2 - k^2(1+t)^2}\right) + \phi(\mathbf{x}), \quad (97)$$

where all the notations are the same as in Sec. IV, and $\phi(\mathbf{x})$ is a function which goes to zero quite rapidly at infinity, and so does not have to be considered.

Unfortunately, Eq. (97) contains the term

$$\frac{d}{d(i\alpha)}\, {}_1F_1(i\alpha, 1; ik(r-z)),$$

which has the asymptotic form

$$(-i\alpha)\frac{e^{-\pi\alpha/2}}{\Gamma(1-i\alpha)}\ln(kr-kz)\left(e^{ikz-i\alpha\ln(kr-kz)} - \alpha\frac{\Gamma(1-i\alpha)}{\Gamma(1+i\alpha)}\frac{e^{ikr+i\alpha\ln(kr-kz)}}{k(r-z)}\right).$$

In other words, the amplitudes of the incoming and outgoing waves will now contain the factor $\ln(kr-kz)$ which diverges at infinity. In this case, therefore, we have difficulty in giving physical meaning to the wave function $\psi(\mathbf{x})$, and so we can calculate neither the cross section nor the renormalization constant Z_2.

The appearance of a logarithmically diverging amplitude may be due to the failure of the expansion in k/ϵ. As we have noted earlier, a calculation of the potential in terms of such an expansion leads, in the lowest-order approximation, to the ordinary Coulomb potential; the next-order approximation contains an $r^{-1}\ln r$ term. This $r^{-1}\ln r$ potential has a much longer range than the ordinary Coulomb potential $1/r$ and so would inevitably lead to an additional distortion of the incoming and outgoing waves. There remains the possibility that we can circumvent these difficulties by avoiding a perturbation-theoretical calculation in k/ϵ. Unfortunately, this general case is extremely difficult to solve.

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Detection and Generation of Gravitational Waves*

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Methods are proposed for measurement of the Riemann tensor and detection of gravitational waves. These make use of the fact that relative motion of mass points, or strains in a crystal, can be produced by second derivatives of the gravitational fields. The strains in a crystal may result in electric polarization in consequence of the piezoelectric effect. Measurement of voltages then enables certain components of the Riemann tensor to be determined. Mathematical analysis of the limitations is given. Arrangements are presented for search for gravitational radiation.

The generation of gravitational waves in the laboratory is discussed. New methods are proposed which employ electrically induced stresses in crystals. These give approximately a seventeen-order increase in radiation over a spinning rod of the same length as the crystal. At the same frequency the crystal gives radiation which is about thirty-nine orders greater than that of a spinning rod.

INTRODUCTION

THE question of gravitational radiation has always been a central issue in the General Theory of Relativity. Long ago, Einstein[1] and Eddington[2] studied the problem and predicted that very small amounts of energy would be radiated by a spinning rod or a double star. A great deal of theoretical work on the radiation problem has appeared, during the past four decades.

Experimental work along these lines now appears possible. Two avenues of approach will be considered.[3] First we should like to detect the presence of gravitational radiation incident on earth from either the sun or outside the solar system. Secondly it would be highly desirable to be able to generate and detect this radiation in a small laboratory.

Devices for detection of the radiation operate essentially by measuring the Fourier transform of the Riemann tensor. These will be discussed first. This will then be followed by proposals for generation of gravitational radiation which may give an increase of many orders over the gravitational radiation from a spinning rod.

DETECTION OF GRAVITATIONAL RADIATION

Suppose we have a system of masses which may interact with each other. We start with the action principle

$$\delta I=\delta\left[-cm\int ds+W\right]=0. \tag{1}$$

In (1) m is the rest mass and W is the part of the action function associated with forces arising from the motion of the mass relative to other masses with which it interacts. The line element ds is given by

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu. \tag{2}$$

For δW we assume a function given by

$$-c\delta W=\int F_\mu \delta x^\mu ds; \tag{3}$$

(3) identifies F_μ as the four-force. The Euler-Lagrange

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[3] A number of the results discussed here were given without proof in the author's Gravity Research Foundation Prize Essays, April 1958 and April 1959, and at the Royaumont Conference on the Relativistic Theories of Gravitation, Royaumont, France, June, 1959 (unpublished).