Double Beta Decay

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Formulas for the probability of double β decay including energy distributions and angular correlations of the emitted electrons according to the theory of Feynman and Gell-Mann are given. A strong dependence on the change of angular momentum exists. The formulas also exhibit interference between different intermediate states. The half-life of ${}_{20}\mathrm{Ca}_{28}$ is calculated using matrix elements of j-j shell-model configurations. It is found to be $t\sim 10^{17}\cdots 10^{20}$ years. Actually, t will be much greater, since the matrix elements used are those of favored transitions.

ACCORDING to the theory of the Fermi interaction of Feynman and Gell-Mann,[1] the probability of allowed double β decay[2-4] from $J_i=0$ to $J_f=0$, 1, 2 becomes

$$d^3\lambda(0\rightarrow 0)=\frac{G^4}{30\pi^7}F(Z,W_1)F(Z,W_2)p_1W_1p_2W_2 \times(\epsilon-W_1-W_2)^5\{|\sum_m \eta_m^{-1}\mathfrak{A}_m|^2 +2[(W_2-W_1)^2+(1/7)(\epsilon-W_1-W_2)^2] \times \mathrm{Re}\,(\sum_m \eta_m^{-1}\mathfrak{A}_m\sum_m \eta_m^{-3}\mathfrak{A}_m^*)\} \times(1-v_1v_2\cos\theta)dW_1dW_2\sin\theta d\theta/2, \quad (1a)$$

$$\mathfrak{A}_m\equiv\langle f\|\mathbf{1}\|m\rangle\langle m\|\mathbf{1}\|i\rangle+3^{\frac{1}{2}}\langle f\|\boldsymbol{\sigma}\|m\rangle\langle m\|\boldsymbol{\sigma}\|i\rangle, \quad (1b)$$

$$d^3\lambda(0\rightarrow 1)=\frac{G^4}{10\pi^7}F(Z,W_1)F(Z,W_2)p_1W_1p_2W_2 \times(\epsilon-W_1-W_2)^5\{(W_2-W_1)^2 \times(1+\tfrac{1}{3}v_1v_2\cos\theta)|\sum_m \eta_m^{-2}\mathfrak{B}_m^+|^2 +(1/7)(\epsilon-W_1-W_2)^2(1-v_1v_2\cos\theta) \times|\sum_m \eta_m^{-2}\mathfrak{B}_m^-|^2\} \times dW_1dW_2\sin\theta d\theta/2, \quad (2a)$$

$$\mathfrak{B}_m^\pm\equiv\langle f\|\mathbf{1}\|m\rangle\langle m\|\boldsymbol{\sigma}\|i\rangle-\langle f\|\boldsymbol{\sigma}\|m\rangle\langle m\|\mathbf{1}\|i\rangle \pm 2^{\frac{1}{2}}\langle f\|\boldsymbol{\sigma}\|m\rangle\langle m\|\boldsymbol{\sigma}\|i\rangle, \quad (2b)$$

$$d^3\lambda(0\rightarrow 2)=0. \quad (3)$$

Here $G=1.41\times10^{-49}$ erg cm$^3=3.00\times10^{-12}$ $mc^2(\hbar/mc)^3$; p_κ, W_κ, v_κ ($\kappa=1$, 2) are the momenta, total energies, and velocities of the electrons; θ is the angle between their directions of propagation; Z is the final nuclear charge number; $F(Z,W_\kappa)$ is the Coulomb factor; $\epsilon\equiv\epsilon_i-\epsilon_f$; $\eta_m\equiv 2\epsilon_m-\epsilon_i-\epsilon_f$; ϵ_i, ϵ_m, ϵ_f are the nuclear energies of the initial, intermediate, and final nuclei, respectively; $\langle f\|\mathbf{1}\|m\rangle$, etc., are reduced matrix elements of the operators

$$\mathbf{1}\equiv\sum_{\nu=1}^{A}O_+(\nu),\quad \boldsymbol{\sigma}\equiv\sum_{\nu=1}^{A}\boldsymbol{\sigma}(\nu)O_+(\nu),$$

given by[5]

$$\langle fM'|\sigma_\mu|mM\rangle=C(J_m1J_f;\,M\mu M')\langle f\|\boldsymbol{\sigma}\|m\rangle,\ \text{etc.}$$

i, m, f represent all the quantum numbers of initial, intermediate, and final nuclei, respectively, except the projection angular momenta M. For single β decay one has

$$\mathfrak{M}_{GT}=\left(\frac{2J_f+1}{2J_i+1}\right)^{\frac{1}{2}}\langle f\|\boldsymbol{\sigma}\|i\rangle,\quad \mathfrak{M}_F=\langle f\|\mathbf{1}\|i\rangle.$$

The formulas are expansions in decreasing powers of η_m, broken off after second-order terms. Necessary and sufficient conditions for this expansion are $(W_2-W_1)^2/\eta_m\eta_{m'}<1$ and $(\epsilon-W_1-W_2)^2/\eta_m\eta_{m'}<1$ for all pairs m, m', including $m'=m$. Since $|W_2-W_1|\leqq\epsilon-2$, $0\leqq\epsilon-W_1-W_2\leqq\epsilon-2$, and $\eta_m>\epsilon-2$, these relations are always satisfied. Breaking off after second-order terms gives sufficiently exact values only for $(W_2-W_1)^2/\eta_m\eta_{m'}\ll1$ and $(\epsilon-W_1-W_2)^2/\eta_m\eta_{m'}\ll1$. The errors for other values of W_1 and W_2, however, are not serious, since then $d^3\lambda$ becomes smaller. For $d^3\lambda(0\rightarrow1)$, no terms in η_m^{-1} occur, so that the energy distribution is not the same as that for $d^3\lambda(0\rightarrow0)$. The angular correlations are also different. Since the contribution of second-order terms is generally small, transitions with vanishing terms in η_m^{-1} may be considered as some kind of forbidden transitions. For $d^3\lambda(0\rightarrow2)$, both first- and sec-

[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[2] M. Goeppert-Mayer, Phys. Rev. **48**, 512 (1935).
[3] E. J. Konopinski, U. S. Atomic Energy Commission Report LAMS-1949 (unpublished).
[4] H. Primakoff and S. P. Rosen, Repts. Progr. in Phys. **22**, 121 (1959); S. P. Rosen, Proc. Phys. Soc. **74**, 350 (1959). These authors give a full treatment of double β decay for the most general choice of coupling constants, including nonconservation of leptons. They also give a summary of the available experimental data and a large list of references. However, only transitions $0\rightarrow0$ are considered. Our calculations, which were made independently, are restricted to the coupling according to Feynman and Gell-Mann, but they also contain transitions $0\rightarrow1$ and $0\rightarrow2$. Closure approximation is not made, instead we get interference between different intermediate states of the nucleus.
[5] M. E. Rose, *Elementary Theory of Angular Momentum*, p. 85 (John Wiley & Sons, Inc., New York; Chapman and Hall, Ltd., London, 1957).

ond-order terms vanish; thus this transition is (at least) "second forbidden."

Squaring the magnitudes of the sums over m gives expressions $\langle f\|O_1\|m\rangle\langle f\|O_2\|m'\rangle^*\langle m\|O_3\|i\rangle\langle m'\|O_4\|i\rangle^*$, where O_1, O_2, O_3, O_4 represents one of the operators $\mathbf{1}$, $\boldsymbol{\sigma}$. Thus there are sixteen possible combinations of reduced matrix elements. If the initial nucleus is non-oriented, combinations with one $\boldsymbol{\sigma}$ do not occur. This is true for any initial angular momentum. Equations (1) and (2) exhibit interference between different intermediate states. This interference was not taken into consideration by Konopinski.[3]

The total decay probability is

$$\lambda(0\to 0)=\frac{G^4}{30\pi^7}\{f_{50}(Z,\epsilon)|\sum_m \eta_m^{-1}\mathcal{A}_m|^2+2[f_{52}(Z,\epsilon)+f_{70}(Z,\epsilon)]\,\mathrm{Re}\,(\sum_m \eta_m^{-1}\mathcal{A}_m\sum_m \eta_m^{-3}\mathcal{A}_m^*)\},\quad (4)$$

$$\lambda(0\to 1)=\frac{G^4}{10\pi^7}\{f_{52}(Z,\epsilon)|\sum_m \eta_m^{-2}\mathcal{B}_m^+|^2+f_{70}(Z,\epsilon)|\sum_m \eta_m^{-2}\mathcal{B}_m^-|^2\},\quad (5)$$

$$\lambda(0\to 2)=0,\quad (6)$$

with

$$f_{50}(Z,\epsilon)\equiv\iint_{W_1\leqq W_2} F(Z,W_1)F(Z,W_2)p_1W_1p_2W_2(\epsilon-W_1-W_2)^5dW_1dW_2$$
$$=\frac{1}{32}\int_2^\epsilon dW(\epsilon-W)^5\int_0^{W-2}dDG(Z,W_1)G(Z,W_2)(W^2-D^2)^2$$
$$\approx\frac{1}{32}[G(Z,\bar{W}/2)]^2\int_2^\epsilon dW(\epsilon-W)^5\int_0^{W-2}dD(W^2-D^2)^2$$
$$=\frac{1}{84}[G(Z,\bar{W}/2)]^2(\epsilon-2)^7\left[1+\frac{1}{2}(\epsilon-2)+\frac{1}{9}(\epsilon-2)^2+\frac{1}{90}(\epsilon-2)^3+\frac{1}{1980}(\epsilon-2)^4\right],\quad (7)$$

$$W\equiv W_1+W_2,\quad D\equiv W_2-W_1,\quad G(Z,W_\kappa)\equiv(p_\kappa/W_\kappa)F(Z,W_\kappa),$$

$$f_{52}(Z,\epsilon)\equiv\iint_{W_1\leqq W_2} F(Z,W_1)F(Z,W_2)p_1W_1p_2W_2(\epsilon-W_1-W_2)^5(W_2-W_1)^2dW_1dW_2$$
$$\approx\frac{1}{3024}[G(Z,\bar{W}/2)]^2(\epsilon-2)^9\left[1+\frac{4}{5}(\epsilon-2)+\frac{12}{55}(\epsilon-2)^2+\frac{1}{55}(\epsilon-2)^3+\frac{1}{1430}(\epsilon-2)^4\right],\quad (8)$$

$$f_{70}(Z,\epsilon)\equiv\frac{1}{7}\iint_{W_1\leqq W_2} F(Z,W_1)F(Z,W_2)p_1W_1p_2W_2(\epsilon-W_1-W_2)^7dW_1dW_2$$
$$\approx\frac{1}{1008}[G(Z,\bar{W}/2)]^2(\epsilon-2)^9\left[1+\frac{2}{5}(\epsilon-2)+\frac{4}{55}(\epsilon-2)^2+\frac{1}{165}(\epsilon-2)^3+\frac{1}{4290}(\epsilon-2)^4\right].\quad (9)$$

The first index of f refers to the power of $\epsilon-W_1-W_2$, the energy of both antineutrinos, and the second to the power of W_2-W_1. $\bar{W}$ is determined in the following way. $(W^2-D^2)^2(\epsilon-W)^5$ has a pronounced maximum at $D=0$, $W=(4/9)\epsilon$. Since the upper limit of integration depends on W, we weight $(W^2-D^2)^2(\epsilon-W)^5$ with the upper limit of integration, $W-2$. The maximum value of $(W^2-D^2)^2(\epsilon-W)^5(W-2)$ is at

$$\bar{W}=\frac{1}{20}\{5(\epsilon-2)+28+[25(\epsilon-2)^2-40(\epsilon-2)+144]^{\frac{1}{2}}\}.\quad (10)$$

$G(Z,W_1)$ and $G(Z,W_2)$ are both replaced by $G(Z,\bar{W}/2)$. This approximation is more exact than substituting $G(Z,0)$ instead of $G(Z,\bar{W}/2)$, as it was done by Konopinski,[3] since $G(Z,W)$ is nearly independent of W only in the region of $Z\sim 40$. $G(Z,W)$ is tabulated.[6]

As an example, the decay probability of ${}_{20}Ca_{28}$ was calculated. Allowed transitions are possible only to the ground state of ${}_{22}Ti_{26}$ according to the level scheme given by Way *et al.*[7] The decay energy is 5.25 Mev. The positions of the intermediate levels with $J_m=0$, 1 are not known. The reduced matrix elements were calculated according to the shell model with j-j

[6] K. Siegbahn, *Beta- and Gamma-Ray Spectroscopy*, app. II (North-Holland Publishing Company, Amsterdam, 1955).

[7] *Nuclear Level Schemes*, $A=40$—$A=92$, compiled by Way, King, McGinnis, and van Lieshout, Atomic Energy Commission Report TID-5300, p. 28 (U. S. Government Printing Office, Washington, D. C., 1955).

coupling. The transition then takes place within the subshell $f_{7/2}$. There is one intermediate state with $J_m=0$ and one with $J_m=1$. The reduced matrix elements between the initial and intermediate states are

$$\langle m_0\|\mathbf{1}\|i\rangle=-8^{\frac{1}{2}},\quad \langle m_1\|\boldsymbol{\sigma}\|i\rangle=(24/7)^{\frac{1}{2}}. \tag{11}$$

In the final state, the six neutrons and the two protons may couple to $J_f=0$ in nine different ways. From these nine functions one linear combination with total isotopic spin $T=4$ and eight with $T=2$ may be built up. Assuming that in the ground-state of ${}_{22}\mathrm{Ti}_{26}$ the neutrons and protons couple to $J_n=J_p=0$, that is $|f\rangle=|00_n0_p\rangle$, we get

$$\langle f\|\mathbf{1}\|m_0\rangle=2^{-\frac{1}{2}},\quad \langle f\|\boldsymbol{\sigma}\|m_1\rangle=-(9/14)^{\frac{1}{2}}. \tag{12}$$

The assumed ground-state function has no sharp isotopic spin. We therefore also take into consideration a linear combination with $T=2$ and expectation values of the neutron and proton angular momenta as small as possible. This combination is

$$|f\rangle=(5/6)^{\frac{1}{2}}|00_n0_p\rangle-(1/6)^{\frac{1}{2}}|02_n2_p\rangle. \tag{13}$$

With Eq. (12) and

$$\langle 02_n2_p\|\boldsymbol{\sigma}\|m_1\rangle=-(17/7)(5/14)^{\frac{1}{2}}, \tag{14}$$

we now get

$$\langle f\|\boldsymbol{\sigma}\|m_1\rangle=-(2/7)(5/21)^{\frac{1}{2}}. \tag{15}$$

The Fermi element vanishes according to the selection rules for the isotopic spin. The calculated half-lives are plotted in Fig. 1.[8]

Konopinski[3] assumes that a half-life of $t\sim10^{18}$ years is perhaps still detectable. The calculated values are certainly not below this limit. Moreover, the calculated matrix elements have the same order of magnitude as favored transitions. However, the allowed single β transitions within the region of nucleon numbers considered here are normally allowed, so their ft values differ by those of the favored transitions by the order of magnitude of 10^2. Consequently, the actual half-life of ${}_{20}\mathrm{Ca}_{28}$ will differ from the calculated one by the order of magnitude of 10^4. λ rises very much with increasing ϵ and is approximately proportional to Z^2 according to Konopinski.[3] If there were a pair of isobars with large Z and with a much greater decay energy than ${}_{20}\mathrm{Ca}_{28}$, double β decay could perhaps be detected. However, the energy difference of pairs of isobars with $\Delta Z=2$ is proportional to A^{-1} according to Weizsäcker's formula; thus pairs of isobars with the required energy difference cannot be expected among nuclei with high Z. Moreover, it is unlikely that the smallness of the matrix elements can be compensated by a great number of intermediate states. About 10^2 intermediate states would be required, whose energies should be close to ϵ_i due to the factor η_m^{-1}. A compilation of double β decay experiments is given by Allen.[9]

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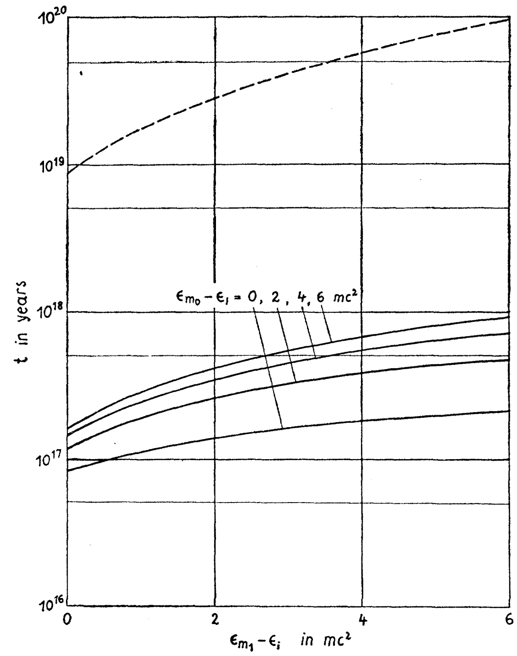

FIG. 1. Half-life of ${}_{20}\mathrm{Ca}_{28}$. Full line: reduced matrix elements of Eqs. (11) and (12). Dashed line: reduced matrix elements of Eqs. (11) and (15). *Note.*—The "mc^2" label in the figure and in the abscissa should read "Mev."

[8] V. B. Belayaev and B. N. Zaharyev [J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 505 (1958)] find $t\approx10^{19}$ years on the basis of a shell-model estimate of the matrix elements, as mentioned in reference 4.

[9] J. S. Allen, *The Neutrino* (Princeton University Press, Princeton, New Jersey, 1958), Chap. 6.