Two-Nucleon Potential from Pion Field Theory with Pseudoscalar Coupling*

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The two-nucleon potential is derived from *ps-ps* pion field theory, using the Tamm-Dancoff method with the subsequent appropriate renormalization of the two-nucleon amplitude, up to orders $g^2(p/\kappa)^2$ and $g^4(p/\kappa)$, g^2 being the equivalent *ps-pv* coupling constant and p, κ the nucleon (relative) momentum and rest mass, respectively. Neglected are the mass and charge renormalizations and the pion-pion scattering term. It is shown that the only quadratic term is $-V_2(\mathbf{r})(p^2/2\kappa^2)+\text{H.c.}$, where $V_2(\mathbf{r})$ is the second order static potential. All remaining terms of order $g^2(p/\kappa)^2$ are converted by a canonical transformation to terms of order $g^4(p/\kappa)$ or a linear combination of a static and an $\mathbf{L}\cdot\mathbf{S}$ potential. In the case of *ps-ps* coupling, in particular, no velocity-dependent potential besides the quadratic one mentioned above follows from diagrams of one- and two-pion exchange without nucleon pairs; thus the nucleon-pair diagrams are the only source of an $\mathbf{L}\cdot\mathbf{S}$ potential, if there is any, up to the orders considered. The nucleon-pair diagrams are also estimated assuming the effective pion-pair interaction Hamiltonian, the static limit of which agrees with the *S*-wave static model of Drell *et al.* The $\mathbf{L}\cdot\mathbf{S}$ potential thus obtained has the right sign in the odd state and changes its sign in the even state, while its magnitude seems in both states somewhat too small. As for the static part of our potential, the new correction $[g^4(\mu/\kappa)]$ cancels out the conventional fourth-order term $[g^4]$ appreciably; the entire static potential becomes quite close to the second-order static potential down to distances of the order of the pion Compton wavelength, except for the central force in the triplet even state. The details are shown on graphs. The *ps-pv* coupling case is treated separately in the following paper.

1. INTRODUCTION

MANY calculations have been done to derive the $\mathbf{L}\cdot\mathbf{S}$ potential from pion field theory.[1] They seem to claim that some $\mathbf{L}\cdot\mathbf{S}$ potential follows if we retain velocity-dependent terms from diagrams of at least the fourth-order.

We here present a similar report. The present work, is, however, the most systematic and exhaustive among all done thus far, though conservative in the basic spirit of deriving the potential.

Our work is conservative in the sense that we assume the adequacy of expanding the two-nucleon potential in powers of both the equivalent *ps-pv* coupling constant g^2 and the nucleon (relative) momentum p over the nucleon rest mass κ. We also simply drop the mass and charge renormalizations and use the renormalized coupling constant and mass throughout the paper. We neglect also the pion-pion scattering term. Insofar as we assume all this, the Tamm-Dancoff method is convenient for deriving the potential (Sec. 2). To get the correct potential we perform wave function renormalization, which takes due account of transition from the Tamm-Dancoff two-nucleon amplitude to the nonrelativistic Schrödinger wave function (Sec. 3).

Our work is, however, systematic and exhaustive since we retain all terms up to $g^2(p/\kappa)^2$ and $g^4(p/\kappa)$. We remark first that we get the second-order static potential if we keep only terms involving g^2, neglecting all nucleon recoils. If we go one step further as regards both g^2 and p/κ, then terms involving g^4 and $g^2(p/\kappa)$ together give the fourth-order static potential; the correction of $g^2(p/\kappa)$ is shown to be equivalent to a g^4-term.[2] It is, therefore, essential to retain terms involving both $g^4(p/\kappa)$ and $g^2(p/\kappa)^2$ to discuss the lowest order velocity-dependent potential. This is exactly what we have done in this paper.

Such a work was done by Okubo and Marshak.[1] Here we are presenting an almost complete answer to the problem outlined above, achieved mainly through the following three improvements. First we here evaluate not only the velocity-dependent term, but also the static potential of orders $g^4(p/\kappa)$ and $g^2(p/\kappa)^2$. This is important because we thus can claim the consistency of our basic assumption [the expansion with respect to g^2 and $(p/\kappa)]^2$ and eventually the reliability of the velocity-dependent potential obtained.

As another improvement, we make use of a canonical transformation (Sec. 5) which remains arbitrary when we define the wave function renormalization. We can show that the only essentially quadratic term is $-V_2(\mathbf{r})\times(p^2/2\kappa^2)+\text{H.c.}$, $V_2(\mathbf{r})$ being the second-order static potential, and all remaining terms involving $g^2(p/\kappa)^2$ are converted, after a canonical transformation is applied, into terms of order $g^4(p/\kappa)$, or a linear combination of a static potential and an L-S potential (Secs. 4 and 5).

We also carefully discriminate between the *ps-ps* and *ps-pv* theories since we found interesting differences be-

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[1] S. Okubo and R. E. Marshak, Ann. Phys. 4, 166 (1958); Tzoar, Raphael, and Klein, Phys. Rev. Letters 2, 433 (1959). Previous works are cited in these articles.

[2] The fourth-order static potential is, therefore, the Taketani-Machida-Onuma potential and not the Brueckner-Watson potential. This is correct as far as we assume the expansion with respect to g^2 and (p/κ). The consistency of this expansion has been criticized by Fukuda, Sawada, and Taketani, Progr. Theoret. Phys. (Kyoto) 12, 156 (1954) and A. Klein, Progr. Theoret. Phys. (Kyoto) 20, 257 (1958). According to the present work, the new correction is shown to cancel the fourth-order potential significantly (see the details given in the final section), suggesting that the expansion is at least better than has long been suspected.

tween predictions of these theories: a large difference in the $\mathbf{L}\cdot\mathbf{S}$ potential, a very small difference in the static potential, and no difference in the quadratic term. (Details are shown in the following paper.[3]) This great difference in the $\mathbf{L}\cdot\mathbf{S}$ potential is due to the fact that no velocity-dependent potential, except for the quadratic one just mentioned, follows from diagrams of one- and two-pion exchange without nucleon pairs in case of *ps-ps* theory (Secs. 5 and 6) while the corresponding diagrams give a large $\mathbf{L}\cdot\mathbf{S}$ potential in the other theory.[3]

Of course, there is considerable question whether we can estimate the contributions from the nucleon pairs in *ps-ps* theory, to which we can hardly apply the method outlined above. We here assume the effective pion-pair interaction Hamiltonian, the static limit of which agrees with the S-wave static model of Drell *et al.*,[4] and then estimate the diagrams including the pion-pair interaction once and twice (Sec. 7). This method, seemingly the only reasonable one though hardly justifiable at the moment, gives an $\mathbf{L}\cdot\mathbf{S}$ potential from the one-pair diagram. This is shown to have the right sign and a strong isospin dependence, though its magnitude seems to be too small. (Detailed comparisons among various $\mathbf{L}\cdot\mathbf{S}$ potentials are given in the following paper.[3])

In the final section the main conclusions are summarized. Our entire potential is shown there graphically.

2. TAMM-DANCOFF AND NONRELATIVISTIC APPROXIMATIONS

The derivation of the two-nucleon potential essentially consists of reducing the relativistic field-theoretical Schrödinger wave equation,

$$(H_0+H')|\Psi\rangle=E|\Psi\rangle, \tag{1}$$

to the nonrelativistic two-body Schrödinger wave equation

$$[(p^2/\kappa)+V(\mathbf{r},\mathbf{p})]\psi(\mathbf{r})=W\psi(\mathbf{r}), \tag{2}$$

where $E=W+2\kappa$.

We do this reduction in the following steps. The first is the customary Tamm-Dancoff expansion:

$$|\Psi\rangle=\int\int c(\mathbf{p}_1\mathbf{p}_2)|\mathbf{p}_1\mathbf{p}_2\rangle d\mathbf{p}_1d\mathbf{p}_2 +\int\int\int c(\mathbf{p}_1\mathbf{p}_2\mathbf{k}_\alpha)|\mathbf{p}_1\mathbf{p}_2\mathbf{k}_\alpha\rangle d\mathbf{p}_1d\mathbf{p}_2d\mathbf{k}_\alpha+\cdots, \tag{3}$$

with subsequent elimination of amplitudes including pions, such as $c(\mathbf{p}_1\mathbf{p}_2\mathbf{k}_\alpha)$, etc. Here we drop all mass and charge renormalization terms. If we retain up to the two-pion exchange terms, we get

$$(2E_\mathbf{p}-E)c(\mathbf{p})+\int U_2(\mathbf{p},\mathbf{k},E)c(\mathbf{p}+\mathbf{k})d\mathbf{k} +\int U_4(\mathbf{p},\mathbf{k},\mathbf{k}',E)c(\mathbf{p}+\mathbf{k}+\mathbf{k}')d\mathbf{k}d\mathbf{k}'=0, \tag{4}$$

where $c(\mathbf{p})$ is $c(\mathbf{p}_1\mathbf{p}_2)$ in the c.m. system, $E_\mathbf{p}$ the relativistic nucleon energy for momentum $\mathbf{p}$, and U_2, U_4 contributions from, respectively, one- and two-pion exchange. Specifically U_2 is given by

$$U_2(\mathbf{p},\mathbf{k},E)=\frac{f^2}{(2\pi)^3}\sum_\alpha\frac{(\bar{u}_\mathbf{p}\gamma_5\tau_\alpha u_{\mathbf{p}+\mathbf{k}})(\bar{u}_{-\mathbf{p}}\gamma_5\tau_\alpha u_{-\mathbf{p}-\mathbf{k}})}{\omega(E_\mathbf{p}+E_{\mathbf{p}+\mathbf{k}}+\omega-E)}, \tag{5}$$

where $u_\mathbf{p}$ is the free positive-energy Dirac spinor and ω the relativistic pion energy for momentum $\mathbf{k}$.

We then make the nonrelativistic approximation, $E_\mathbf{p}=\kappa+p^2/2\kappa$, everywhere in (4) and (5):

$$(\bar{u}_\mathbf{p}\gamma_5\tau_\alpha u_{\mathbf{p}+\mathbf{k}})=-\tau_\alpha\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{2\kappa}-\frac{\boldsymbol{\sigma}\cdot\mathbf{k}[\mathbf{p}^2+(\mathbf{p}+\mathbf{k})^2]}{8\kappa^3} +\frac{\boldsymbol{\sigma}\cdot(2\mathbf{p}+\mathbf{k})[\mathbf{p}^2-(\mathbf{p}+\mathbf{k})^2]}{16\kappa^3}\right], \tag{6}$$

$$\frac{1}{E_\mathbf{p}+E_{\mathbf{p}+\mathbf{k}}+\omega-E}=\frac{1}{\omega}\left[1-\frac{\mathbf{p}^2+(\mathbf{p}+\mathbf{k})^2-2\kappa W}{2\kappa\omega} +\left(\frac{\mathbf{p}^2+(\mathbf{p}+\mathbf{k})^2-2\kappa W}{2\kappa\omega}\right)^2\right], \tag{7}$$

$$2E_\mathbf{p}-E=(p^2/\kappa)-W. \tag{8}$$

These equations are correct to order κ^{-2} and are sufficient to derive the potential to orders $g^2(p/\kappa)^2$ and $g^4(p/\kappa)$.

The remaining two steps are the transition from $c(\mathbf{p})$ to $\psi(\mathbf{r})$ in (2) and the elimination of E or W from U_2 and U_4. These are explained in the following sections separately.

3. WAVE-FUNCTION RENORMALIZATION

Since the Schrödinger wave function $\psi(\mathbf{r})$ in (2) is supposed to be a substitute for Ψ in (1), it is most appropriate to define $\psi(\mathbf{r})$ such that

$$\int|\psi(\mathbf{r})|^2d\mathbf{r} =\langle\Psi|\Psi\rangle=\int\int|c(\mathbf{p}_1\mathbf{p}_2)|^2d\mathbf{p}_1d\mathbf{p}_2 +\int\int\int|c(\mathbf{p}_1\mathbf{p}_2\mathbf{k}_\alpha)|^2d\mathbf{p}_1d\mathbf{p}_2d\mathbf{k}_\alpha+\cdots. \tag{9}$$

[3] M. Sugawara and S. Okubo, Phys. Rev., following paper [Phys. Rev. **117**, 611 (1959)].
[4] Drell, Friedman, and Zachariasen, Phys. Rev. **104**, 236 (1956).

Upon eliminating $c(\mathbf{p}_1\mathbf{p}_2\mathbf{k}_\alpha)$, we can show that to orders $g^2(p/\kappa)$ and g^4,

$$\langle\Psi|\Psi\rangle=\int\phi^*(\mathbf{r})[1-v_2+v_4]\phi(\mathbf{r})d\mathbf{r}, \tag{10}$$

where $\phi(\mathbf{r})$ is the coordinate-space transform of $c(\mathbf{p})$ and

$$v_2=\int(1/\omega)U_2(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}. \tag{11}$$

Here

$$V_2(\mathbf{r})=\int U_2(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k} \tag{12}$$

is the second-order static potential. v_4 is a certain Hermitian term involving g^4 the details of which do not have to be specified for the following arguments. The term of order $g^2(p/\kappa)$ on the right-hand side of (10) disappears under the integral.

We now define the wave-function renormalization:

$$\varphi(\mathbf{r})=[1-(v_2/2)+(v_2{}^2/8)+v_4']\phi(\mathbf{r}), \tag{13}$$

where $v_4'=(v_4/2)-(v_2/2)^2$. The operator in (13) is just the square root of the operator occurring under the integral in (10). This transforms

$$[(p^2/\kappa)-W+V_2+V_4]\phi(\mathbf{r})=0 \tag{14}$$

into

$$\begin{aligned}\{(p^2/\kappa)-W+V_2+[(p^2/2\kappa),v_2]_-+V_4&\\+[[(p^2/8\kappa),v_2]_-,v_2]_-&\\+(\tfrac{1}{2})[V_2,v_2]_--[(p^2/\kappa),v_4']_-\}\varphi(\mathbf{r})&=0,\end{aligned} \tag{15}$$

where $[A,B]_-\equiv AB-BA$. It is now clear that terms up to $g^2(p/\kappa)$ and g^4 need be retained in (10) to derive the potential up to $g^2(p/\kappa)^2$ and $g^4(p/\kappa)$.

Two comments are added. Firstly, our defining equation (9) does not uniquely specify the wave function renormalization; (13) may be multiplied by an arbitrary unitary transformation. This canonical transformation is exploited in the next section to simplify the velocity dependence of the potential.

Secondly, our wave function renormalization (13) induces imaginary terms in (15), since the transformation is not unitary. However, V_2 and V_4 in (14) already include imaginary terms, which come in upon eliminating E (or W) from U_2 and U_4 in (4). In the next section we show that all imaginary terms proportional to g^2 are cancelled out exactly after the transformation (13) is done. We can prove generally that the Hermiticity of the potential is secured independent of the perturbation expansion as far as we define $\psi(\mathbf{r})$ by (9).[2,5]

4. SECOND-ORDER POTENTIAL

We remark that $U_2(\mathbf{p},\mathbf{k},E)$ (5) includes, after the nonrelativistic approximations (6) and (7) are made, two characteristic factors, $\mathbf{p}^2+(\mathbf{p}+\mathbf{k})^2$ and $\mathbf{p}^2-(\mathbf{p}+\mathbf{k})^2$.

[5] S. Okubo, Progr. Theoret. Phys. (Kyoto) 12, 603 (1954).

We can show the following useful relations:

$$\begin{aligned}\int[\mathbf{p}^2+(\mathbf{p}+\mathbf{k})^2]f(\mathbf{p},\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}&=\{p^2,g(\mathbf{p},\mathbf{r})\}_+,\\ \int[\mathbf{p}^2-(\mathbf{p}+\mathbf{k})^2]f(\mathbf{p},\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}&=[p^2,g(\mathbf{p},\mathbf{r})]_-,\end{aligned} \tag{16}$$

which are correct for an arbitrary $f(\mathbf{p},\mathbf{k})$ [$g(\mathbf{p},\mathbf{r})$ being its coordinate transform]. In (16), $\{A,B\}_+\equiv AB+BA$, $[A,B]_-\equiv AB-BA$, and $\mathbf{p}\equiv-i\nabla$ is the momentum operator. It is, therefore, seen that W occurs in the potential only in the combination $(p^2/\kappa)-W$ and also only in the anticommutator form.

It is now straightforward to eliminate W through the customary iteration procedure[6] using the identities

$$\begin{aligned}&\{(p^2/\kappa)-W,A\}_+=[p^2/\kappa,A]_-+2A((p^2/\kappa)-W),\\&\{(p^2/\kappa)-W,\{(p^2/\kappa)-W,B\}_+\}_+\\&\quad=[p^2/\kappa,[p^2/\kappa,B]_-]_-\\&\qquad+4[p^2/\kappa,B]_-((p^2/\kappa)-W)+4B((p^2/\kappa)-W)^2.\end{aligned} \tag{17}$$

This elimination gives many imaginary terms. However, if we do the wave-function renormalization (13), all the imaginary terms proportional to g^2 are exactly cancelled out. The potential in the equation satisfied by $\varphi(\mathbf{r})$[15] becomes, if we retain terms up to $g^2(p/\kappa)^2$ and $g^4(p/\kappa)$ consistently,

$$\begin{aligned}V(\mathbf{r},\mathbf{p})=V_2(\mathbf{r})&-\{(p^2/2\kappa^2),V_2(\mathbf{r})\}_+\\&+i[(p^2/8\kappa^2),V_2(\mathbf{r})(\mathbf{r}\cdot\mathbf{p})+\text{H.c.}]_-\\&+v_2V_2(\mathbf{r})+[[(p^2/8\kappa),v_2]_-v_2]_-\\&\qquad+V_4-[(p^2/\kappa),v_4']_-.\end{aligned} \tag{18}$$

The second term comes from the second term in (6). The third term is due to the last terms in both (6) and (7). It is readily seen from (16) and (17) that this term is of the commutator form. All remaining terms are of fourth-order, the first of which originates from the second term in (7) and the well-known difference between the potentials of Taketani-Machida-Onuma and of Brueckner-Watson.[2] Other terms are the same as those in (15).

5. CANONICAL TRANSFORMATION

The third term in (18) consists, if the commutator is actually evaluated, of many velocity-dependent terms quadratic with respect to $\mathbf{p}$. Some of these are explicitly

[6] We do this iterational elimination of W in coordinate space, rather than in momentum space, even though we can show they are equivalent. The reason for this deliberate procedure is to show explicitly that our elimination method of W does not mix the unphysical singularities near the origin exhibited by $V_2(\mathbf{r})$. We, therefore, do not agree with Brueckner and Watson's argument on this point in Phys. Rev. 92, 1023 (1953). The present work shows that the convergence of this procedure is at least better than suspected before (see the reference 2).

given in the paper by Okubo and Marshak.[1] The trouble is that they are too complicated. We here propose a completely different approach to these quadratic terms.

As we have remarked in Sec. 3, an arbitrary unitary transformation is left undetermined when we define the wave function renormalization, (13). We, therefore, define $\psi(\mathbf{r})$ finally as

$$\psi(\mathbf{r})=[1+iS_2-\tfrac{1}{2}S_2S_2+iS_4+\cdots]\varphi(\mathbf{r}), \qquad (19)$$

where S_2, S_4 are Hermitian and of orders g^2 and g^4, respectively, and we require that they be of short range so that $\psi(\mathbf{r})$ and $\varphi(\mathbf{r})$ agree asymptotically.

The canonical transformation of this type causes a term in the potential which is of the commutator form with p^2 to transform into a term of the higher order in g^2 but of the lower order in (p/κ). Thus the only essentially quadratic term is the second term in (18).

If we choose S_2 as $(1/8\kappa)[V_2(\mathbf{r})(\mathbf{r}\cdot\mathbf{p})+\text{H.c.}]$, the third term in (18) is totally converted into a fourth-order term,

$$-i[V_2(\mathbf{r}),S_2]_-=(1/8\kappa)(\mathbf{r}\cdot\nabla)V_2^2(\mathbf{r}), \qquad (21)$$

which is purely static. Thus the one-pion exchange diagram does not produce any velocity-dependence other than the second term in (18), up to $g^4(p/\kappa)$ and $g^2(p/\kappa)^2$. This is one of the characteristics of *ps-ps* theory.[3]

6. FOURTH-ORDER POTENTIAL (NO-PAIR TERMS)

The correction of $g^4(p/\kappa)$ comes only from the energy denominator expansion. It is again very convenient to utilize relations analogous to (16). Because of these we can drop from $U_4(\mathbf{p},\mathbf{k},\mathbf{k}',E)$ in (4) [or its nonrelativistic approximation] those terms which include either $[\mathbf{p}^2+(\mathbf{p}+\mathbf{k}+\mathbf{k}')^2-2\kappa W]$ or $[\mathbf{p}^2-(\mathbf{p}+\mathbf{k}+\mathbf{k}')^2]$: Since such terms eventually reduce to commutators between p^2 and some functions except for higher-order terms, they are dropped through the canonical transformation (19), if they are Hermitian, or are cancelled by the last term in (18) due to the wave-function renormalization if they are imaginary.[5]

Using these lemmas, we can show that the entire correction of order $g^4(p/\kappa)$ from pion-crossing diagrams including no nucleon pairs is purely static, and the entire relevant correction from pion-uncrossing diagrams without nucleon pairs becomes a velocity-dependent term which exactly cancels out the fifth term in (18), which is due to wave-function renormalization (13). We thus reach the conclusion that no $\mathbf{L}\cdot\mathbf{S}$ potential results from diagrams including no nucleon pairs, in case of *ps-ps* theory, up to $g^4(p/\kappa)$ and $g^2(p/\kappa)^2$; the only velocity-dependent term is $-V_2(\mathbf{r})(p^2/2\kappa^2)+\text{H.c.}$

We finally give our entire potential:

$$\begin{aligned}
&V_{ps\text{-}ps}(\text{no-pair})\\
&=V_2(\mathbf{r})-\{(p^2/2\kappa^2),V_2(\mathbf{r})\}_++V_4(\mathbf{r})\\
&+\frac{g^4\mu^2}{(4\pi)^2\kappa}\Bigg[\left\{\frac{3}{4x}+\frac{3}{4x^2}+\frac{6}{x^3}+\frac{21}{x^4}+\frac{27}{x^5}+\frac{27}{2x^6}\right\}\\
&+(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\left\{\frac{3}{2x}+\frac{11}{2x^2}+\frac{20}{x^3}+\frac{46}{x^4}+\frac{54}{x^5}+\frac{27}{x^6}\right\}\\
&-\frac{9+2(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)}{4}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\left\{\frac{4}{3x^2}+\frac{16}{3x^3}+\frac{32}{3x^4}+\frac{12}{x^5}+\frac{6}{x^6}\right\}\\
&+\frac{9+2(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)}{4}S_{12}\left\{\frac{2}{3x^2}+\frac{11}{3x^3}+\frac{28}{3x^4}+\frac{12}{x^5}+\frac{6}{x^6}\right\}\Bigg]e^{-2x}, \qquad (22)
\end{aligned}$$

where $g\equiv(\mu/2\kappa)f$ is the equivalent *ps-pv* coupling constant, $x\equiv\mu r$, and $V_4(\mathbf{r})$ is the Taketani-Machida-Onuma potential.[2] The final term of (22) is the sum of (21) and a term involving $g^4(\mu/\kappa)$ resulting from the pion crossing diagrams. These are found to be of comparable magnitude.

7. FOURTH-ORDER POTENTIAL (PAIR TERMS)

Since we do not know any other reasonable methods to estimate diagrams including nucleon pairs, we assume here the effective interaction Hamiltonian

$$H'=\frac{\lambda_1}{\mu}\bar{\psi}\psi\boldsymbol{\phi}^2+i\frac{\lambda_2}{\mu^2}\bar{\psi}\gamma_\mu\boldsymbol{\tau}\psi\boldsymbol{\phi}\times\frac{\partial\boldsymbol{\phi}}{\partial x_\mu}, \qquad (23)$$

where $\partial\boldsymbol{\phi}/\partial t$ is the canonical conjugate to $\boldsymbol{\phi}$. This is just the relativistic generalization of the S-wave static model proposed by Drell *et al.*[4] We, therefore, assume that a choice

$$\lambda_1\approx\lambda_2\approx0.4 \qquad (24)$$

is adequate. We then evaluate the two-pion exchange diagrams which include (23) once and twice; these are called the one-pair term and the two-pair term, respectively.

The two-pair term is shown to be purely static up to (p/κ). The result is

$$\begin{aligned}
&V_{ps\text{-}ps}(\text{two-pair})\\
&=-\frac{\lambda_1^2\mu}{(4\pi)^2}\frac{6}{\pi}\frac{K_1(2x)}{x^2}\\
&\qquad+\frac{\lambda_2^2\mu}{(4\pi)^2}(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\frac{2}{\pi}\left[\frac{K_0(2x)}{x^3}+\frac{K_1(2x)}{x^4}\right]. \qquad (25)
\end{aligned}$$

The one-pair term is shown to include an $\mathbf{L}\cdot\mathbf{S}$ potential:

The result is

$V_{ps\text{-}ps}$(one-pair)

$$=\frac{\lambda_1 g^2\mu}{(4\pi)^2}6\left(\frac{1}{x}+\frac{1}{x^2}\right)^2 e^{-2x}+\frac{\lambda_2 g^2\mu}{(4\pi)^2}\frac{8(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)}{\pi}$$
$$\times\left[\frac{5}{2x^3}K_0(2x)+\left(\frac{1}{x^2}+\frac{5}{2x^4}\right)K_1(2x)\right]-\frac{\lambda_1 g^2\mu^2}{(4\pi)^2\kappa}$$
$$\times\frac{12}{\pi}\left[\left(\frac{1}{x}+\frac{23}{4x^4}\right)K_0(2x)+\left(\frac{3}{x^2}+\frac{23}{4x^4}\right)K_1(2x)\right]$$
$$-\frac{\lambda_2 g^2\mu^2}{(4\pi)^2\kappa}2(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\left[\left(\frac{1}{x^2}+\frac{4}{x^3}+\frac{10}{x^4}+\frac{12}{x^5}+\frac{6}{x^6}\right)\right.$$
$$-(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\left(\frac{4}{3x^3}+\frac{10}{3x^4}+\frac{4}{x^5}+\frac{2}{x^6}\right)$$
$$\left.+S_{12}\left(\frac{2}{3x^3}+\frac{8}{3x^4}+\frac{4}{x^5}+\frac{2}{x^6}\right)\right]e^{-2x}$$
$$-\left[\frac{\lambda_1 g^2\mu^2}{(4\pi)^2\kappa}\frac{12}{\pi}\left(\frac{2}{x^3}K_0(2x)+\frac{3}{x^4}K_1(2x)\right)\right.$$
$$\left.+\frac{\lambda_2 g^2\mu^2}{(4\pi)^2\kappa}8(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\left(\frac{1}{x^2}+\frac{1}{x^3}\right)^2 e^{-2x}\right]\mathbf{L}\cdot\mathbf{S}. \quad (26)$$

To get (26), we have used the *ps-ps* vertices twice and the pion-pair term (23) once. It might be conjectured that we should better use the *ps-pv* vertices to have a better estimate of the one-pair term. If we do so, the first term of the $\mathbf{L}\cdot\mathbf{S}$ potential [proportional to λ_1] is dropped, and a very minor correction has to be made in the third term of (26) [with λ_1 again].[7] However, those corrections with λ_1 are shown, as far as (24) is adequate, to be relatively small in magnitude. Thus the main feature is well represented by (26).

8. QUANTITATIVE DISCUSSIONS AND CONCLUSIONS

Our entire potential [(22), (25), and (26)] is plotted in Figs. 1, 2, and 3, assuming $g^2/4\pi=0.08$ and (24), in units of the pion rest mass against $x=\mu r$, $1/\mu=1.414\times10^{-13}$ cm being the pion Compton wavelength. Curve 1 is the second-order static potential, $V_2(\mathbf{r})$ in (22), and curve 2 is the sum of the second- and fourth-order[2] static potentials, $V_2(\mathbf{r})+V_4(\mathbf{r})$ in (22). Curve 3 shows $V_{ps\text{-}ps}$(no-pair) given by (22), which includes our new correction of order $g^4(\mu/\kappa)$. Curve 4 includes, in addition, the pair-term contributions, $V_{ps\text{-}ps}$(one-pair) and $V_{ps\text{-}ps}$(two-pair) given by (26) and (25), respectively.

It is well known that $V_4(\mathbf{r})$ is too large a correction to $V_2(\mathbf{r})$ even at distances larger than the pion Compton wavelength. This leads one to suspect that the expansion with respect to g^2 and (p/κ) is badly convergent or even that the potential concept is wrong.[2,5] According to our figures, however, our new correction of order $g^4(\mu/\kappa)$

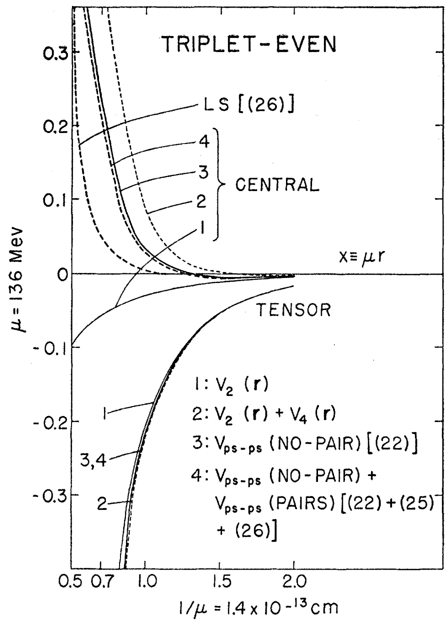

FIG. 1. Our entire potential [(22), (25), and (26)] is plotted in units of μ against $x\equiv\mu r$, for the triplet even state. Curves 1, 2, 3, and 4 correspond, respectively, to the second-order potential, the sum of potentials up to the fourth-order, the sum of all potentials except pair terms, and the entire sum of all potentials.

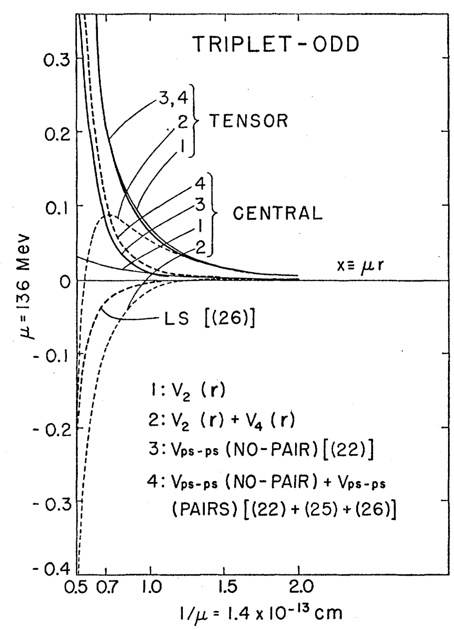

FIG. 2. Our entire potential [(22), (25), and (26)] is plotted in units of μ against $x\equiv\mu r$, for the triplet odd state. Curves 1, 2, 3, and 4 correspond, respectively, to the second-order potential, the sum of potentials up to the fourth-order, the sum of all potentials except pair terms, and the entire sum of all potentials.

[7] The explicit expression is given in the following paper (reference 3).

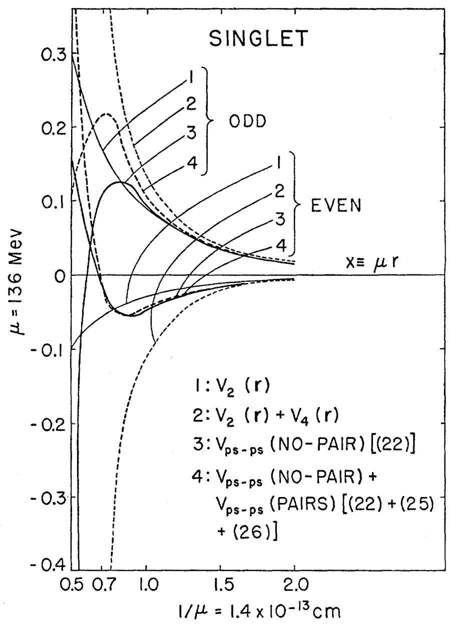

FIG. 3. Our entire potential [(22), (25), and (26)] is plotted in units of μ against $x\equiv\mu r$, for the singlet states. Curves 1, 2, 3, and 4 correspond, respectively, to the second-order potential, the sum of potentials up to the fourth-order, the sum of all potentials except pair terms, and the entire sum of all potentials.

always cancels out $V_4(\mathbf{r})$, making the entire potential (curves 3 or 4) quite close to the second-order potential $V_2(\mathbf{r})$ (curve 1) down to distances of the order of the pion Compton wavelength, as should be if our procedure is self-consistent. The central force in the triplet even state is the only exception. Here curves 3 and 4 deviate appreciably from curve 1 at $x=1$, though a significant improvement is noticed.

These results suggest that our basic presumption [the expansion with respect to g^2 and (p/κ)] is better than suspected in the past.[2,6] This statement is not only interesting itself, but is also important since the velocity-dependent terms obtained in this paper can thus get a sounder theoretical basis because they have the same origin as those static corrections which improve the situation appreciably.

It is interesting to remark that the higher-order corrections are almost purely central; in the case of the tensor force, curves 3 and 4 stay quite close to curve 1 at all points on the figures, while various central curves fluctuate among themselves especially at distances smaller than $x=1$.

Regarding the velocity-dependence of the two-nucleon potential, we have shown that the only essentially quadratic term is $-V_2(\mathbf{r})(p^2/2\kappa^2)+\text{H.c.}$ Furthermore, no other velocity-dependent term follows from diagrams of one-pion and two-pion exchange including no nucleon pairs; diagrams including nucleon pairs are the only source of an $\mathbf{L}\cdot\mathbf{S}$ potential, if there is any.

According to our estimate of the nucleon-pair contributions they have a minor effect on the static potential (the close behavior of curves 3 and 4) and give rise to a strongly isospin-dependent $\mathbf{L}\cdot\mathbf{S}$ potential. The sign of this $\mathbf{L}\cdot\mathbf{S}$ potential for the odd state is the right one; the sign changes for the even state. The magnitude seems, however, somewhat too small compared with potentials proposed phenomenologically.[8]

Of course, it is an open question how reliable is our estimate of the nucleon-pair diagrams. Assuming the estimate is reliable *ps-ps* theory is characterized by an $\mathbf{L}\cdot\mathbf{S}$ potential which has different sign in even and odd states and has a very short range (practically vanishing outside the pion Compton wavelength).

The final comment concerns the higher-order terms neglected here. According to the argument given in the introduction, terms of order g^6 must be kept together with terms of orders $g^4(p/\kappa)$ and $g^2(p/\kappa)^2$. This, of course, affects only the static part of our potential. This modification could be important since g^6 terms include, besides three-pion exchange contributions, those two-pion exchange contributions in which either pion-nucleon rescattering or the pion-pion scattering term occur as intermediate processes, even if we drop renormalization diagrams. The various existing calculations[9] on the pion-nucleon rescattering effect agree in that an appreciable attractive central force seems to be expected, though these calculations are only to be trusted qualitatively. There does not seem to be any work available on the possible effect of the pion-pion scattering term on the nuclear force.

According to some recent works,[10] pion-nucleon rescattering could also be a source of a large $\mathbf{L}\cdot\mathbf{S}$ potential. We simply point out here that these calculations do not agree even in the sign of the $\mathbf{L}\cdot\mathbf{S}$ potential,[10] and the reliability of their estimates is yet an open question.

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[8] Detailed comparisons among various $\mathbf{L}\cdot\mathbf{S}$ potentials are given in the following paper (reference 3).

[9] Matsumoto, Hamada, and Sugawara, Progr. Theoret. Phys. (Kyoto) **10**, 199 (1953); **12**, 553 (1954). Konuma, Miyazawa, and Otsuki, Phys. Rev. **107**, 320 (1957) and Progr. Theoret. Phys. (Kyoto) **19**, 17 (1958). R. E. Cutkosky, Phys. Rev. **112**, 1027 (1958) and a forthcoming paper (to be published).

[10] S. Okubo and S. Sato, Progr. Theoret. Phys. (Kyoto) **21**, 383 (1959). Paper by Klein *et al.* of reference 1. An error was found in the latter paper, but the favorable sign of a large $\mathbf{L}\cdot\mathbf{S}$ potential in the odd state remains unchanged. [Tzoar, Raphael, and Klein, Phys. Rev. Letters **3**, 145(E) (1959).]