Contribution to the Electrostatic Self-Energy of a Charged Liquid Drop

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The electrostatic self-energy of the liquid-drop-model nucleus has been divided into two parts, one of which has been evaluated. The problems arising due to difficulties in choosing limits of integration—difficulties which occur due to the presence in the integrand of the factor r_{12}^{-1}—are all contained in the formulation of that part of the self-energy which we have obtained. The evaluation of the other part of the self-energy poses no such limit problems.

The shapes which we consider for the drop are the shapes adopted by a nucleus undergoing symmetric fission.

The self-energy is expressed as a multiple-power series in the deformation parameters a_n. The part of the self-energy which we calculate is zero through the general seventh-power term in the multiple-power series for the self-energy.

It has been shown in a recent paper[1] that in calculating the electrostatic self-energy for the liquid-drop model of the nucleus one finds it convenient to formulate two contributions to the energy. In this paper we shall consider in detail one of the two contributions.

CHARACTERIZATION OF THE DEFORMED DROP

The drop is allowed to have deformations which preserve an axis of symmetry (the z-axis) and a plane of symmetry (the x,y-plane). For the equation of the surface of any such drop, the distance from the origin to a point in the surface can be represented by

$$R(\cos\vartheta)=R(\mu)=R_0\left[1+a_0+\sum_{n=2,\text{even}}^{\text{finite}} a_nP_n(\mu)\right], \quad (1)$$

where ϑ is the colatitude ($\mu=\cos\vartheta$), P_n is the Legendre polynomial of degree n, and a_n $[=a_n(t)]$ is a distortion parameter. (A set of values a_n defines a given shape.) We further demand that any spherical surface constructed with its center at the origin of coordinates either has no intersection with the drop's surface or intersects this surface twice: at $\vartheta=\vartheta_0$ and at $\vartheta=\pi-\vartheta_0$.

In Fig. 1 we have the profile of a deformed drop which satisfies these conditions. The largest sphere which can be placed inside the drop has radius b. A somewhat larger spherical surface with radius r intersects the drop twice. The shaded region outside the drop's profile represents an annular region interior to the sphere of radius r and exterior to the drop.

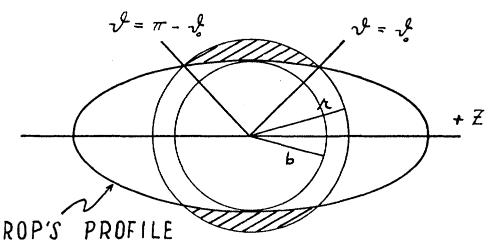

FIG. 1. Profile of a deformed drop (with constructions).

FORMULATION OF THE PROBLEM FOR ELECTROSTATIC SELF-ENERGY

The electrostatic self-energy of the drop is

$$U^c=\tfrac{1}{2}\int\int\rho_1\frac{d\tau_1\rho_2d\tau_2}{r_{12}}, \quad (2)$$

with $\rho_1=\rho_2=\rho=Ze/(4\pi R_0^3/3)$ for a drop with charge Ze. In terms of the electrostatic potential this becomes

$$U^c=\tfrac{1}{2}\int\rho V d\tau_1, \quad (3)$$

in which

$$V=\int\rho d\tau_2/r_{12}. \quad (4)$$

We expand r_{12}^{-1} in terms of $P_l(\mu_{12})$, where μ_{12} is the cosine of the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, to get

$$V(r_1,\mu_1)=\rho\int_0^{2\pi}d\varphi_2\int_{-1}^{1}d\mu_2\left[\int_0^{r_1}dr_2\,r_2^2\sum_{l=0}^{\infty}\frac{r_2^l}{r_1^{l+1}}P_l(\mu_{12}) +\int_{r_1}^{R(\mu_2)}dr_2\,r_2^2\sum_{l=0}^{\infty}\frac{r_1^l}{r_2^{l+1}}P_l(\mu_{12})\right]. \quad (5)$$

We now note that the equation for $V(r_1,\mu_1)$ is valid only when $r_1<b$. (See Fig. 1.) For $r_1>b$ the limits of integration include regions outside the drop. We may use this expression for the potential for the entire drop provided we then compensate for the error introduced by using it for regions where it is incorrect. Using this potential for the entire drop, we get a contribution U_0^c to the electrostatic energy of the drop. We then calculate a correction, U_1^c, to the energy such that

$$U^c=U_0^c+U_1^c. \quad (6)$$

For regions where $r_1>b$ we add to the above potential a potential correction ΔV

$$\Delta V=\rho\int_0^{2\pi}d\varphi_2\int_{-\mu_0(r_1)}^{\mu_0(r_1)}d\mu_2\left[-\int_{R(\mu_2)}^{r_1}dr_2\,r_2^2\sum_{l=0}^{\infty}\frac{r_2^l}{r_1^{l+1}}P_l(\mu_{12}) -\int_{r_1}^{R(\mu_2)}dr_2\,r_2^2\sum_{l=0}^{\infty}\frac{r_1^l}{r_2^{l+1}}P_l(\mu_{12})\right]. \quad (7)$$

[1] W. D. Foland and R. D. Present, Phys. Rev. 113, 613 (1959).

where $\mu_0(r_1)$ defines the intersection of the drop's surface with a sphere of radius r_1; i.e., $R(\mu_0)=r_1$. One sees that this potential corrects for the error introduced by using $V(r_1,\mu_1)$ in the regions for which $r_1>b$. The correction ΔV is to be added then when $r_1>b$. Thus

$$U_1{}^c=\tfrac{1}{2}\rho\int_0^{2\pi}d\varphi_1\int_{-1}^{1}d\mu_1\int_b^{R(\mu_1)}dr_1\,r_1^2\Delta V$$

$$=2\pi^2\rho^2\sum_{l=0}^{\infty}\int_{-1}^{1}d\mu_1\,P_l(\mu_1)\int_b^{R(\mu_1)}dr_1\,r_1^2$$

$$\times\int_{-\mu_0(r_1)}^{\mu_0(r_1)}d\mu_2\,P_l(\mu_2)\int_{R(\mu_2)}^{r_1}dr_2\,r_2^2\left(\frac{r_1^l}{r_2^{l+1}}-\frac{r_2^l}{r_1^{l+1}}\right) \quad (8)$$

$$=2\pi^2\rho^2\sum_{l=0}^{\infty}\int_{-1}^{1}d\mu_1\,P_l(\mu_1)\int_{-|\mu_1|}^{|\mu_1|}d\mu_2\,P_l(\mu_2)$$

$$\times\int_{R(\mu_2)}^{R(\mu_1)}dr_1\,r_1^2\int_{R(\mu_2)}^{r_1}dr_2\,r_2^2\times\left(\frac{r_1^l}{r_2^{l+1}}-\frac{r_2^l}{r_1^{l+1}}\right), \quad (9)$$

as one sees upon using the addition theorem for $P_l(\mu_{12})$ and then interchanging the order of integration for r_1 and μ_2. Our surface demands have required that $R(\mu)$ be an even function of μ, so the equation for $U_1{}^c$ can finally be written as

$$U_1{}^c=8\pi^2\rho^2\sum_{\text{even }l}^{\infty}\int_0^1 d\mu_1\,P_l(\mu_1)\int_0^{\mu_1}d\mu_2\,P_l(\mu_2)$$

$$\times\int_{R(\mu_2)}^{R(\mu_1)}dr_1\,r_1^2\int_{R(\mu_2)}^{r_1}dr_2\,r_2^2\left(\frac{r_1^l}{r_2^{l+1}}-\frac{r_2^l}{r_1^{l+1}}\right). \quad (10)$$

Equation (10) has already been given in reference 1 where it was shown that $U_1{}^c$ vanishes through terms in a_2^3. It is this contribution to the electrostatic self-energy of the drop that we are considering now in greater detail and to higher orders of approximation.

EVALUATION OF THE CONTRIBUTION $U_1{}^c$

For any given nuclide Z and R_0 are fixed, and $U_1{}^c$ is a multiple-power series in the parameters a_0, a_2, a_4, $\cdots$.[2] In order to establish the coefficients of the terms in this series, we shall differentiate Eq. (10) the requisite number of times for each term. We find it convenient to reduce the problems of notation by defining

(1) ϵ as $\epsilon\equiv U_1{}^c/8\pi^2\rho^2$, (11)

(2) R_0 as the unit of length [see Eq. (1)],

and (3) $G(r_1,\mu_1,r_2,\mu_2)$ as

$$G(r_1,\mu_1,r_2,\mu_2)\equiv\sum_{l=0,\,\text{even}}^{N}P_l(\mu_1)P_l(\mu_2)\left[\frac{r_1^{l+2}}{r_2^{l-1}}-\frac{r_2^{l+2}}{r_1^{l-1}}\right]. \quad (12)$$

We then have the problem of evaluating the function

$$\epsilon=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\int_{R(\mu_2)}^{R(\mu_1)}dr_1\int_{R(\mu_2)}^{r_1}dr_2\,G(r_1,\mu_1,r_2,\mu_2), \quad (13)$$

after which we let $N\to\infty$.

We can use a general k-fold differentiation of ϵ to find the coefficients of all terms of kth power[3] in the multiple-power series. In order to obtain these general differentiations and the resulting general terms we introduce two devices in notation: (1) To denote an arbitrary a_n we shall use the notation a_{i_j}. The index i is used to keep the a_n's arbitrary; the index j is used to allow a distinction between the arbitrary a_n's. Thus $a_{i_1}a_{i_2}a_{i_3}\cdots a_{i_k}$ will represent any general kth power member of the power series (including a_2^k, $a_2^{k-1}a_4^1$, etc.). (2) To denote any general k-fold differentiation we shall use the symbol $\partial^k/(\partial a_i)^k$. This will include any arbitrary choice of the a_{i_j} [$\partial^3/(\partial a_i)^3$ includes $\partial^3/\partial a_{i_1}^3$, $\partial^3/\partial a_{i_1}^2\partial a_{i_2}$, $\partial^3/\partial a_{i_1}\partial a_{i_2}^2$, and $\partial^3/\partial a_{i_1}\partial a_{i_2}\partial a_{i_3}$].

The first general differentiation of ϵ is

$$\frac{\partial\epsilon}{\partial a_{i_1}}=\frac{\partial}{\partial a_{i_1}}\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\int_{R(\mu_2)}^{R(\mu_1)}dr_1\int_{R(\mu_2)}^{r_1}dr_2\,G(r_1,\mu_1,r_2,\mu_2)$$

$$=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\left\{\left[\int_{R(\mu_2)}^{r_1}dr_2\,G(r_1,\mu_1,r_2,\mu_2)\right]_{r_1=R(\mu_1)}\frac{\partial R(\mu_1)}{\partial a_{i_1}}-\left[\int_{R(\mu_2)}^{R(\mu_1)}dr_1\,G(r_1,\mu_1,r_2,\mu_2)\right]_{r_2=R(\mu_2)}\frac{\partial R(\mu_2)}{\partial a_{i_1}}\right\}, \quad (14)$$

or

$$\frac{\partial\epsilon}{\partial a_{i_1}}=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\int_{R(\mu_2)}^{R(\mu_1)}\left\{dr_2\,G(R(\mu_1),\mu_1,r_2,\mu_2)\frac{\partial R(\mu_1)}{\partial a_{i_1}}-dr_1\,G(r_1,\mu_1,R(\mu_2),\mu_2)\frac{\partial R(\mu_2)}{\partial a_{i_1}}\right\}. \quad (15)$$

Upon interchanging r_1 and $R(\mu_2)$ in $G(r_1,\mu_1,R(\mu_2),\mu_2)$ in the second term of the integrand and then interchanging the names of r_1 and r_2 in the resulting term, one finds that Eq. (15) becomes

$$\frac{\partial\epsilon}{\partial a_{i_1}}=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\int_{R(\mu_2)}^{R(\mu_1)}dr_2\left[G(R(\mu_1),\mu_1,r_2,\mu_2)\frac{\partial R(\mu_1)}{\partial a_{i_1}}+G(R(\mu_2),\mu_1,r_2,\mu_2)\frac{\partial R(\mu_2)}{\partial a_{i_1}}\right]. \quad (16)$$

[2] The parameter a_0 is ultimately obtained as a function of the other a_n by using the condition of constant volume; we are here giving it (seemingly) independent status in $U_1{}^c$. For the operations which we shall perform this status for a_0 is desirable.

[3] By a kth power term we shall mean a term for which the sum of exponents of the a_n is k: $a_1^{p_1}a_m^{p_m}a_n^{p_n}\cdots a_s^{p_s}$ is a kth power term if $p_1+p_m+p_n+\cdots+p_s=k$.

Here we have used the antisymmetry of G with respect to an interchange of r_1 and r_2 [see Eq. (12)]. The equation for $R(\mu)$ is linear in the a_{ij}. We write Eq. (16), then, as

$$\frac{\partial\epsilon}{\partial a_{i_1}}=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\int_{R(\mu_2)}^{R(\mu_1)}dr_2\,[G(R(\mu_1),\mu_1,r_2,\mu_2)P_{i_1}(\mu_1)+G(R(\mu_2),\mu_1,r_2,\mu_2)P_{i_1}(\mu_2)]. \tag{17}$$

Let $\mathcal{G}(r_1,\mu_1,r_2,\mu_2)$ be an indefinite integral of $\int dr_2\, G(r_1,\mu_1,r_2,\mu_2)$; then Eq. (17) becomes

$$\frac{\partial\epsilon}{\partial a_{i_1}}=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\,\{[\mathcal{G}(R(\mu_1),\mu_1,R(\mu_1),\mu_2)-\mathcal{G}(R(\mu_1),\mu_1,R(\mu_2),\mu_2)]P_{i_1}(\mu_1)$$
$$+[\mathcal{G}(R(\mu_2),\mu_1,R(\mu_1),\mu_2)-\mathcal{G}(R(\mu_2),\mu_1,R(\mu_2),\mu_2)]P_{i_1}(\mu_2)\}. \tag{18}$$

We now differentiate n additional times

$$\frac{\partial^{n+1}\epsilon}{(\partial a_i)^{n+1}}=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\left\{\left[\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_1),\mu_1,R(\mu_1),\mu_2)-\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_1),\mu_1,R(\mu_2),\mu_2)\right]P_{i_1}(\mu_1)\right.$$
$$\left.+\left[\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_2),\mu_1,R(\mu_1),\mu_2)-\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_2),\mu_1,R(\mu_2),\mu_2)\right]P_{i_1}(\mu_2)\right\}. \tag{19}$$

When the indicated differentiations have been performed and the a_{ij} remaining in the integrand have been set equal to zero, the resulting expression is proportional to the coefficient of the general $(n+1)$-power term in ϵ. The particular coefficient being evaluated is determined by the choice of the $n+1$ a_{ij}'s. Let the coefficient of the term $a_{i_1}a_{i_2}a_{i_3}\cdots a_{i_{n+1}}$ in ϵ be N_{n+1}. (This coefficient obviously depends on the choice of the a_{ij}.) We define κ as[4]

$$\kappa=\frac{\partial^{n+1}}{(\partial a_i)^{n+1}}a_{i_1}a_{i_2}\cdots a_{i_{n+1}}, \tag{20}$$

and M_{n+1} as κN_{n+1}. Then we have

$$M_{n+1}=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\left\{\left[\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_1),\mu_1,R(\mu_1),\mu_2)-\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_1),\mu_1,R(\mu_2),\mu_2)\right]P_{i_1}(\mu_1)\right.$$
$$\left.+\left[\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_2),\mu_1,R(\mu_1),\mu_2)-\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_2),\mu_1,R(\mu_2),\mu_2)\right]P_{i_1}(\mu_2)\right\}. \tag{21}$$

[Set all $R(\mu_1)=1=R(\mu_2)$.]

For the differentiations arising we have

$$\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_1),\mu_1,R(\mu_1),\mu_2)=\sum_{s=0}^{n}\binom{n}{s}\frac{\partial^{n-s}}{\partial r_1^{n-s}}\frac{\partial^s}{\partial r_2^s}\mathcal{G}\prod_{j=2}^{n+1}P_{i_j}(\mu_1),$$
$$\frac{\partial^n}{(\partial a_i)^n}\mathcal{G}(R(\mu_1),\mu_1,R(\mu_2),\mu_2)=\sum_{s=0}^{n}\frac{\partial^{n-s}}{\partial r_1^{n-s}}\frac{\partial^s}{\partial r_2^s}\mathcal{G}\sum_{\alpha_n}\prod_{j=2}^{n-s+1}P_{i_j}(\mu_1)\prod_{k=n-s+2}^{n+1}P_{i_k}(\mu_2), \tag{22}$$

where $\sum_{\alpha_n}$ sums all possible, nonrepeating permutations of indices i_j and i_k. Some care must be excised in forming the α_n summation. (1) The summation must be performed before the actual choice of the a_{ij} is made—the permutations are made with the a_{i_1}, a_{i_2}, a_{i_3}, etc., and not on the a_2, a_4, a_6, etc. (2) A permutation, to be accepted, must give a term that has not already been included: interchanging indices among either the μ_1, or μ_2 polynomials does not produce new terms for the summation—for example $P_{i_2}(\mu_1)P_{i_3}(\mu_1)P_{i_4}(\mu_2)$ is the same as $P_{i_3}(\mu_1)P_{i_2}(\mu_1)P_{i_4}(\mu_2)$, and of the two expressions only one is included in an α_n summation.

We now note that no term arising from a differentiation $(\partial^0/\partial r_2^0)\mathcal{G}$ can contribute to the value of M_{n+1}: These terms arise from differentiations $\int dr_2\,(\partial^n G/\partial r_1^n)$; and upon entering the limits for the M_{n+1} evaluation, one finds these terms, $\int_{+1}^{+1}dr_2\,(\partial^n G/\partial r_1^n)$, are all zero. This result means that all summations $\sum_{s=0}^n$ in Eqs. (22) can be replaced, for our purposes, by $\sum_{s=1}^n$ summations. The fact that 1 is the lowest necessary value of s allows us to replace $\partial^s\mathcal{G}/\partial r_2^s$ by $\partial^{s-1}G/\partial r_2^{s-1}$—our indefinite integral $\mathcal{G}$ is, then, not needed as an explicit function.

[4] The value of κ ranges from 1—the case for which none of the a_{ij} are chosen to be the same—to $(n+1)!$—the case for which $a_{i_1}=a_{i_2}=\cdots=a_{i_{n+1}}$.

The integrand for M_{n+1} has become

$$\sum_{s=1}^{n}\binom{n}{s}\frac{\partial^{n-s}}{\partial r_1^{n-s}}\frac{\partial^{s-1}}{\partial r_2^{s-1}}G(R(\mu_1),\mu_1,R(\mu_1),\mu_2)P_{i1}(\mu_1)\prod_{j=2}^{n+1}P_{ij}(\mu_1)$$
$$-\sum_{s=1}^{n}\binom{n}{s}\frac{\partial^{n-s}}{\partial r_1^{n-s}}\frac{\partial^{s-1}}{\partial r_2^{s-1}}G(R(\mu_2),\mu_1,R(\mu_2),\mu_2)P_{i1}(\mu_2)\prod_{j=2}^{n+1}P_{ij}(\mu_2)$$
$$-\sum_{s=1}^{n}\frac{\partial^{n-s}}{\partial r_1^{n-s}}\frac{\partial^{s-1}}{\partial r_2^{s-1}}G(R(\mu_1),\mu_1,R(\mu_2),\mu_2)P_{i1}(\mu_1)\sum_{\alpha_n}\prod_{j=2}^{n-s+1}P_{ij}(\mu_1)\prod_{k=n-s+2}^{n+1}P_{ik}(\mu_2)$$
$$+\sum_{s=1}^{n}\frac{\partial^{n-s}}{\partial r_1^{n-s}}\frac{\partial^{s-1}}{\partial r_2^{s-1}}G(R(\mu_2),\mu_1,R(\mu_1),\mu_2)P_{i1}(\mu_2)\sum_{\alpha_n}\prod_{j=2}^{n-s+1}P_{ij}(\mu_2)\prod_{k=n-s+2}^{n+1}P_{ik}(\mu_1). \quad (23)$$

When one has performed the differentiations in expression (23), he then sets $R(\mu_1)=1=R(\mu_2)$ (this is equivalent to setting all a_{ij} equal to zero). Upon entering these limits, one finds a single expression for the four types of differentiation demanded in expression (23):

$$\frac{\partial^{n-s}}{\partial r_1^{n-s}}\frac{\partial^{s-1}}{\partial r_2^{s-1}}G(1,\mu_1,1,\mu_2)=(-1)^{s-1}\sum_{l=0,\text{even}}P_l(\mu_1)P_l(\mu_2)\frac{(l+2)!}{(l-2)!}\left(\frac{[l-2+(s-1)]!}{[l+2-(n-s)]!}+(-1)^n\frac{[l-2+(n-s)]!}{[l+2-(s-1)]!}\right). \quad (24)$$

With Eq. (24) one can write for M_{n+1}

$$M_{n+1}=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\sum_{s=1}^{n}\frac{\partial^{n-s}}{\partial r_1^{n-s}}\frac{\partial^{s-1}}{\partial r_2^{s-1}}G(1,\mu_1,1,\mu_2)\Bigg\{\binom{n}{s}\left[\prod_{j=1}^{n+1}P_{ij}(\mu_1)-\prod_{j=1}^{n+1}P_{ij}(\mu_2)\right]$$
$$-P_{i1}(\mu_1)\sum_{\alpha_n}\prod_{j=2}^{n-s+1}P_{ij}(\mu_1)\prod_{k=n-s+2}^{n+1}P_{ik}(\mu_2)+P_{i1}(\mu_2)\sum_{\alpha_n}\prod_{j=2}^{n-s+1}P_{ij}(\mu_2)\prod_{k=n-s+2}^{n+1}P_{ik}(\mu_1)\Bigg\}. \quad (25)$$

EXPLICIT EXPRESSIONS FOR M_1 THROUGH M_7

One can see from Eq. (13) that ϵ will have no general first or second-power terms in the a_{ij}: the ranges of integration for r_1 and r_2 are of first power in the a_{ij}, and $G(r_1,\mu_1,r_2,\mu_2)$ is of first or higher power in the a_{ij}. We thus see that

$$M_1=0=M_2. \quad (26)$$

Of the higher orders in the M_{n+1} we have found the integrals for M_3 through M_9 and have evaluated these integrals for M_3 through M_7. The integrals for M_3 through M_7 are:

$$M_3=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\sum_{l=0,\text{even}}(2l+1)P_l(\mu_1)P_l(\mu_2)\prod_{j=1}^{3}[P_{ij}(\mu_1)-P_{ij}(\mu_2)], \quad (27)$$

$$M_4=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\sum_{l=0,\text{even}}2(2l+1)P_l(\mu_1)P_l(\mu_2)$$
$$\times\left\{2\left[\prod_{j=1}^{4}P_{ij}(\mu_2)-\prod_{j=1}^{4}P_{ij}(\mu_1)\right]+\sum_{\alpha_4'}[P_{i1}(\mu_1)-P_{i1}(\mu_2)]\left[\prod_{j=2}^{4}P_{ij}(\mu_2)+\prod_{j=2}^{4}P_{ij}(\mu_1)\right]\right\},\text{[5]} \quad (28)$$

$$M_5=\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\sum_{l=0,\text{even}}l(l+1)(2l+1)P_l(\mu_1)P_l(\mu_2)\prod_{j=2}^{5}[P_{ij}(\mu_1)-P_{ij}(\mu_2)]$$
$$+\int_0^1 d\mu_1\int_0^{\mu_1}d\mu_2\sum_{l=0,\text{even}}2(2l+1)P_l(\mu_1)P_l(\mu_2)\Bigg\{-P_{i1}(\mu_1)\sum_{\alpha_4}\prod_{j=2}^{3}P_{ij}(\mu_1)\prod_{k=4}^{5}P_{ik}(\mu_2)$$
$$+P_{i1}(\mu_2)\sum_{\alpha_4}\prod_{j=2}^{3}P_{ij}(\mu_2)\prod_{k=4}^{5}P_{ik}(\mu_1)+P_{i1}(\mu_1)\sum_{\alpha_4}P_{i2}(\mu_1)\prod_{k=3}^{5}P_{ik}(\mu_2)-P_{i1}(\mu_2)\sum_{\alpha_4}P_{i2}(\mu_2)\prod_{j=3}^{5}P_{ij}(\mu_1)\Bigg\}, \quad (29)$$

$$M_6=0. \quad (30)$$

[5] The $\sum_{\alpha_4'}$ is a sum of permutations different from the α permutation: for α' one permutes the index on the two functions in the first factor—this index takes, one at a time, all four values i_1, i_2, i_3, and i_4; and the index j takes the remaining three of the four values for any summand. The α' permutation allows the negative term $-P_{i1}(\mu_2)P_{i2}(\mu_2)P_{i3}(\mu_2)P_{i4}(\mu_2)$ and the positive term $P_{i1}(\mu_1)P_{i2}(\mu_1)\times P_{i3}(\mu_1)P_{i4}(\mu_1)$ to occur four times each in the summation; the α permutation would have allowed only one of each.

(All fourth-order derivatives of G are zero when $r_1=1=r_2$.)

$$M_7=\int_0^1 d\mu_1 \int_0^{\mu_1} d\mu_2 \sum_{l=0,\,\mathrm{even}} (2l+1)P_l(\mu_1)P_l(\mu_2)l(l+1)(l+2)(l-1)\prod_{j=1}^{7}[P_{i_j}(\mu_1)-P_{i_j}(\mu_2)]. \tag{31}$$

We have not been able to evaluate M_8 and M_9; we have formulated the integrals for these two quantities, using Eq. (25) and the results of the differentiations. For M_8 we have the differentiation

$$\frac{\partial^6 G}{\partial r_1^{6-p}\partial r_2^{p}}=(-1)^p\sum_{l=0,\,\mathrm{even}}(2l+1)P_l(\mu_1)P_l(\mu_2)(l+2)(l+1)l(l-1)f(p)$$
$$[f(0)=-6=-f(6);\, f(1)=-4=-f(5);\, f(2)=-2=-f(4);\, f(3)=0], \tag{32}$$

and for M_9 the differentiation

$$\frac{\partial^7 G}{\partial r_1^{7-p}\partial r_2^{p}}=(-1)^p\sum_{l=0,\,\mathrm{even}}(2l+1)P_l(\mu_1)P_l(\mu_2)l(l+1)(l+2)(l-1)[l(l+1)+g(p)]$$
$$[g(0)=36=g(7);\, g(1)=18=g(6);\, g(2)=6=g(5);\, g(3)=0=g(4)]. \tag{33}$$

We have not considered M_8 and M_9 to be of sufficient importance to merit numerical calculations: in performing calculations for the liquid-drop model, one terminates his power series after a few, low-order terms. We find that M_3 through M_7 are zero. [Equation (26) gave $M_1=0=M_2$.]

Three general integrals are evaluated in an appendix to this paper. With these three integrals we can evaluate M_3, M_4, M_5, and M_7. These integrals are

$$\mathop{L}_{N\to\infty}\int_0^1 dx\int_0^x dy \sum_{n=0,\,\mathrm{even}}^{N}(2n+1)P_n(x)P_n(y)(x^2-y^2)F(x,y)=0, \tag{34}$$

$$\mathop{L}_{N\to\infty}\int_0^1 dx\int_0^x dy \sum_{n=0,\,\mathrm{even}}^{N}(2n+1)n(n+1)P_n(x)P_n(y)(x^2-y^2)^3F(x,y)=0, \tag{35}$$

$$\mathop{L}_{N\to\infty}\int_0^1 dx\int_0^x dy \sum_{n=0,\,\mathrm{even}}^{N}(2n+1)n(n+1)(n-1)(n+2)P_n(x)P_n(y)(x^2-y^2)^5F(x,y)=0. \tag{36}$$

These integrals are valid for any $F(x,y)$ whose power-series expansion (in x and y): (1) contains only even-integral, positive powers of x and y and (2) terminates with some finite power for each of x and y. Equation (34) shows that M_3 and M_4 and the second integral in M_5 are zero.[6] The first integral in M_5 is shown to be zero by Eq. (35), and M_7 is shown to be zero by Eq. (36) alone. We thus see that $M_{n+1}=0$ for $0\leq n\leq 6$. This statement tells us that ϵ and thus U_1^c has no term of order less than eight in its multiple-power series.

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APPENDIX

By twice using the recurrence formula for Legendre polynomials on each of $P_n(x)$ and $P_n(y)$, one can show that

$$(2n+1)(x^2-y^2)P_n(x)P_n(y)=(n+1)(n+2)/(2n+3)[P_{n+2}(x)P_n(y)-P_{n+2}(y)P_n(x)]$$
$$-(n-1)n/(2n-1)[P_n(x)P_{n-2}(y)-P_n(y)P_{n-2}(x)].$$

From this expression one obtains immediately

$$\sum_{n=0,\,\mathrm{even}}^{N}(2n+1)(x^2-y^2)P_n(x)P_n(y)=(N+1)(N+2)/(2N+3)[P_{N+2}(x)P_N(y)-P_{N+2}(y)P_N(x)].$$

[6] That the integrands of M_{n+1}'s meet the requirements needed for the validity of Eqs. (34), (35), and (36) is assured by the requirement of finite n in Eq. (1)—the $F(x,y)$ can be found in every case.

We shall now use this summation to show that

$$\underset{N\to\infty}{L}\int_0^1 dx\int_0^x dy \sum_{n=0,\text{even}}^{N}(2n+1)(x^2-y^2)P_n(x)P_n(y)F(x,y)=0 \tag{I}$$

provided that $F(x,y)$ is an even-multiple power series of x and y with only finite powers of either occurring. So

$$F(x,y)=\sum_{\mu}^{M}\sum_{\lambda}^{\Lambda} a_{\lambda,\mu}x^{2\mu}y^{2\lambda}\qquad \begin{matrix} M,\Lambda & \text{both finite} \\ \mu,\lambda & \text{positive, integral.}\end{matrix}$$

Use the above summation in this integrand, and consider one of the two terms of the summation

$$\frac{(N+1)(N+2)}{2N+3}\int_0^1 dx\, P_{N+2}(x)\int_0^x dy\, P_N(y)F(x,y).$$

Express $F(x,y)$ in terms of Legendre polynomials

$$F(x,y)=\sum_l\sum_m A(l,m)P_{2l}(x)P_{2m}(y),$$

and consider the contribution of one of these terms to the last integral above

$$\frac{(N+1)(N+2)}{2N+3}A(j,k)\int_0^1 dx\, P_{N+2}(x)P_{2j}(x)\int_0^x dy\, P_N(y)P_{2k}(y).$$

Remove the products of polynomials by using the summation formula, and consider one of the terms arising

$$\frac{(N+1)(N+2)}{2N+3}A(j,k)B\int_0^1 dx\, P_{\mathfrak{N}}(x)\int_0^x dy\, P_{\mathfrak{N}'}(y).$$

Perform the y integration

$$\frac{(N+1)(N+2)}{2N+3}A(j,k)B\frac{1}{2\mathfrak{N}'+1}\int_0^1 dx\, P_{\mathfrak{N}}(x)[P_{\mathfrak{N}'+1}(x)-P_{\mathfrak{N}'-1}(x)].$$

We now note that (1) $A(j,k)$ is independent of N and is some finite number, (2) B is a number arising from products $P_{N+2}(x)P_{2j}(x)$ and $P_N(y)P_{2k}(y)$, and for large N is of zero order[7] in N, and (3) $\mathfrak{N}$ and $\mathfrak{N}'$ are both even numbers, and both are nearly equal[7] to N. As

$$\underset{N\to\infty}{L}\int_0^1 dx\, P_{\mathfrak{N}}(x)P_{\mathfrak{N}'+1}(x)$$

behaves like N^{-1} as $N\to\infty$, each of the two terms of the integral being considered becomes zero in the limit.

We have considered now only two of many terms of the original integral; these terms have a zero limit. In each of the four cases where we have considered a single term of a number of terms, the number of terms was finite—i.e., independent of N. We can thus add all terms to show that

$$\underset{N\to\infty}{L}\int_0^1 dx\int_0^x dy \sum_{n=0,\text{even}}^{N}(2n+1)(x^2-y^2)P_n(x)P_n(y)F(x,y)\to \underset{N\to\infty}{L}N^{-1}$$

which establishes Eq. (I).

We shall now use Eq. (I) to show that

$$\underset{N\to\infty}{L}\int_0^1 dx\int_0^x dy \sum_{n=0,\text{even}}^{N}(2n+1)n(n+1)P_n(x)P_n(y)(x^2-y^2)^3Q(x,y)=0 \tag{II}$$

provided $Q(x,y)$ satisfies the demands placed on $F(x,y)$ in Eq. (I).

[7] The demand for finite powers of x and y in $F(x,y)$ allows one to choose N so large that $2k$ and $2j$ are both quite small compared with N.

Rewrite the left side of Eq. (II) as

$$-\int_0^1 dx \int_0^x dy \sum_{n=0,\text{even}}^{N} (2n+1)P_n(x)\frac{d}{dy}\left[(1-y^2)\frac{d}{dy}P_n(y)\right](x^2-y^2)^3Q(x,y)$$

Integrate once by parts (on y)

$$+\int_0^1 dx \int_0^x dy \sum_{n=0,\text{even}}^{N} (2n+1)P_n(x)\frac{d}{dy}P_n(y)(1-y^2)\frac{d}{dy}[(x^2-y^2)^3Q(x,y)],$$

and now a second time, to obtain

$$\int_0^1 dx \int_0^x dy \sum_{n=0,\text{even}}^{N} (2n+1)n(n+1)P_n(x)P_n(y)(x^2-y^2)^3Q(x,y)$$
$$=-\int_0^1 dx \int_0^x dy \sum_{n=0,\text{even}}^{N} (2n+1)P_n(x)P_n(y)\frac{d}{dy}\left\{(1-y^2)\frac{d}{dy}[(x^2-y^2)^3Q(x,y)]\right\}.$$

The integrand on the right has a factor $(x^2-y^2)-(x^2-y^2)^3$—is differentiated at most two times—and

$$(d/dy)\{(1-y^2)(d/dy)[(x^2-y^2)^3Q(x,y)]\}$$

can be written as $(x^2-y^2)\cdot F(x,y)$ for some $F(x,y)$ satisfying the demands on $F(x,y)$ of Eq. (I). Thus Eq. (II) is established.

We finally use Eq. (II) to establish

$$\underset{N\to\infty}{L} \int_0^1 dx \int_0^x dy \sum_{n=0,\text{even}}^{N} (2n+1)n(n+1)(n-1)(n+2)P_n(x)P_n(y)(x^2-y^2)^5S(x,y)=0, \qquad \text{(III)}$$

provided that $S(x,y)$ satisfies the demands placed on $F(x,y)$ in Eq. (I).

Replace $(n-1)(n+2)$ by $n(n+1)-2$ and note that

$$\int_0^1 dx \int_0^x dy \sum_{n=0,\text{even}}^{N} (2n+1)n(n+1)(n)(n+1)P_n(x)P_n(y)(x^2-y^2)^5S(x,y)$$

has the same value as the left side of Eq. (III): the difference between the two is shown to be zero by Eq. (II).

Repetition of the partial integrations performed in establishing Eq. (II) for this last integral shows that

$$\int_0^1 dx \int^x dy \sum_{n=0,\text{even}}^{N} (2n+1)n(n+1)n(n+1)P_n(x)P_n(y)(x^2-y^2)^5S(x,y)$$
$$=-\int_0^1 dx \int_0^x dy \sum_{n=0,\text{even}}^{N} (2n+1)n(n+1)P_n(x)P_n(y)\frac{d}{dy}\left\{(1-y^2)\frac{d}{dy}[(x^2-y^2)^5S(x,y)]\right\}.$$

The integrand on the right has a factor $(x^2-y^2)^3$ and

$$(d/dy)\{(1-y^2)(d/dy)[(x^2-y^2)^5S(x,y)]\}$$

can be written as $(x^2-y^2)^3Q(x,y)$ for some $Q(x,y)$ satisfying the demands on $Q(x,y)$ of Eq. (II). We thus have established the validity of Eq. (III).