Magnetic Moment of the Deuteron*†

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A contribution to the deuteron magnetic moment which results from the altered expectation values of nucleon core spins in the bound state is calculated. The adiabatic approximation, in which the π-meson clouds of a static-nucleon model are assumed to follow the orbital motion of the sources, is used. The two-nucleon states are expanded in Heitler-London states; the expectation value of the deuteron magnetic moment operator is related to single-nucleon matrix elements by means of an expansion corresponding to exchange of various numbers of mesons between the nucleons. The single-nucleon matrix elements are evaluated using the Chew-Low-Wick fixed-source theory. If an orbital wave function with a relatively large D-state probability is used, the one-meson exchange terms give an increase in the deuteron magnetic moment of about one percent. The two-meson exchange terms are considerably smaller than the one-meson exchange terms.

I. INTRODUCTION

THE work to be described is an attempt to determine the consequences of a simple meson theory for the magnetic moment of the deuteron.

The first attempts to account for the deuteron magnetic moment ignored the internal structure of the nucleons and simply postulated a nucleon-nucleon potential which is partly central and partly tensor potential, as suggested by the existence of the deuteron electric quadrupole moment. Such a potential leads to a deuteron wave function which is a mixture of S and D states, and hence to a contribution to the magnetic moment associated with the orbital motion. It is easy to show that the magnetic moment of the deuteron is then

$$\mu_d=\mu_s-\tfrac{3}{2}(\mu_s-\tfrac{1}{2})p_D, \tag{1.1}$$

where p_D is the D-state probability and μ_s is the sum of the proton and neutron moments; $\mu_s=\mu_n+\mu_p$. A value of p_D of about 4% gives the observed value of μ_d, although meson-theoretic calculations of the tensor potential indicate that p_D may be considerably larger than 4%.[1–3]

It has been pointed out[4–6] that there are relativistic effects, resulting from the fact that the interaction of the nucleons with an electromagnetic field depends on their kinetic energies. Because of uncertainty about the transformation properties of the nucleon-nucleon potential, the magnitudes and even signs of these corrections are not known with any certainty.

If the nucleon-nucleon interaction contains spin-orbit potentials [7,8] there is an additional electromagnetic interaction.[9,10] At present, however, there is no clear evidence, either theoretical or experimental, concerning the magnitude or sign of the spin-orbit potential for nucleons in $T=0$ states.[11,12]

A third effect, the one with which the present work is concerned, is the modification of the internal structure of the nucleons resulting from their binding in the deuteron. Previous attempts to calculate mesonic contributions to the deuteron magnetic moment have all been based on some form of perturbation theory, usually a Tamm-Dancoff method.[5,6,13,14] These methods suffer from the difficulty that the renormalization of the coupling constant, and the elimination of the self-energies and other unobservable quantities, must all be done explicitly. The results of such calculations are not quantitatively very reliable.

In the present work the deuteron moment corrections are related to observable properties of single nucleons.[15] First, a model for the nucleon is used which consists of a "rigid core" having a spin and a magnetic moment, surrounded by a cloud of π mesons. The contribution to the magnetic moment which we have calculated results from the fact that the probability that a nucleon

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core spin has a given orientation may be different in the bound state from its value for an isolated nucleon.

To calculate this contribution a fixed-source theory has been used, in which all nucleon recoil effects are ignored. The "adiabatic approximation" is used, in which the meson cloud resulting from the fixed sources is assumed to follow them if they are given some slow orbital motion. The deuteron magnetic moment is then made up of two parts: the contribution from the nucleon cores, which is calculated using static nucleons, and a part due to the slow orbital motion of the cores.

In the charge-symmetric theory the π-meson current itself does not contribute to the deuteron magnetic moment because it is an isotopic spin vector, while the state of the deuteron has total isotopic spin $T=0$.

II. MODEL

A. Hamiltonian

In the present model the nucleon is considered to consist of a nucleon core surrounded by a π-meson cloud. The fixed, extended core is the source of the meson field. No nonlinear interactions (such as meson-meson interactions) are included.

The nucleon core itself has an internal structure, resulting from creation of virtual nucleon pairs, heavy mesons and hyperons, and so forth. We assume, however, that the structure of the core is not affected by the change in the meson cloud resulting from interaction of two nucleons, and thus the internal coordinates of the nucleon core may be ignored. We use the Hamiltonian[16,17]

$$H=\sum_k Ka_k^*a_k-E_0(\alpha_x^*\alpha_x+\alpha_y^*\alpha_y)+\sum_k(a_k+a_{-k}^*)(V_{xk}+V_{yk}). \quad (2.1)$$

We use units in which $\hbar$, c, and the meson mass are all equal to unity. A capital letter denotes the energy of a meson whose momentum is the corresponding small letter, e.g., $K=(k^2+1)^{\frac{1}{2}}$. The meson creation and destruction operators are a_k^* and a_k, respectively, where the index k includes both momentum and isotopic spin state; α_x^* and α_x are creation and destruction operators for a nucleon core, where x includes position, spin, and isotopic spin of the nucleon core. The value of E_0 is chosen to make the energy of an isolated physical nucleon zero. The interaction operators are

$$V_{xk}=\alpha_x^*[f_0v_k(2K)^{-\frac{1}{2}}\tau_x^k i\boldsymbol{\sigma}^x\cdot\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{x})]\alpha_x, \quad (2.2)$$

where f_0 is the rationalized but unrenormalized coupling constant, and v_k is the source function.

B. Magnetic Moments

The nucleon core has spin one-half and isotopic spin one-half. Its magnetic moment depends on its charge state, so the magnetic moment operator for the core can be written

$$\boldsymbol{\mu}_c=\tfrac{1}{2}(\alpha+\beta\tau_3)\boldsymbol{\sigma}, \quad (2.3)$$

where α and β are constants, and $\boldsymbol{\sigma}$ and τ_3 are operators for the *core* spin and isotopic spin, respectively. The total magnetic moment of the nucleon is the sum of this moment and the contribution from the meson current. Thus we have

$$\mu_p=\tfrac{1}{2}(\alpha/\rho_1+\beta/\rho_2)+\langle\mu_\pi\rangle \text{ and } \mu_n=\tfrac{1}{2}(\alpha/\rho_1-\beta/\rho_2)-\langle\mu_\pi\rangle,$$

where $1/\rho_1=\langle\sigma_z\rangle$,[18] $1/\rho_2=\langle\sigma_z\tau_3\rangle$, and $\langle\mu_\pi\rangle$ is the contribution of the π-meson cloud. We note that $\mu_s=\alpha/\rho_1$.

In the deuteron the total magnetic moment is the expectation value of the operator

$$\tfrac{1}{2}\alpha(\sigma_z^1+\sigma_z^2)+\tfrac{1}{2}\beta(\sigma_z^1\tau_3^1+\sigma_z^2\tau_3^2)+\mu_\pi+\mu_{\rm orb}, \quad (2.4)$$

where $\langle\mu_{\rm orb}\rangle=\frac{3}{4}p_D$ is the contribution from the orbital motion of the cores. However, because the deuteron is a $T=0$ state, neither the meson current nor the part of the core moment containing τ_3 contributes to this expectation value. Hence, defining the total core spin as $\mathbf{S}=\frac{1}{2}(\boldsymbol{\sigma}^1+\boldsymbol{\sigma}^2)$, we have for the total magnetic moment

$$\mu_d=\alpha\langle S_z\rangle+\langle\mu_{\rm orb}\rangle. \quad (2.5)$$

In the calculation, $\langle S_z\rangle$ will be proportional to $1/\rho_1$, so that only the ratio $\alpha/\rho_1=\mu_s$ appears in the final result. The experimental value of this quantity will be used.

In order to describe how $\langle S_z\rangle$ is calculated, we first discuss the wave function $\psi(r)$ which would be used if the nucleons had no internal degrees of freedom. We use a function containing a mixture of S and D states, with total angular momentum $J=1$ and $J_z=1$. Since we shall use a fixed-source model, it is convenient to represent this function using spin functions referred to an axis system in which the nucleon cores are at rest and which rotates with respect to the fixed axis system. We call this the deuteron axis system, and denote quantities referred to it with a bar. We take the nucleons to lie on the $\bar{z}$ axis, and describe the orientation of this axis with respect to the fixed z axis by means of the usual spherical coordinate angles θ and ϕ. We then find

$$S_z=2^{-\frac{1}{2}}\sin\theta e^{-i\phi}\bar{S}_+ +\cos\theta\bar{S}_0-2^{-\frac{1}{2}}\sin\theta e^{i\phi}\bar{S}_-, \quad (2.6)$$

where the spherical components $\bar{S}_0=\bar{S}_z$,

$$\bar{S}_\pm=\mp(\bar{S}_x\pm i\bar{S}_y)/\sqrt{2}$$

are used, and

$$\psi_1=r^{-1}(4\pi)^{-\frac{1}{2}}[(u+w/\sqrt{2})\cos^2\tfrac{1}{2}\theta\bar{\chi}_1-(u-\sqrt{2}w)\times\sin\theta e^{i\phi}\bar{\chi}_0+(u+w/\sqrt{2})\sin^2\tfrac{1}{2}\theta e^{2i\phi}\bar{\chi}_{-1}], \quad (2.7)$$

where the functions $\bar{\chi}_m$ are eigenfunctions of $\bar{S}_0$, and u and w are the usual S and D radial functions.

Now, making use of the adiabatic approximation, we generalize the meaning of the spin functions so that they represent also the internal degrees of freedom of the nucleons. We replace $\bar{\chi}_m$ by a normalized eigenstate

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Ψ_m of the static theory which contains two nucleon cores lying along the $\bar{z}$ axis (separated by a distance r) and an associated meson cloud. In the next section we show how the states Ψ_m can be constructed.

The deuteron state vector is

$$\boldsymbol{\Psi}_1 = (2.7 \text{ with } \Psi_m \text{ instead of } \bar{\chi}_m); \tag{2.8}$$

our objective is to calculate

$$\langle S_z \rangle = \langle \boldsymbol{\Psi}_1, S_0 \boldsymbol{\Psi}_1 \rangle. \tag{2.9}$$

This depends on the matrix elements

$$(\Psi_m, \bar{S}_M \Psi_{m'}).$$

Only two of these are independent, because the value of any matrix element must be invariant with respect to a rotation about the $\bar{z}$ axis and a reflection in the $\bar{x}$–$\bar{y}$ plane. This symmetry leads to the following relations:

$$\begin{aligned} p &= (\Psi_1, \bar{S}_0 \Psi_1) = -(\Psi_{-1}, S_0 \Psi_{-1}), \\ q &= (\Psi_1, \bar{S}_+ \Psi_0) = (\Psi_0, \bar{S}_+ \Psi_{-1}) \\ &= -(\Psi_0, \bar{S}_- \Psi_1) = -(\Psi_{-1}, \bar{S}_- \Psi_0), \\ &(\Psi_0, \bar{S}_0 \Psi_0) = 0. \end{aligned} \tag{2.10}$$

When these expressions are substituted into (2.9) and the angular integrations are performed, the result is

$$\langle S_z \rangle = \int_0^\infty \left[\tfrac{1}{3}(u+w/\sqrt{2})^2 p - \tfrac{2}{3}(u+w/\sqrt{2})(u-\sqrt{2}w) q\right] dr. \tag{2.11}$$

In our fixed-source calculation, p and q are functions of the internuclear distance r.

C. Heitler-London States

The principal problem in this calculation is to evaluate the matrix elements p and q, defined by Eq. (2.10). It is here that our method differs significantly from those used in previous attempts to calculate the deuteron magnetic moment. Rather than expand the state of the deuteron in states containing two "bare nucleons" and various numbers of mesons, we use a method[15] described in A to expand the deuteron state in states containing two *physical* (clothed) nucleons and various numbers of meson in scattering states.

As in A, we define operators which, operating on the vacuum state, create physical nucleons. For a nucleon at x, $\mathfrak{F}_x^* |0\rangle = |x\rangle$. We specify that each $\mathfrak{F}^*$ shall contain only meson creation operators and one nucleon core creation operator.

The Heitler-London states are defined as combinations of states of the form $\mathfrak{F}_x^* \mathfrak{F}_y^* |0\rangle$. The states used in the present work are linear combinations of the states constructed in A, chosen to be eigenstates of total angular momentum and total isotopic spin.

Assuming that the Heitler-London (H-L) states form a complete set, we can expand the state vector Ψ_m of the deuteron in these states:

$$\Psi_m = \Phi_m + \sum_{km'} \chi_{m'm}(k) \Phi_{m'k} + \sum_{klm'} \chi_{m'm}(kl) \Phi_{m'kl} + \cdots. \tag{2.12}$$

Equations for the amplitudes can be obtained from the variational principle

$$\delta[\Psi_m, (H-E)\Psi_m] = 0. \tag{2.13}$$

In general the H-L states are not eigenstates of H, and they are not orthonormal. At sufficiently large distances, however, it is expected that the amplitudes $\chi_{m'm}(k)$ will be small and $\chi_{m'm}(kl)$ smaller, so that approximate calculations of them will be sufficient. Assuming that the H-L states are approximately orthonormal and eigenstates of H, we obtain approximately:

$$\chi_{m'm}(k) = -K^{-1} V_{m'm}(k;), \tag{2.14}$$

$$\chi_{m'm}(kl) = -\tfrac{1}{2}(K+L)^{-1} V_{m'm}(kl;), \tag{2.15}$$

where $V_{m'm}(k\cdots;) = (\Phi_{m'k\ldots}, H\Phi_m)$. The norms of the H-L states, and the various matrix elements of H, can be expanded in terms of single-nucleon matrix elements. These expansions are given in A. Similar expansions for the matrix elements of $\bar{S}^\lambda$ ($\lambda = 0, \pm 1$) can all be obtained easily from corresponding expansions for the norms of the states. Because we deal always with symmetric spin states,

$$(\Phi_{m'}, \bar{S}^\lambda \Phi_m) = (\Phi_{m'}, \sigma^{\lambda,x} \Phi_m), \tag{2.16}$$

and it is the latter expression which is evaluated in most of the following calculations. For example,

$$\begin{aligned} (\Phi_{m'}, \bar{S}^\lambda \Phi_m) = \sum_{N=0} \sum_{\nu=0}^{N} \sum_{k_i} [\nu!(N-\nu)!]^{-1} \\ \times \langle x' | a_1^* \cdots a_\nu^* \sigma^\lambda a_{\nu+1} \cdots a_N | x \rangle \\ \times \langle y' | a_N^* \cdots a_{\nu+1}^* a_\nu \cdots a_1 | y \rangle. \end{aligned} \tag{2.17}$$

It is convenient to separate contributions to p and q corresponding to different numbers of exchanged mesons. We therefore write

$$\begin{aligned} p &= \frac{S_0^0 + S_1^0 + S_2^0}{A_0^+ + A_1^+ + A_2^+}; \\ q &= \frac{S_0^+ + S_1^+ + S_2^+}{(A_0^+ + A_1^+ + A_2^+)^{\frac{1}{2}} (A_0^0 + A_1^0 + A_2^0)^{\frac{1}{2}}}, \end{aligned} \tag{2.18}$$

where the numerator is the matrix element of S in the state (2.12), and the denominator corrects for the fact that Ψ_m as given by (2.12) is not normalized to unity. The subscripts refer to the number of exchanged mesons; the superscripts on S denote the component of S, and those on A denote the value of m for the corresponding state. In calculating the various contributions to these quantities we shall use an additional subscript in parentheses to denote their origin.

In evaluating these expressions, all the terms corresponding to zero- or one-meson exchange will be included, but only the two-meson terms which involve the ground states or the $\frac{3}{2}$–$\frac{3}{2}$ excited states of the nucleons. Only the basic H-L state contributes to the zero-meson terms; therefore $A_0^0=A_0^+=1$, and $S_0^0=-S_0^+=1/\rho_1$. Thus the zero-meson terms give just the values of p and q for noninteracting nucleons. The one- and two-meson exchange terms give corrections to these values. Making Taylor series expansions of the denominators in (2.18), we find

$$\begin{aligned} p&=S_0^0+(S_1^0-S_0^0A_1^+)+(S_2^0-S_0^0A_2^+)\\ &\quad -A_1^+(S_1^0-S_0^0A_1^+),\\ q&=S_0^++S_1^+-\tfrac{1}{2}S_0^+(A_1^0+A_1^+)+S_2^+\\ &\quad -\tfrac{1}{2}S_0^+(A_2^0+A_2^+)+\tfrac{3}{8}S_0^+((A_1^0)^2+(A_1^+)^2)\\ &\quad +\tfrac{1}{4}S_0^+A_1^0A_1^+-\tfrac{1}{2}S_1^+(A_1^0+A_1^+). \end{aligned} \tag{2.19}$$

III. ONE-MESON EXCHANGE TERMS

We calculate first the one-meson terms from the zero-meson Heitler-London state. These will then be combined with the one-meson terms from the one-meson H-L states, which have a very similar form.

A. Zero-Meson Heitler-London State

The $N=1$ terms in the expansions for the zero-meson H-L states are

$$\begin{aligned} \mathbf{S}_{1(0)}&=\textstyle\sum_k [\langle x'|\boldsymbol{\sigma}a_k|x\rangle\langle y'|a_k^*|y\rangle\\ &\quad +\langle x'|a_k^*\boldsymbol{\sigma}|x\rangle\langle y'|a_k|y\rangle]\\ &=\textstyle\sum_k K^{-1}\langle x'|\boldsymbol{\sigma}(H+K)^{-1}V_k^*\\ &\quad +V_k^*(H+K)^{-1}\boldsymbol{\sigma}|x\rangle\langle y'|V_k|y\rangle, \end{aligned} \tag{3.1}$$

$$\begin{aligned} A_{1(0)}&=\textstyle\sum_k [\langle x'|a_k|x\rangle\langle y'|a_k^*|y\rangle\\ &\quad +\langle x'|a_k^*|x\rangle\langle y'|a_k|y\rangle]\\ &=\textstyle\sum_k 2K^{-2}\langle x'|V_k^*|x\rangle\langle y'|V_k|y\rangle. \end{aligned} \tag{3.2}$$

In Eq. (3.1) the matrix element containing $\boldsymbol{\sigma}$ can be approximated by making a closure expansion using a complete set of one-nucleon states which are eigenstates of the total Hamiltonian. The first term in this expansion gives

$$\begin{aligned} &\textstyle\sum_k K^{-2}\sum_{x''} [\langle x'|\boldsymbol{\sigma}|x''\rangle\langle x''|V_k^*|x\rangle\\ &\quad +\langle x'|V_k^*|x''\rangle\langle x''|\boldsymbol{\sigma}|x\rangle]\langle y'|V_k|y\rangle. \end{aligned} \tag{3.3}$$

These matrix elements for the physical nucleon states can now be expressed in terms of matrix elements for the corresponding "bare nucleon" states.[18] We denote by $|x)$ the bare nucleon state corresponding to the physical nucleon state $|x\rangle$. The operator V_k is understood to contain the unrenormalized coupling constant f_0 when it appears with physical nucleon states, and the renormalized coupling constant f when it appears with bare states. Expression (3.3) then becomes

$$\begin{aligned} &\textstyle\sum_k K^{-2}\rho_1^{-1}\sum_{x''} [(x'|\boldsymbol{\sigma}|x'')(x''|V_k|x)\\ &\quad +(x'|V_k|x'')(x''|\boldsymbol{\sigma}|x)](y'|V_k|y), \end{aligned} \tag{3.4}$$

where x'' ranges over the four bare-nucleon states. The product $S_0A_{1(0)}$ may be written in the form

$$\begin{aligned} \mathbf{S}_0A_{1(0)}&=\rho_1^{-1}\textstyle\sum_k K^{-2}\sum_{x''} [(x'|\boldsymbol{\sigma}|x'')(x''|V_k^*|x)\\ &\quad +(x'|V_k^*|x'')(x''|\boldsymbol{\sigma}|x)](y'|V_k|y), \end{aligned} \tag{3.5}$$

because for either σ_0 or σ_+, only one of the matrix elements $(x''|\boldsymbol{\sigma}|x)$ is different from zero.

In this approximation, then,

$$S_{1(0)}{}^0-S_0^0A_{1(0)}{}^+=0, \tag{3.6}$$

and

$$S_{1(0)}{}^+-\tfrac{1}{2}S_0^+(A_{1(0)}{}^0+A_{1(0)}{}^+)=0; \tag{3.7}$$

the first term in the closure expansion for $S_{1(0)}$ is exactly cancelled by the change in the normalization.

The next term in the closure expansion for (3.1) is

$$\begin{aligned} &\textstyle\sum_k \sum_{x''p} K^{-1}(K+P)^{-1}[\langle x'|\boldsymbol{\sigma}|x''p\rangle\langle x''p|V_k^*|x\rangle\\ &\quad +\langle x'|V_k^*|x''p\rangle\langle x''p|\boldsymbol{\sigma}|x\rangle]\langle y'|V_k|y\rangle, \end{aligned} \tag{3.8}$$

where the $|x''p\rangle$ are the meson scattering states with one meson in a plane wave. In this calculation it is not necessary to specify whether the complete set with incoming waves or the complete set with outgoing waves is used. It is assumed that one or the other is used consistently.

B. One-Meson Heitler-London State

We now proceed to the calculations of the one-meson exchange contribution from the one-meson H-L states $\Phi_{m'',p}$. The amplitude of the one-meson H-L state is given by Eq. (2.14). An expansion for $V_{m''m}(p)$; is given in A; keeping only terms corresponding to exchange of one meson, we have

$$\begin{aligned} V_{m''m}(p;)&=\textstyle\sum_k [K^{-1}+(P+K)^{-1}][\langle x''p|V_k^*|x\rangle\\ &\quad \times\langle y''|V_k|y\rangle+\langle x''|V_k^*|x\rangle\langle y''p|V_k|y\rangle]. \end{aligned} \tag{3.9}$$

In the expansion of $A_{m'm}(;k)$ at least one additional meson is exchanged; thus the one-meson H-L state does not contribute to A_1. Furthermore, only the first term of the expansion for $(\Phi_{m'},\mathbf{S}\Phi_{mk})$, which is $\langle x'|\boldsymbol{\sigma}|xk\rangle\langle y'|y\rangle$, should be included in the one-meson exchange contributions. Therefore, the contribution of the one-meson H-L state to $\mathbf{S}_1$ is

$$\begin{aligned} \mathbf{S}_{1(1)}&=\sum_{kpx''}\frac{1}{P}\left[\frac{1}{K}+\frac{1}{K+P}\right][\langle x'|\boldsymbol{\sigma}|x''p\rangle\langle x''p|V_k^*|x\rangle\\ &\quad +\langle x'|V_k^*|x''p\rangle\langle x''p|\boldsymbol{\sigma}|x\rangle]\langle y'|V_k|y\rangle. \end{aligned} \tag{3.10}$$

C. Evaluation of One-Meson Exchange Terms

The result (3.10) has exactly the same form as (3.8) except for the expressions containing the energies. Combining (3.8) and (3.10), we find

$$\begin{aligned} \mathbf{S}_1&=\sum_{kpx''}\frac{2}{KP}[\langle x'|\boldsymbol{\sigma}|x''p\rangle\langle x''p|V_k^*|x\rangle\\ &\quad +\langle x'|V_k^*|x''p\rangle\langle x''p|\boldsymbol{\sigma}|x\rangle]\langle y'|V_k|y\rangle. \end{aligned} \tag{3.11}$$

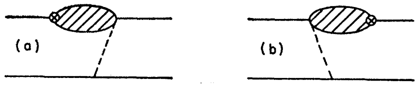

FIG. 1. Diagrams corresponding to one-meson exchange terms.

The higher terms in the closure expansion used in (3.1) can be shown to combine in an analogous way with corresponding terms for the multimeson H-L states. Thus the sum of all the one-meson exchange contributions is

$$\mathbf{S}_1-\mathbf{S}_0A_1=\sum_{x''k}\sum_n 2K^{-1}E_n^{-1}[\langle x'|\boldsymbol{\sigma}|x''n\rangle \times\langle x''n|V_k^*|x\rangle+\langle x'|V_k^*|x''n\rangle \times\langle x''n|\boldsymbol{\sigma}|x\rangle]\langle y'|V_k|y\rangle, \quad (3.12)$$

where the sum on n includes *all* multimeson states but not the unexcited nucleon states $|x\rangle$. However, since not enough is known about the states which contribute to (3.12), we must resort to some approximation methods.

The expression (3.12) has the form of a closure expansion for the quantity

$$2\langle E^{-1}\rangle \sum_k K^{-1}\langle x'|\boldsymbol{\sigma}V_k^*+V_k^*\boldsymbol{\sigma}|x\rangle\langle y'|V_k|y\rangle, \quad (3.13)$$

where $\langle E^{-1}\rangle$ is the average reciprocal energy of the n-meson states, except that the zero-meson term of the expansion is missing. We therefore replace (3.12) by

$$\mathbf{S}_1-\mathbf{S}_0A_1=2\langle E^{-1}\rangle\{\sum_k K^{-1}\langle x'|\boldsymbol{\sigma}V_k^*+V_k^*\boldsymbol{\sigma}|x\rangle\langle y'|V_k|y\rangle -\sum_{kx''}K^{-1}[\langle x'|\boldsymbol{\sigma}|x''\rangle\langle x''|V_k^*|x\rangle +\langle x'|V_k^*|x''\rangle\langle x''|\boldsymbol{\sigma}|x\rangle]\langle y'|V_k|y\rangle\}. \quad (3.14)$$

This expression can be represented pictorially by the diagrams of Fig. 1, in which solid lines represent physical nucleons, dashed lines represent mesons, the cross represents the operator $\boldsymbol{\sigma}$, and the bubbles represent excited states of the nucleons.

The matrix elements in (3.14) can be related immediately to the corresponding bare-nucleon matrix elements:

$$\mathbf{S}_1-\mathbf{S}_0A_1=2\langle E^{-1}\rangle\rho_1^{-1}[(f_0/f)-1] \times\sum_k K^{-1}(x'|\boldsymbol{\sigma}V_k^*+V_k^*\boldsymbol{\sigma}|x)(y'|V_k|y). \quad (3.15)$$

Inserting the explicit expression (2.2) for V_k, we find

$$\mathbf{S}_1-\mathbf{S}_0A_1=2\langle E^{-1}\rangle\rho_1^{-1}(ff_0-f^2) \times\sum_{\mathbf{k}\lambda}\tfrac{1}{2}v_k^2K^{-2}\exp[i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})] \times(x'|(\boldsymbol{\sigma}\boldsymbol{\sigma}\cdot\mathbf{k}+\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma})\tau_\lambda^x|x) \times(y'|\boldsymbol{\sigma}\cdot\mathbf{k}\tau_\lambda^y|y), \quad (3.16)$$

where the isotopic spin index is now written explicitly. We use the spherical components of $\boldsymbol{\sigma}$ and $\mathbf{k}$:

$$\boldsymbol{\sigma}\cdot\mathbf{k}=\sigma_0k_0-\sigma_+k_--\sigma_-k_+.$$

In the integration over $\mathbf{k}$, the only terms which survive contain the factors k_+k_- or k_0^2. The result can be expressed in terms of the integrals

$$Z(r)=aF,\quad Y(r)=bF, \quad (3.17)$$

where

$$F(r)=4\pi\int\frac{d^3k}{(2\pi)^3}\frac{v_k^2}{K^2}e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (3.18)$$

and a and b are differential operators: $a=d^2/dr^2$ and $b=-r^{-1}d/dr$. The factor k_0^2 leads to a factor $-a$, and the factor k_+k_- to a factor $-b$.[2] The results (in $T=0$ states) are

$$\Delta p_1=6\langle E^{-1}\rangle\rho_1^{-1}[ff_0-f^2](4\pi)^{-1}aF(r), \quad (3.19)$$

$$\Delta q_1=6\langle E^{-1}\rangle\rho_1^{-1}[ff_0-f^2](4\pi)^{-1}bF(r). \quad (3.20)$$

An alternative procedure for evaluating (3.11) is given by a method which is essentially equivalent to fourth-order perturbation theory, except that the resulting expression is completely renormalized. We use a formula for the one-meson scattering states[17]:

$$|xp_\pm\rangle=a_p^*|x\rangle-(H-P\pm i\epsilon)^{-1}V_p|x\rangle. \quad (3.21)$$

We insert this into (3.11) and make closure expansions where needed, keeping only the zero-meson term in each. This is a reasonable procedure because the $\frac{3}{2}$–$\frac{3}{2}$ resonant state does not appear in the expansions. Equation (3.11) then becomes

$$\mathbf{S}_1=\rho_1^{-1}\sum_{kp}2(KP^3)^{-1}(x'|[(\boldsymbol{\sigma}V_p-V_p\boldsymbol{\sigma}) \times(V_p^*V_k^*-V_k^*V_p^*)+(V_p^*V_k^*-V_k^*V_p^*) \times(\boldsymbol{\sigma}V_p-V_p\boldsymbol{\sigma})]|x)(y'|V_k|y). \quad (3.22)$$

The operators a and b associated with the p-integration operate on a function $H(r)$ which is related to $F(r)$ by having an extra factor $1/P^2$ in the integrand. Furthermore, because the meson p is emitted and reabsorbed by the same nucleon instead of being exchanged, we evaluate the function at $r=0$ after applying the operators. We obtain

$$\Delta p_1=\rho_1^{-1}(f^2/4\pi)^2 12[bH(r)]_{r=0}aF(r), \quad (3.23)$$

$$\Delta q_1=\rho_1^{-1}(f^2/4\pi)^2 12[bH(r)]_{r=0}bF(r), \quad (3.24)$$

where

$$[bH]_{r=0}=\frac{1}{3}\frac{(4\pi)^2}{(2\pi)^3}\int_0^\infty\frac{k^4v_k^2}{(k^2+1)^2}dk. \quad (3.25)$$

The dependence of Δp_1 and Δq_1 on r is the same as in (3.19) and (3.20), but the numerical coefficient is $\rho_1^{-1}(f^2/4\pi)^2 12[bH(r)]_{r=0}$ instead of $6\langle E^{-1}\rangle\rho_1^{-1}(ff_0-f^2)\times(4\pi)^{-1}$. The values of these coefficients turn out to be remarkably similar.

IV. TWO-MESON EXCHANGE TERMS

The two-meson terms which will be calculated are of two kinds: the $N=2$ terms in the expansion (2.17) for the zero-meson H-L states, and contributions from the two-meson H-L states. The two-meson terms from the one-meson H-L states are not included because they all involve matrix elements for inelastic meson scattering which are assumed to be small compared with the

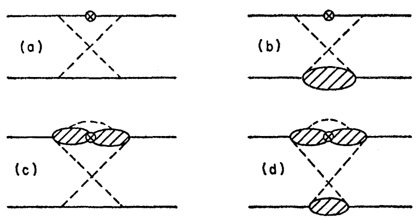

FIG. 2. Diagrams corresponding to two-meson exchange terms.

matrix elements for the ground states and $\frac{3}{2}$-$\frac{3}{2}$ resonant states which appear in the other two-meson terms.

A. Zero-Meson Heitler-London State

The $N=2$ terms of (2.17) are

$$\mathbf{S}_{2(0)}=\sum_{kl}\left[\tfrac{1}{2}\langle x'|a_k^*a_l^*\boldsymbol{\sigma}|x\rangle\langle y'|a_la_k|y\rangle+\langle x'|a_k^*\boldsymbol{\sigma}a_l|x\rangle\langle y'|a_l^*a_k|y\rangle+\tfrac{1}{2}\langle x'|\boldsymbol{\sigma}a_ka_l|x\rangle\langle y'|a_l^*a_k^*|y\rangle\right]. \quad (4.1)$$

In evaluating these, one must make closure expansions. It can be shown that in the first and third terms the zero-meson term in a closure expansion inserted next to the $\boldsymbol{\sigma}$ operator, is exactly cancelled by the corresponding term of the normalization factors A_2. Furthermore, the one-meson terms in the closure expansion do not contain the $\frac{3}{2}$-$\frac{3}{2}$ states. Hence in the approximations used in this calculation the first and third terms of (4.1) do not contribute anything.

The remaining term of (4.1) is

$$\mathbf{S}_{2(0)}=\sum_{kl}\langle x'|V_k(H+K)^{-1}[1]\boldsymbol{\sigma}[2](H+L)^{-1}V_l^*|x\rangle\times\langle y'|V_l(H+L)^{-1}[3](H+K)^{-1}V_k^*|y\rangle. \quad (4.2)$$

The symbols $[n]$ indicate points at which closure expansions are needed. The following combinations will be considered: (1) zero-meson terms in [1], [2], and [3] [Fig. 2(a)], (2) zero-meson terms in [1] and [2]; one-meson term in [3] [Fig. 2(b)], (3) one-meson terms in [1] and [2]; zero-meson term in [3] [Fig. 2(c)], and (4) one-meson terms in [1], [2], and [3] [Fig. 2(d)]. In the one-meson terms, only the $\frac{3}{2}$-$\frac{3}{2}$ states will be included; combinations other than the above are excluded because they involve either $T=\frac{1}{2}$ states or inelastic scattering matrix elements.

We calculate combination (1) first:

$$\mathbf{S}_{2(1)}=\frac{1}{\rho_1}\sum_{kl}\frac{1}{K^2L^2}(x'|V_k\boldsymbol{\sigma}V_l^*|x)(y'|V_lV_k^*|y)$$

$$=\frac{(f^2)^2}{\rho_1}\sum_{kl}\frac{1}{K^2L^2}\frac{v_k^2v_l^2}{(2K)(2L)}\exp[i(\mathbf{k}-\mathbf{l})\cdot\mathbf{r}]$$

$$\times(x'|\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\boldsymbol{\sigma}\cdot\mathbf{l}\tau_x{}^\lambda\tau_x{}^\mu|x)(y'|\boldsymbol{\sigma}\cdot\mathbf{l}\boldsymbol{\sigma}\cdot\mathbf{k}\tau_y{}^\mu\tau_y{}^\lambda|y). \quad (4.3)$$

Using the same procedure as for the one-meson exchange terms, we define a function

$$G(r)=4\pi\int\frac{d^3k}{(2\pi)^3}\frac{v_k^2}{K^3}e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (4.4)$$

in terms of which

$$S_{2(1)}{}^0=-\tfrac{3}{4}\gamma^2\rho_1{}^{-1}(a^2-4b^2)G(r)G(r), \quad (4.5)$$

where $\gamma=f^2/4\pi$, and we use the convention that whenever a product of two operators a or b appears, one of them operates on each of the two functions of r following. The corresponding contributions to A_2 is derived in the same way. The contribution of combination (1) to Δp is

$$\Delta p_{2(1)}=6\gamma^2\rho_1{}^{-1}b^2GG, \quad (4.6)$$

and the contribution to Δq is

$$\Delta q_{2(1)}=-\tfrac{3}{2}\gamma^2\rho_1{}^{-1}(a-b)^2GG. \quad (4.7)$$

B. Resonant Intermediate States

We next calculate combination (2) in the closure expansions for (4.2):

$$\mathbf{S}_{2(2)}=\frac{1}{\rho_1}\sum_{klpy''}\frac{1}{KL(K+P)(L+P)}\langle x'|V_k\boldsymbol{\sigma}V_l^*|x\rangle\times\langle y'|V_l|y''p\rangle\langle y''p|V_k^*|y\rangle. \quad (4.8)$$

Following Chew and Low,[16] we define an operator $T_q(p)$ such that

$$\langle y'p|V_q|y\rangle=(y'|T_q(p)|y). \quad (4.9)$$

It is shown in reference 16 that $T_q(p)$ can be expressed as

$$T_q(p)=-\frac{v_pv_q}{(2P)^{\frac{1}{2}}(2Q)^{\frac{1}{2}}}4\pi\sum_\alpha P_\alpha(pq)\times\frac{\exp[i\delta_\alpha(P)]\sin\delta_\alpha(p)}{p^3v_p{}^2}, \quad (4.10)$$

where $P_\alpha(pq)$ is a projection operator for the state with total angular momentum and isotopic spin denoted by α, and δ_α is the phase shift for meson-nucleon scattering in this state.

We assume as before that the only important contribution comes from the $\frac{3}{2}$-$\frac{3}{2}$ state, and in this state only from energies near the resonance energy ω_0. After replacing the energy denominators $(K+P)^{-1}$ and $(L+P)^{-1}$ by $(K+\omega_0)^{-1}$ and $(L+\omega_0)^{-1}$, we perform the integration on p and obtain

$$\sum_p(y'|T_l{}^\dagger(p)T_k(p)|y)=(4KL)^{-\frac{1}{2}}v_kv_l\exp[i(\mathbf{l}-\mathbf{k})\cdot\mathbf{y}]\times\tfrac{4}{3}\pi\gamma\alpha_3(y'|P_3(lk)|y), \quad (4.11)$$

where

$$\alpha_3\gamma=\frac{3}{\pi}\int_1^\infty\frac{\sin^2\delta_3(P)dP}{v_p{}^2p^3}. \quad (4.12)$$

Equation (4.8) then becomes

$$\mathbf{S}_{2(2)}=\frac{\gamma\alpha_3}{3\rho_1}\sum_{kl}(4\pi)^2\frac{v_k^2v_l^2}{4K^2L^2(K+\omega_0)(L+\omega_0)}\exp(i\mathbf{l}\cdot\mathbf{r})$$
$$\times\exp(-i\mathbf{k}\cdot\mathbf{r})(x'|\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\boldsymbol{\sigma}\cdot\mathbf{l}\tau_x{}^{\lambda}\tau_x{}^{\mu}|x)$$
$$\times(y'|(3\mathbf{l}\cdot\mathbf{k}-\boldsymbol{\sigma}\cdot\mathbf{l}\boldsymbol{\sigma}\cdot\mathbf{k})(\delta_{\lambda\mu}-\tfrac{1}{3}\tau_y{}^{\mu}\tau_y{}^{\lambda})|y), \quad (4.13)$$

from which we find

$$S_{2(2)}{}^0=\tfrac{2}{3}\gamma^2\alpha_3\rho_1{}^{-1}(a^2-b^2)G_1G_1, \quad (4.14)$$

where

$$G_1=4\pi\int\frac{d^3k}{(2\pi)^3}\frac{v_k^2}{K^2(K+\omega_0)}. \quad (4.15)$$

The quantities $S_{2(2)}{}^+$, $A_{2(2)}{}^0$, and $A_{2(2)}{}^+$ are obtained in an analogous way. The final results for combination (2) are

$$\Delta p_{2(2)}=-\tfrac{4}{3}\gamma^2\alpha_3\rho_1{}^{-1}b^2G_1G_1, \quad (4.16)$$

$$\Delta q_{2(2)}=\tfrac{4}{3}\gamma^2\alpha_3\rho_1{}^{-1}(a^2+ab+b^2)G_1G_1. \quad (4.17)$$

The calculation of combinations (3) and (4) proceeds similarly. The only additional complication is the appearance of the operator $\boldsymbol{\sigma}$ between two projection operators. We make the additional approximation $\langle xp|\boldsymbol{\sigma}|x'q\rangle=\langle x|\boldsymbol{\sigma}|x'\rangle\delta_{pq}$. The results of these combinations are

$$\begin{aligned}\Delta p_{2(3)}&=S_{2(3)}{}^0-S_0{}^2A_{2(3)}{}^+\\&=-(4/9)\gamma^2\alpha_3\rho_1{}^{-1}(a^2+2b^2)G_1G_1,\\ \Delta q_{2(3)}&=S_{2(3)}{}^+-\tfrac{1}{2}S_0{}^+(A_{2(3)}{}^0+A_{2(3)}{}^+)\\&=(2/9)\gamma^2\alpha_3\rho_1{}^{-1}(a^2-2ab+3b^2)G_1G_1,\end{aligned} \quad (4.18)$$

$$\begin{aligned}\Delta p_{2(4)}&=S_{2(4)}{}^0-S_0{}^0A_{2(4)}{}^+\\&=-(1/81)\gamma^2\alpha_3{}^2\rho_1{}^{-1}(4a^2+2b^2)G_2G_2,\\ \Delta q_{2(4)}&=S_{2(4)}{}^+-\tfrac{1}{2}S_0{}^+(A_{2(4)}{}^++A_{2(4)}{}^0)\\&=(1/81)\gamma^2\alpha_3{}^2\rho_1{}^{-1}(2a^2+2ab+6b^2)G_2G_2,\end{aligned} \quad (4.19)$$

where

$$G_2(r)=4\pi\int\frac{d^3k}{(2\pi)^3}\frac{v_k^2}{K(K+\omega_0)^2}e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (4.20)$$

The remaining two-meson terms from the zero-meson H-L state are products of one-meson exchange terms in (2.19):

$$\Delta p_2'=-A_1{}^+(S_1{}^0-S_0{}^0A_1{}^+), \quad (4.21)$$

$$\Delta q_2'=\tfrac{3}{8}S_0{}^+((A_1{}^0)^2+(A_1{}^+)^2)+\tfrac{1}{4}S_0{}^+A_0{}^+A_0{}^0$$
$$-\tfrac{1}{2}S_1{}^+(A_1{}^++A_1{}^0). \quad (4.22)$$

We evaluate these using only the zero-meson terms in the necessary closure expansions because, as has been pointed out, the $\tfrac{3}{2}$-$\tfrac{3}{2}$ states do not appear. In this approximation

$$S_1{}^0-S_0{}^0A_1{}^+=0, \quad S_1{}^+-\tfrac{1}{2}S_0{}^+(A_1{}^0+A_1{}^+)=0,$$
$$\Delta p_2'=0, \quad (4.23)$$

$$\Delta q_2'=\tfrac{1}{8}S_0{}^+(A_1{}^0-A_1{}^+)^2=-(9/2)\gamma^2\rho_1{}^{-1}(a+b)^2GG. \quad (4.24)$$

C. Two-Meson Heitler-London States

The remaining two-meson exchange terms come from the two-meson H-L state and are denoted by the subscript (II). The amplitude of this state is given by (2.15); the expansions for $S_{2(\mathrm{II})}$ and $A_{2(\mathrm{II})}$ are obtained by methods already described. Discarding all terms corresponding to exchange of more than two mesons, we find

$$\sum_{klp}\frac{1}{2(K+L)}\left(\frac{1}{K+P}+\frac{1}{L+P}\right)\langle x'|\boldsymbol{\sigma}|x''kl\rangle$$
$$\times[\langle x''k|V_p|x\rangle\langle y''l|V_p{}^*|y\rangle$$
$$+\langle x''l|V_p|x\rangle\langle y''k|V_p{}^*|y\rangle]$$
$$+\sum_{klpq}\frac{1}{(K+L)}\left(\frac{1}{K+P}+\frac{1}{L+P}\right)[\langle x'|a_q{}^*\boldsymbol{\sigma}|x''k\rangle$$
$$\times\langle y'|a_q|y''l\rangle+\langle x'|\boldsymbol{\sigma}a_q|x''k\rangle\langle y'|a_q{}^*|y''l\rangle]$$
$$\times\langle x''k|V_p|x\rangle\langle y''l|V_p{}^*|y\rangle$$
$$=\rho_1{}^{-1}\sum_{klp}\frac{1}{(K+L)^2}\left(\frac{1}{K+P}+\frac{1}{L+P}\right)$$
$$\times(x'|[\boldsymbol{\sigma}T_l{}^\dagger(k)-T_l{}^\dagger(k)\boldsymbol{\sigma}]T_p(k)|x')(y'|T_p(l)|y)$$
$$+\rho_1{}^{-1}\sum_{klpq}\frac{1}{(K+L)}\left(\frac{1}{K+P}+\frac{1}{L+P}\right)$$
$$\times\left[\frac{1}{K+Q}\frac{1}{Q-L}(x'|T_q{}^\dagger(k)\boldsymbol{\sigma}T_p(k)|x)\right.$$
$$\left.+\frac{1}{L+Q}\frac{1}{Q-K}(x'|\boldsymbol{\sigma}T_q{}^\dagger(k)T_p(k)|x)\right]$$
$$\times(y'|T_q{}^\dagger(l)T_p(l)|y)+\text{c.c.} \quad (4.25)$$

for the terms arising from the cross term between the zero-meson H-L state and the two-meson state.

It can be shown that the entire last term of (4.25) is cancelled by the normalization terms. The singularity $1/(Q-L)$ in the second term can be eliminated by making a partial fraction expansion and then using the Low equation. We write

$$\frac{1}{K+L}\left(\frac{1}{K+P}+\frac{1}{L+P}\right)\frac{1}{K+Q}\frac{1}{Q-L}=\frac{1}{Q-L}Z+Y, \quad (4.26)$$

where Z is independent of L, and neither Y nor Z contain a singularity. The Low equation, including only zero- and one-meson states, is

$$T_p(q)=\frac{V_q{}^*V_p-V_pV_q{}^*}{Q}$$
$$+\sum_l\left[\frac{T_q{}^\dagger(l)T_p(l)}{Q-L}-\frac{T_p{}^\dagger(l)T_q(l)}{Q+L}\right]. \quad (4.27)$$

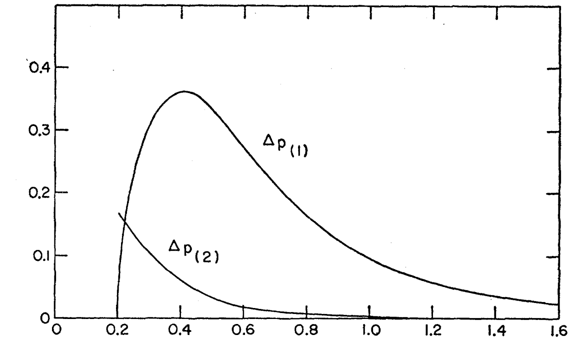

FIG. 3. One- and two-meson exchange contributions to Δp as a function of r (in units of $\lambda = 1.4\times 10^{-13}$ cm).

Using this, we obtain

$$S_{2(\mathrm{II})} = \rho_1^{-1} \sum_{kpq} Z(x'|T_q^\dagger(k)\boldsymbol{\sigma} T_p(k)|x) \times \left(y'\left|\left[\frac{V_pV_q^* - V_q^*V_p}{Q} + \sum_l \frac{T_p^\dagger(l)T_q(l)}{Q+L}\right]\right|y\right) + \rho_1^{-1}\sum_{klpq} Y(x'|T_q^\dagger(k)\boldsymbol{\sigma}T_p(k)|x) \times (y'|T_q^\dagger(l)T_p(l)|y) + \mathrm{c.c.} \quad (4.28)$$

The remaining calculations are straightforward. The energy denominators are represented approximately as products of factors, each of which contains only one energy K or L, so that the functions $F_i(r)$ and $G_i(r)$ can be used (see also B). Finally, the contributions of the two-meson H-L states are:

$$\begin{aligned} \Delta p_{\mathrm{II}} &= -(14/27)\gamma^2\alpha_3\rho_1^{-1}(2a^2+4b^2)G_1G_1 \\ &\quad -(2/243)\gamma^2\alpha_3^2(\rho_1\omega_0^2)^{-1}(38a^2+55b^2)F_1F_1, \\ \Delta q_{\mathrm{II}} &= (14/27)\gamma^2\alpha_3\rho_1^{-1}(a^2-2ab+3b^2)G_1G_1 \\ &\quad +(2/243)\gamma^2\alpha_3^2(\rho_1\omega_0^2)^{-1} \\ &\quad \times(19a^2-17ab+57b^2)F_1F_1. \end{aligned} \quad (4.34)$$

V. NUMERICAL RESULTS AND CONCLUSIONS

A. Parameters and Wave Functions

The parameters f, f_0, α_3, and ω_0 which appear in the model are chosen to fit low-energy π-meson–nucleon scattering data. The values used are[2]:

$$f^2/4\pi = \gamma = 0.08, \quad \alpha_3 = 2.6,$$
$$f_0^2/4\pi = 0.25, \quad \omega_0 = 2.1.$$

The cutoff function used is $v_k = (\lambda^2-1)/(\lambda^2+k^2)$, with $\lambda = 7$. This function is normalized so that $v=1$ when $K=0$, rather than when $k=0$. This modifies the coupling constant by an insignificant (2%) amount.

For μ_s we use the experimental value 0.8796 nm. For the average reciprocal energy $\langle E^{-1}\rangle$ appearing in the one-meson exchange calculations, we assume that the most important contribution to the closure expansion is from the one-meson terms, so $\langle E^{-1}\rangle$ is approximately the average reciprocal energy of the one-meson states. Since the excited states of the nucleon which enter are $T=\frac{1}{2}$ states, for which there is no strong resonant interaction at low energies, this average energy is estimated by calculating the average reciprocal energy of a virtual meson in the physical nucleon state, which is given by[2]

$$\langle K^{-1}\rangle = \frac{\sum_k \langle x|K^{-1}a_k^*a_k|x\rangle}{\sum_k \langle x|a_k^*a_k|x\rangle} = \frac{1}{6.1}.$$

Therefore we take $\langle E^{-1}\rangle = 1/6.1$.

This procedure is equivalent to calculating the average energy by perturbation theory. In fact, the integral which appears in the zero-meson approximation for $\sum_k \langle x|K^{-1}a_k^*a_k|x\rangle$ is proportional to the quantity $[bH(r)]_{r=0}$, Eq. (3.25), which appears in the alternate method for the evaluation of the one-meson exchange terms.

The coefficients of the one-meson terms, Eqs. (3.19) and (3.23), have remarkably similar numerical values:

$$6\langle E^{-1}\rangle(ff_0-f^2)/4\pi = 0.059,$$
$$12(f^2/4\pi)^2[bH(r)]_{r=0} = 0.055.$$

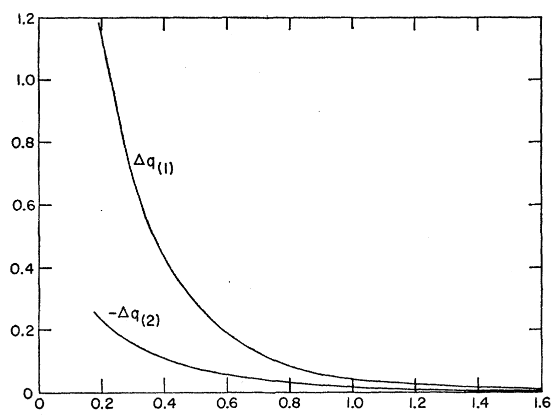

FIG. 4. One- and two-meson exchange contributions to Δq as a function of r (in units of $\lambda = 1.4\times 10^{-13}$ cm). $\Delta q_{(2)}$ is negative.

The functions $F_i(r)$ and $G_i(r)$, Eqs. (3.18), (4.4), (4.15), (4.20), and their derivatives were obtained by numerical integration.[19] The one- and two-meson contributions to Δp, not including those from the two-meson H-L state, are shown in Fig. 3. One- and two-meson contributions to Δq, not including the two-meson H-L state, are shown in Fig. 4. The contributions to Δp and Δq from the two-meson H-L state are shown separately in Fig. 5. It should be noted that the two-meson contributions are much smaller than the one-meson contributions except at very small distances. In

[19] The numerical values used are the same as those used in B; the calculations were made by S. H. Vosko and R. E. Cutkosky on an IBM-650 computer.

Figs. 3–5, ρ_1 has been taken to be 1. Therefore the quantities plotted are multiplied by $\alpha/\rho_1=\mu_s$ in calculating $\Delta\mu$.

The radial wave functions $u(r)$ and $w(r)$ are given, in principle, by solution of the Schrödinger equation. Since, however, the potentials are not known sufficiently well to obtain ψ by this method, we use a phenomenological approach.

We represent the radial functions by

$$u(r)=\cos\alpha' v(r), \quad w(r)=\sin\alpha' v(r), \tag{5.1}$$

in which α' is a function of r. The usefulness of such a representation is suggested by the "radially adiabatic eigenstates" discussed in B. In the outer region we determine α' from a phenomenological wave function calculated by Iwadare *et al.*[3]; this procedure guarantees that the function will give approximately the proper quadrupole moment. In the inner region we choose values of α' consistent with meson-theoretical calcu-

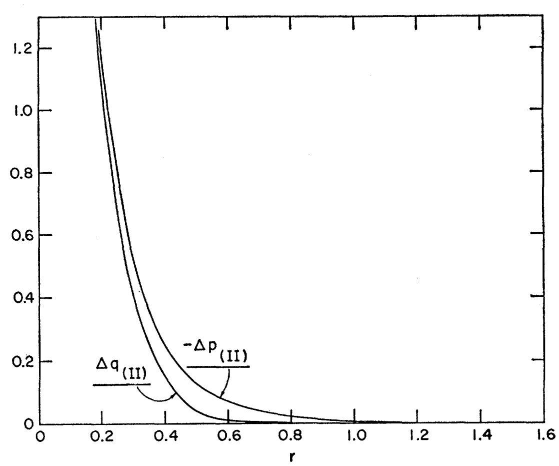

FIG. 5. Contributions to Δp and Δq from two-mesons H-L state as a function of r (in units of $\lambda\!\!\!^{-}=1.4\times10^{-13}$ cm). $\Delta p_{(\mathrm{II})}$ is negative.

lations of the tensor potential, as discussed in B, but giving various D-state probabilities. Values of α' used are shown in Fig. 6.

The function $v(r)$ used has an inner part corresponding to a potential with a repulsive core.[1,3] We used a modified Hulthén function,[20]

$$v(r)=N\{\exp[-a(r-r_c)]-\exp[-b(r-r_c)]\} \quad \text{for } r\geqslant r_c;$$
$$=0 \quad \text{for } r<r_c,$$

with $r_c=0.2$, $a=0.323$, $b=2.26$. The value of b was chosen to fit approximately the triplet effective range.

B. Numerical Results

The expression (2.11) has been evaluated numerically, using the wave functions and values of Δp and Δq described above. Contributions from the two-meson

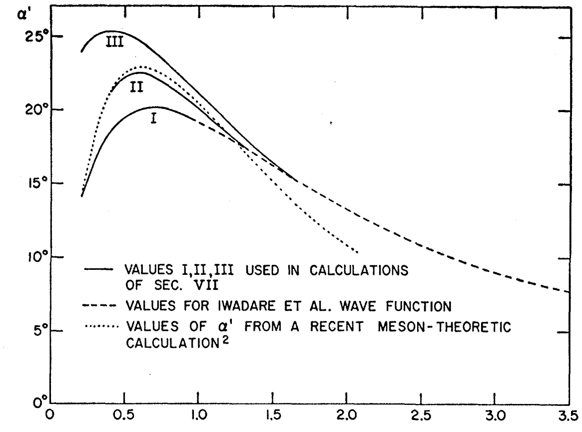

FIG. 6. Values of α' (in degrees) as a function of r (in units of $\lambda\!\!\!^{-}=1.4\times10^{-13}$ cm).

H-L states are not included, for reasons to be discussed. The results for $\Delta\mu'$, the additional correction calculated in this paper, are summarized in Table I, along with the corresponding values of p_D and $\Delta\mu_0=-\frac{3}{2}(\mu_s-\frac{1}{2})p_D$. We also show the result obtained when one uses the same radial function $v(r)$, but with $\alpha'=0$ everywhere, i.e., a pure S-state. The sign of $\Delta\mu'$ is opposite in this case to that for the other cases. The totals in the last column should be compared with the experimental value $\mu_d-\mu_s=-0.0222$.[21]

The effect of the contributions from the two-meson H-L states has been calculated separately for wave function II; the result is

$$\Delta\mu'=-0.0005 \text{ nm},$$

which is considerably smaller in magnitude than either the one-meson or the other two-meson contributions.

These results (except those from the two-meson H-L state) are not excessively sensitive to the choice of core radius in the radial wave function.

C. Conclusions

In interpreting the results of this calculation, it should be pointed out that it is not intended to be a complete and definitive calculation of all significant effects which

TABLE I. Contributions to deuteron magnetic moment.

	p_D (%)	$\Delta\mu'$ (nm)	$\Delta\mu_0$ (nm)	$\Delta\mu'+\Delta\mu_0$ (nm)
I	5.9	0.0074	−0.0330	−0.0256
II	6.4	0.0091	−0.0355	−0.0264
III	6.9	0.0112	−0.0380	−0.0268
IV	0	−0.0083	0	−0.0083

[20] Numerical calculations of this function were made by H. N. Pendleton.

[21] These results may be compared with those of reference 13, in which the $\Delta\mu'$ effect was also examined (with a different method). In contrast to the values shown in Table I, reference 13 gives $\Delta\mu'\sim+0.01$ for a pure S state, and $\Delta\mu'\sim+0.002$ for a potential having a dominant tensor force of the one-meson exchange type.

contribute to the magnetic moment of the deuteron. It is, rather, an application and extension of a simple model which has had some success in correlating in a semiquantitative way low-energy phenomena involving π mesons and nucleons. Therefore one should not expect exact agreement between the result of this calculation and the experimental value of $\mu_s-\mu_d$.

The calculation does show that the contribution to μ_d resulting from $\Delta\langle S_0\rangle$ is positive, opposite in sign to the elementary D-state contribution, and is relatively large, of the order of 1% of μ_s. Because both of these corrections increase in magnitude with increasing p_D, their sum is less sensitive to p_D than one might expect. The results are consistent (considering the approximate nature of the model) with values of p_D of 6%, and show that the observed value of the deuteron magnetic moment permits considerable freedom in the choice of p_D, much more than would be thought possible if this $\Delta\langle S_0\rangle$ effect were not included.[6,22]

The limitations on the meaning of our results are largely the limitations of the model used. Some of these are the omission of all relativistic-kinematic effects, the use of the adiabatic approximation, the use of a rigid core, the neglect of nucleon recoil, and the omission of meson-meson interactions and nucleon-pair effects.

The uncertainty of the relativistic effects was mentioned in the introduction. We further remark only that these effects are one of the reasons that one should not expect exact agreement between this calculation and the experimental value of μ_s, and that they introduce further uncertainty into the value of the D-state probability of the wave function as determined from μ_d.

The validity of the adiabatic approximation is difficult to assess. It depends on the relative frequencies of the meson field and the orbital motion of the nucleon cores, which in turn depend on the energies involved. In the region where the nucleons interact strongly, their kinetic energy is about equal to the meson rest energy, and the average energy of the virtual mesons is not very much larger than this.

It seems likely that the adiabatic approximation will affect the amplitude of the two-meson H-L state more than that of the one-meson state. The two-meson state makes its entire contribution in the region of strong interaction and correspondingly large nucleon kinetic energy. Also, the average meson energy is smaller; it is of the order of twice the resonance energy (or about 4.2) for the two-meson state, compared to about 6.1 for the one-meson states. For these reasons, it is felt that the contribution of the two-meson H-L state is too unreliable to be included in the results. If it *were* added, it would not change the results significantly.

The other limitations of the model mentioned above are all related. Two phenomenological features of the model, the use of an anomalous magnetic moment for the nucleon core, and the use of wave functions corresponding to a "hard-core" potential, are intended to compensate partly for these limitations.

By using an extended nucleon core with an anomalous magnetic moment one hopes to include in a phenomenological way some of the additional currents associated with nucleon recoil, nucleon pairs, K-mesons and hyperons, and so on. This is a valid procedure as long as the core of one nucleon is not distorted by the presence of another. At small distances (say $r\sim0.3$) this is certainly not the case. The resonances which have been observed recently[23,24] in π-meson-nucleon photoproduction and scattering in $T=\frac{1}{2}$ states may be associated with the internal structure of the nucleon cores.

The strong repulsion between nucleons at small distances is assumed to be due at least in part to these same effects. The effect of this on the orbital wave function of the deuteron is partly included through the use of a function which goes to zero at a certain "core radius."

The fact that in this calculation the principal effect comes from the one-meson exchange contributions is encouraging. This result suggests that it may be possible to make a more definitive calculation using just the one-meson exchange terms but with a better model for the $T=\frac{1}{2}$ excited states of the nucleons than is now available.

[22] M. Sugawara, Phys. Rev. **99**, 1601 (1955).

[23] Burrowes, Caldwell, Frisch, Hill, Ritson, Schluter, and Wahlig, Phys. Rev. Letters **2**, 119 (1959).

[24] Crittenden, Scandrett, Shephard, Walker, and Ballam, Phys. Rev. Letters **2**, 121 (1959).