observed fact suggests that the strong interaction has invariance under C and hence under P if only CP invariance is assumed. Also, the necessity for exchange antisymmetry in the baryon-antibaryon model of the boson implies rather simple charge space properties for the interaction. On this compound model the statement that strong interactions conserve strangeness is a tautology.

The interaction that does not distinguish fermions from antifermions can be—and presumably is—universal to both baryons and leptons. It must be weak enough not to form bound boson states, or free leptons would not be observed; according to the argument above it must also contain $(1\pm\gamma_5)$ factors and be parity-violating. The choice of a charge space vector form for the interaction, required to avoid revealing relative leptonness through $(\mu e)\pi$ interactions, implies $T=CP=-1$ rather than invariance and also a lower degree of charge symmetry than in the strong interaction.

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Inconsistency of Cubic Boson-Boson Interactions

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It is shown that there does not exist a ground state for a system of spin zero bose fields coupled only by local interactions involving three powers of the fields. Thus these interactions alone are not suitable for a model of interacting fields.

I. INTRODUCTION

It is necessary, if a field theory is to give a consistent description of physical reality, that there be a state of lowest energy. If the energy spectrum has no lower bound, then the system can undergo a "radiative" collapse. We shall be concerned in this paper with a group of local interactions of spin zero bose fields that do not give rise to a state of lowest energy. These are interactions that involve three powers of the fields, i.e., for one field, ϕ, interacting with itself,[1]

$$H_{\text{int}}=\lambda\int\phi^3(x)d^3x, \tag{1}$$

for two fields, ϕ and χ,

$$H_{\text{int}}=\lambda\int\chi(x)\phi^2(x)d^3x, \tag{2}$$

and for three fields, ϕ, χ, and θ,

$$H_{\text{int}}=\lambda\int\chi(x)\theta(x)\phi(x)d^3x. \tag{3}$$

The field operators are taken to be Hermitean and thus λ must be a real constant, with dimensions of an inverse length. These three interactions are the so-called "super-renormalizable" ones.[2]

For each of the interactions given above we shall show that the assumption of the existence of a ground state leads to a contradiction, and hence that these interactions alone between the fields are not physically realizable.

II. PROOF OF THE NONEXISTENCE OF THE GROUND STATE

First consider the case of one scalar field with the cubic self-interaction (1). The dynamics are described by a Lagrangian

$$L=\tfrac{1}{2}\int d^3x[\phi(x)(\partial_\mu\partial^\mu-m^2)\phi(x)-\lambda\phi^3(x)], \tag{4}$$

where

$$\partial_\mu\partial^\mu=\nabla^2-(\partial/\partial t)^2,$$

and m^2 is taken to be a finite real number which may be zero. The requirement of invariance of L under spatial reflection implies that ϕ must be a scalar field. From L is derived the form of the energy operator, P^0, for the field

$$P^0=\tfrac{1}{2}\int d^3x[(\phi^0)^2+(\phi^k)^2+m^2\phi^2+\lambda\phi^3], \tag{5}$$

where

$$\phi^\mu(x)=\partial^\mu\phi(x).$$

Integrating by parts and introducing the real symmetric

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[1] M. Fierz has already pointed out that in the classical limit this interaction (1), does not lead to a positive definite energy. *Proceedings of the Fifth Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1955), p. 67.

[2] W. Thirring, Helv. Phys. Acta **26**, 33 (1953).

operator

$$\omega^2=-\nabla^2+m^2,$$

we may write

$$P^0=\tfrac{1}{2}\int[(\phi^0)^2+\phi\omega^2\phi+\lambda\phi^3]d^3x. \tag{6}$$

One may also derive the commutation relations

$$[\phi^0(x),\phi(x')]=i\delta(x-x'),$$
$$[\phi^0(x),\phi^0(x')]=0=[\phi(x),\phi(x')], \tag{7}$$

for x and x' having space-like separation.

The eigenvalues of P^0 are the possible energy values of the system. Assume now that there exists a state of lowest energy, which we call the vacuum, and denote by $\rangle$. Then the vacuum energy is given by $\langle P^0\rangle$. Furthermore, if U is a unitary operator, then U operating on the vacuum cannot lead to a state with a still lower energy expectation value. Thus

$$\langle U^{-1}P^0U\rangle-\langle P^0\rangle\geqslant 0. \tag{8}$$

Also, if U is of the form

$$U=\exp\left[i\int d^3xa(x)G(x)\right],$$

where $a(x)$ is a real function, and $G(x)$ is an Hermitean operator, the requirement that the vacuum energy be an extremum implies that

$$\delta/\delta a(x)\,(\langle U^{-1}P^0U\rangle)_{a(x)\equiv 0}=0, \tag{9}$$

where $\delta/\delta a(x)$ denotes the variational derivative with respect to the function $a(x)$. This is just the statement that

$$\langle[G,P^0]\rangle=0.$$

The commutation relations (7) imply that the unitary operator

$$U=\exp\left[i\int d^3x'a(x')\phi^0(x')\right], \tag{10}$$

satisfies

$$[\phi^0(x),U]=0,\quad [\phi(x),U]=a(x)U, \tag{11}$$

so that

$$U^{-1}\phi(x)U=\phi(x)+a(x),$$

provided that $t=t'$. Since however P^0 is a constant of the motion, we may always evaluate P^0 and the integral occurring in (10) at the same time. Furthermore, the commutation relations are still valid if one takes derivatives in space-like directions, so that

$$U^{-1}\phi^k(x)U=\phi^k(x)+\partial^k a(x),\quad (k=1,2,3). \tag{12}$$

Then for the U in (10) it is easy to calculate that

$$\langle U^{-1}P^0U\rangle-\langle P^0\rangle$$
$$=\tfrac{1}{2}\int d^3x(2a\omega^2\langle\phi\rangle+a\omega^2a+3\lambda a\langle\phi^2\rangle$$
$$+3\lambda a^2\langle\phi\rangle+\lambda a^3)\geqslant 0. \tag{13}$$

Condition (9) implies that the coefficient of the term linear in $a(x)$ vanishes pointwise. Hence

$$2\omega^2\langle\phi(x)\rangle+3\lambda\langle\phi^2(x)\rangle=0. \tag{14}$$

Also, $\langle\phi(x)\rangle$ and $\langle\phi^2(x)\rangle$ are independent of x, since the vacuum is an energy-momentum eigenstate, so that

$$2m^2\langle\phi\rangle+3\lambda\langle\phi^2\rangle=0. \tag{15}$$

[This result may also be arrived at by taking the vacuum expectation of the equation of motion derived from (4).] Thus (13) becomes

$$2\int d^3x\,(a\omega^2a+\lambda a^3)\geqslant 9(\lambda^2/m^2)\langle\phi^2\rangle\int d^3x\,a^2. \tag{16}$$

By choosing $a(x)$ arbitrarily small, arbitrarily slowly varying, and such that the integrals in (16) exist, we can conclude that

$$2m^4\geqslant 9\lambda^2\langle\phi^2\rangle. \tag{17}$$

Thus the assumption that m^2 is finite implies that $\langle\phi^2\rangle$ is also finite. [If $m^2=0$, then one infers from (15) directly that $\langle\phi^2\rangle=0$.] Now the positive nature of the right-hand side of (16), compared with the dominating odd function $a^3(x)$ on the left-hand side establishes the contradiction. Hence for the case of interaction (1) there is no ground state.

The reason that no ground state exists is as follows: Bose fields may be given arbitrarily large excitations, so that the nonpositive definite cubic term in the energy operator will, at large field excitations, in general dominate the positive quadratic terms that are present from the terms in the energy operator independent of λ.

Interactions (2) and (3) are treated similarly. For two fields coupled together

$$L=\tfrac{1}{2}\int d^3x[\phi(\partial_\mu\partial^\mu-m^2)\phi+\chi(\partial_\mu\partial^\mu-\mu^2)\chi-\lambda\chi\phi^2]. \tag{18}$$

Again λ is real, and m^2 and μ^2 are finite real constants. χ must be a scalar field, but ϕ may be either scalar or pseudoscalar, since

$$\phi(x)\rightarrow-\phi(x) \tag{19}$$

is a symmetry of L.

It follows that

$$P^0=\tfrac{1}{2}\int[(\phi^0)^2+\phi\omega_n{}^2\phi+(\chi^0)^2+\chi\omega_\mu{}^2\chi+\lambda\chi\phi^2]d^3x, \tag{20}$$

where $\omega_m{}^2=-\nabla^2+m^2$, and $\omega_\mu{}^2=-\nabla^2+\mu^2$. All commutators at equal times involving one field χ and one field ϕ vanish, and one has (7) obeyed by χ and ϕ individually. Using the same procedure as in the case of one field, taking now

$$U=\exp\left\{i\int[a(x')\phi^0(x')+b(x')\chi^0(x')]d^3x'\right\}, \tag{21}$$

one derives

$$\int [2a\omega_m^2\langle\phi\rangle + a\omega_m^2 a + 2b\omega_\mu^2\langle\chi\rangle + b\omega_\mu^2 b + \lambda b\langle\phi^2\rangle + 2\lambda a\langle\chi\phi\rangle + 2\lambda ab\langle\phi\rangle + \lambda a^2\langle\chi\rangle + \lambda ba^2] d^3x \geqslant 0. \quad (22)$$

It follows from (19) that $\langle\phi\rangle = \langle\chi\phi\rangle = 0$. Also, taking the variational derivative of (22) with respect to $b(x)$, evaluated at $a(x)\equiv b(x)\equiv 0$, one has

$$2\mu^2\langle\chi\rangle + \lambda\langle\phi^2\rangle = 0. \quad (25)$$

One then arrives at the analog of (17)

$$\int (a\omega_m^2 a + b\omega_\mu^2 b + \lambda ba^2) d^3x \geqslant \frac{1}{2}\frac{\lambda^2}{\mu^2}\langle\phi^2\rangle \int a^2 d^3x. \quad (24)$$

As in (17), we find

$$2m^2\mu^2 \geqslant \lambda^2\langle\phi^2\rangle.$$

Again the right-hand side of (24) is positive, but the left-hand side may take either sign for appropriate $a(x)$ and $b(x)$. This contradiction establishes the nonexistence of the ground state for fields coupled by $\lambda\chi\phi^2$.

The discussion for three bose fields interacting via (3), where one field must be scalar and the other two either both scalar or both pseudoscalar, is strictly analogous to the above discussions and therefore need not be given. In this situation, the symmetries

$$\begin{aligned} &\chi\to-\chi, \quad \phi\to-\phi, \quad \theta\to\theta, \\ &\chi\to\chi, \quad \phi\to-\phi, \quad \theta\to-\theta, \\ &\chi\to-\chi, \quad \phi\to\phi, \quad \theta\to-\theta, \end{aligned} \quad (25)$$

guarantee the vanishing of expectation values that occur in

$$\langle U^{-1}P^0U\rangle - \langle P^0\rangle,$$

such as $\langle\chi\rangle$, $\langle\chi\phi\rangle$, etc.

III. DISCUSSION

We have thus shown that if the only interactions between the bose fields are of total power equal to three in the fields, then one is led to a system in which there is no ground state. It is easily verified that any combination of such interactions, or the inclusion of nonspace-time degrees of freedom in no way alters the above conclusion.

As we mentioned, there is no ground state due to the dominance of the cubic term in the energy operator, for large field excitations. The presence in H_{int} of a positive definite fourth-power structure in any of the fields, such as $g\phi^4$ or $g\phi^2\chi^2$ with an arbitrarily small positive coupling constant g, insures however that there will be a ground state. This implies that in order for the interactions of the kind discussed in this paper to be realized physically for π and K mesons, such positive definite quartic terms must be included in the interaction Hamiltonian.

The proof given here will not work, of course, for a Yukawa type interaction since two of the fields occurring in the interaction are fermi fields which have only limited excitations. Also, the electromagnetic interaction of charged spin zero mesons has an $e^2A^2\phi^2$ term in it which essentially insures the positive definiteness of the energy.

Several authors have used cubic boson-boson interactions alone in considering the convergence of the perturbation expansion of the S matrix.[3] The arguments presented here are intended to show the inconsistencies involved in using these interactions alone in a model of a field theory.

Finally, we note that these arguments may be extended to the case of an interaction involving the fourth power of bose fields in which at least one of the fields occurs to an odd power.

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[3] W. Thirring, reference 2; C. A. Hurst, Proc. Cambridge Phil. Soc. **48**, 625 (1952); A. Petermann, Phys. Rev. **89**, 1160 (1953); R. Utiyama and T. Imamura, Progr. Theoret. Phys. (Kyoto) **9**, 431 (1953); also, J. C. Ward, Phys. Rev. **79**, 406 (1950).