Geometrical Representation of the Maxwell Field in Minkowski Space

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The electromagnetic field tensor of a classical charged particle is associated with the orientation and density of a family of two-dimensional surfaces radially distributed about the world line of the particle in Minkowski space.

I. TENSOR FORMULATION

FIRST we give a brief statement of the law of motion of classical charged particles. The motion of the particles may be described by giving their world lines in the four-dimensional Minkowski space of special relativity. Consider a particle at the point P with 4-velocity v_i, rest mass m, and charge e_1. Its motion is determined by the Lorentz force law,

$$m(dv_i/d\tau)=e_1 f_{ij} v_j. \tag{1}$$

The f_{ij} describe the electromagnetic field at P generated by the other particles in the space. To obtain the contribution to the field at any point P due to the field of one of the particles, we find first the point Q, of intersection of the past light cone of P with the world line of the particle. Let its charge be e. Let the 4-velocity of the particle at Q be u_i. We will take the velocity of the light as 1, so that $u_k u_k=-1$. Let the 4-acceleration of the particle at Q be a_i. Then we have $u_k a_k=0$. Denote the null vector $x_i(P)-x_i(Q)$ as c_i. Then $c_k c_k=0$. The contribution of the particle to the field at P is

$$f_{ij}=\frac{e[u_i+(u_i\times a_k)c_k]\times c_j}{4\pi(u_m c_m)^3}. \tag{2}$$

The antisymmetric tensor $x_i y_j - x_j y_i$ is denoted here by $x_i \times y_j$.

This is sufficient to predict the motion of a set of classical charged particles. Formula (2) is equivalent to the expressions of Liénard and Wiechert, and it may readily be shown to satisfy Maxwell's equations in a vacuum:

$$\frac{\partial f_{ij}}{\partial x_j}=0,\quad \frac{\partial f_{ij}}{\partial x_k}+\frac{\partial f_{jk}}{\partial x_i}+\frac{\partial f_{ki}}{\partial x_j}=0.$$

Formula (2), though one of the most fundamental in classical physics, occurs very rarely in textbooks.[1]

II. GEOMETRIC FORMULATION

A geometrical interpretation of the classical electromagnetic field was given by Page and Adams.[2] They pictured a charged particle as a small sphere bristling with short emitters, each of which acts like a machine gun firing a steady stream of bullets, which travel at the speed of light. If we draw a space-time diagram of this scheme, we see that the history of each "bullet" is represented by a null ray starting at the world line of the particle and projecting along a future light cone. Further, all the rays associated with the "bullets" from a single emitter may be grouped to form a two-dimensional surface. With suitable assumptions about the distribution and motion of the emitters about the particle, we can show that the electromagnetic tensor associated with the field generated by the charge is simply a measure of the orientation and density of the set of two-dimensional surfaces radially distributed about the world line (Fig. 1).

The field is completely described by the structure of subsets into which the events of the space are grouped. This structure is defined by the following assumptions about the family of surfaces, or "sheets of force" emanating from the world line of a particle:

(A) The number of surfaces is very large, and proportional to the charge of the particle.

(B) The intersection of a surface with the future light cone of a point Q on the world line is a straight line.

(C) A plane tangent to a surface at a point Q on the world line is also tangent to the world line.

(D) All planes tangent to the same surface at points near Q on the world line intersect the hyperplane orthogonal to the world line at Q in a common direction.

(E) When the future light cone of Q intersects any hyperplane orthogonal to the world line at Q, in a sphere, the points of intersection of the sphere with the sheets of force are uniformly distributed over the sphere.

Assumptions (D) and (E) correspond, respectively, to Page and Adams' assumptions that, in the frame of reference in which a particle is motionless, the emitters

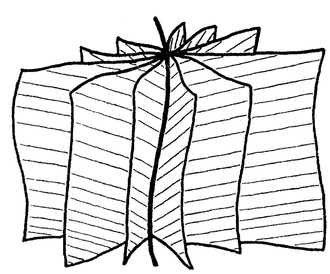

FIG. 1. The world line of a charged particle with sections of a few of the doubly infinite family of surfaces which represent the associated electromagnetic field.

[1] See, however, C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1952), p. 150.

[2] L. Page and N. I. Adams, *Electrodynamics* (D. Van Nostrand Company, Inc., New York, 1940).

have no angular velocity and that they are uniformly distributed in direction about the particle. This family of surfaces was first investigated by Whittaker,[3] who referred to them as magnetopotential surfaces.

III. THE EQUIVALENCE OF THE TWO REPRESENTATIONS OF THE FIELD

In four-dimensional vector analysis, any antisymmetric tensor of the form

$$g_{ij}=x_i\times y_j=x_iy_j-x_jy_i$$

may be used to represent an oriented area in the space. The orientation is defined by the plane which contains the directions of the two vectors x_i and y_i, and the area is that of the parallelogram defined by the line segments of the two vectors. The area A, represented by g_{ij}, is given by

$$A^2=x_k^2y_k^2-(x_ky_k)^2=\tfrac{1}{2}g_{ij}g_{ij}.$$

The tensor g_{ij} may also be used to represent the density within a unit area, as is done here, rather than an area.

The tensor f_{ij} given by (2) may thus represent the orientation of a surface, which has tangent to it at P the directions $u_i+(u_i\times a_k)c_k$ and c_i. The magnitude of f_{ij} may represent the density of a family of such surfaces as they intersect the plane orthogonal to the surfaces near P.

The condition that an antisymmetric tensor have the form $x_i\times y_j$ is that its determinant must vanish. For an electromagnetic tensor, this condition is met if the electric and magnetic vectors are perpendicular.

We will now identify the orientation and density of the sheets of force at a point P with the electromagnetic field tensor given by (2). By (A), the number of surfaces is very large so we may assume that one of them, S, passes through the arbitrary point P. Let Q be the point on the world line whose future light cone includes P. By (B), the tangent plane T_1 to the sheet S at Q contains the line segment QP which we represent by the vector c_i (Fig. 2). By (C), T_1 is then spanned by the vectors u_i and c_i. It therefore contains $(u_i\times c_k)u_k$, which is a linear combination of u_i and c_i orthogonal to $\mathbf{u}_i$.

If we move along the world line a small distance to M, where the line segment QM is represented by the vector $u_i\Delta\tau$, then the 4-velocity of the particle at M is $u_i'=u_i+a_i\Delta\tau$. The future light cone of M intersects the surface

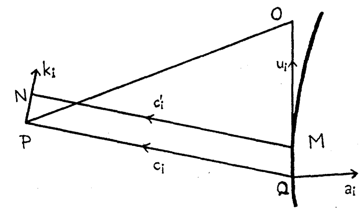

FIG. 2. Construction of the vector k_i.

S in a straight line and we let c_i' be a vector of arbitrary length projecting M into N along the line. The plane T_2 tangent to surface S at M contains the directions u_i' and c_i', and therefore it contains $(u_i'\times c_k')u_k$ which is orthogonal to u_i.

For $\Delta\tau$ small enough we have by (D), that $(u_i'\times c_k')u_k$ and $(u_i\times c_k)u_k$ have the same direction. The length of c_i' is arbitrary so we may fix it by setting

$$(u_i'\times c_k')u_k=(u_i\times c_k)u_k.$$

This requires that

$$c_i'-c_i=(u_i\times a_k)c_k\Delta\tau.$$

The line segment PN may then be represented by the vector

$$k_i\Delta\tau=[u_i+(u_i\times a_k)c_k]\Delta\tau.$$

The vectors c_i and k_i then define the orientation of the surface S at P.

Let H_1 be an hyperplane orthogonal to u_i passing through P. Let O be a point in H_1 on the line tangent to the world line at Q. We can show that H_1 intersects the light cone of Q in a sphere with center O, radius $-u_kc_k$, and surface area $4\pi(u_kc_k)^2$. By (A), the number of surfaces radiating from the world line is proportional to e, the charge of the particle. We may choose our units so that the number of surfaces is equal to the charge e. It will be necessary to imagine a negative number of sheets for a negatively charged particle. The density of the points of intersection of the sphere with the sheets of force is uniform over the sphere by (E). The density at all points including P will be

$$D_1=e/4\pi(u_kc_k)^2.$$

The plane T_3 tangent to the sphere at P is orthogonal to the lines QO and OP, and therefore to any linear combination of these directions which includes u_i and c_i.

We will now compute the density D_2 of the sheets of force near P in the plane T_4 orthogonal to the sheets themselves, that is, orthogonal to c_i and k_i. Both T_3 and T_4 are orthogonal to c_i so they must both lie in the hyperplane H_2 orthogonal to c_i and they must intersect in a line. The only linear combination of c_i and k_i which is orthogonal to c_i has direction c_i, so the sheets of force intersect H_2 in a set of lines with direction c_i at P. We may compare the densities of the sheets of force through T_3 and T_4 near P by comparing the areas of two small rectangles, one in each plane, which bound the same bundle of lines having direction c_i. The two rectangles have a common side on the line of inter-

[3] J. L. Synge, Proc. Edinburgh Math. Soc. **2**, Part 1, p. 39 (1958); E. T. Whittaker, Proc. Roy. Soc. Edinburgh **42**, 1–23 (1921). The Liénard-Wiechert field satisfies the invariant condition that the electric and magnetic fields are everywhere perpendicular. This is the condition for the existence of Whittaker's magnetopotential surfaces. A superposition of Liénard-Wiechert fields will not in general satisfy this condition and will not therefore permit this representation in terms of a single family of surfaces. However, according to Whittaker even the most general electromagnetic field can be described in terms of "tubes of force" of a relativistically invariant character. Therefore it would be of interest to see this tubes-of-force representation of the Liénard-Wiechert field worked out and related to the surfaces of Fig. 1.

section of the two planes. The relative areas are then given by the relative lengths of two sides of a small triangle Δn_i and Δm_i in H_2 where the third side has direction c_i. Then

$$\Delta m_i = \Delta n_i + h c_i,$$

and

$$\Delta m_i^2 = \Delta n_i^2 + 2h \Delta n_i c_i + h^2 c_i^2 = \Delta n_i^2.$$

Therefore the sides of the triangles, the areas of the rectangles, and the densities of points of intersection are equal in T_3 and T_4.

When f_{ij} given by (2) operates on any vector, it produces a linear combination of c_i and k_i. Therefore it represents the sheets of force at P in orientation. The magnitude of f_{ij} is given by

$$|\tfrac{1}{2} f_{ij} f_{ij}|^{\frac{1}{2}} = \frac{e}{4\pi (u_k c_k)^2},$$

which equals the density of the sheets in the plane orthogonal to themselves at P.

The electromagnetic field is then consistently represented by a family of surfaces defined by the above assumptions.

The physical theory is completed by giving the Lorentz force law in an analogous form.

(F) Each family of sheets intersects the hyperplane H_3 in a set of lines, where H_3 is orthogonal to the world line of a charged particle at P on the world line. Each family of sheets exerts a force on the particle at P which has the direction of the lines of intersection, and magnitude which is proportional to their density in H_3, and to the charge of the particle.

The force given by Eq. (1) fulfills these requirements in that it is proportional to the charge e_1, orthogonal to the direction of the world line v_i, and it is a linear combination of c_i and k_i, and therefore it lies on a sheet of force through P. We must also show that the magnitude of this vector is proportional to the density of the sheets of force at P intersecting the plane T_5 which is orthogonal to $f_{ij} v_j$, and v_i. This may be established by noting that both T_4 and T_5 lie in the hyperplane orthogonal to $f_{ij} v_j$, so we may compare two small areas which enclose a bundle of lines of intersection in the same way as was done between T_3 and T_4.

IV. DISCUSSION

It would seem a step forward in understanding the four-dimensional character of the physical world to translate the electromagnetic field from a set of arrays of numbers which have different values in different frames of reference into the geometry of families of surfaces radially distributed about the world lines of the source particles.

It does not appear to be known at present whether or not other physical fields may be similarly analyzed.

Note added in proof.—The general principal of relativity requires that all reference systems be equivalent with respect to the formulation of the fundamental laws of physics.[4] The use of tensor or spinor equations restricts us to coordinate transformations which are continuous, and differentiable, with nonvanishing Jacobian. If it is possible to do so, it may be best to describe the fundamental fields as structures of subsets of points in the space, as is attempted here, rather than as tensor and spinor functions. This would permit a completely arbitrary labeling of the points in the space in conformity with the spirit of relativity in its broadest sense.

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[4] See reference 1, p. 220. Other authors are more restrictive.