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Semiphenomenological Analysis of the Process $p+p \rightarrow d+\pi^+$ near Threshold*

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A semiphenomenological calculation of the amplitudes for the process $p+p\rightarrow d+\pi^+$ near threshold is presented, using phenomenological nuclear wave functions. The meson is produced by the usual static p-wave interaction and the resulting Galilean invariant s-wave interaction. P-wave and s-wave rescattering of the meson are included. Agreement with experiment is obtained for both p-wave production amplitudes. The Galilean-invariant interaction is insufficient to yield the observed s-wave production but the contribution of the s-wave rescattering, while not well determined, is of sufficient magnitude.

I. INTRODUCTION

THE production of pions near threshold by the reaction $p+p\rightarrow d+\pi^+$ can proceed only from the initial diproton states ${}^1S_0$, ${}^1D_2$ (producing a p-wave meson) and ${}^3P_1$ (s-wave meson). If the energy dependence of $\eta^{(2l+1)/2}$ (where η is the center-of-mass momentum of the pion in units of its mass, and l the angular momentum of the pion with respect to the center of mass) predicted by the phenomenological model[1] is assumed for the production amplitudes, the specification of five experimental parameters suffices to determine these three complex amplitudes up to a common phase factor. This number of parameters has been determined experimentally (the cross section, angular distribution, excitation function, and pion asymmetry using a polarized proton beam,[2] and the deuteron polarization using an unpolarized beam.[3] An ambiguity which remains in the determination of the relative phases[3] is removed by comparison with the phase-shift analysis of the 310-Mev proton-proton scattering data,[4] since the scattering phases are related to the relative phases in the production process according to[5]

$$\begin{aligned}\tau_0&=\delta_0-\delta_2+n\pi,\\ \tau_1&=\delta_1-\delta_2+(n'+\tfrac{1}{2})\pi,\end{aligned}\tag{1}$$

where τ_0, τ_1 are the relative phases of the ${}^1S_0$ and ${}^3P_1$ production amplitudes, $a({}^1S_0)$ and $a({}^3P_1)$, to the ${}^1D_2$ amplitude $a({}^1D_2)$, respectively; δ_0, δ_1, and δ_2 are the proton-proton phase shifts for the initial states, and n and n' are integers. The resulting experimental values of the production amplitudes are

$$\begin{aligned}a({}^1S_0)&=(0.60\pm0.2)\exp[(2.6_{-0.2}^{+0.4})i]\eta^{\frac{3}{2}}\text{ mb}^{\frac{1}{2}},\\ a({}^1D_2)&=(1.93\pm0.1)\eta^{\frac{3}{2}}\text{ mb}^{\frac{1}{2}},\\ a({}^3P_1)&=(0.74\pm0.04)e^{(1.15\pm0.2)i}\eta^{\frac{1}{2}}\text{ mb}^{\frac{1}{2}}.\end{aligned}\tag{2}$$

These amplitudes are defined as in Mandl and Regge,[6] so that the cross section is

$$\begin{aligned}4\pi d\sigma/d\Omega
&=\tfrac{1}{4}(\{|a({}^1S)|^2+\tfrac{1}{2}|a({}^1D)|^2+\sqrt{2}\mathrm{Re}[a({}^1S)^*a({}^1D)]\\
&\quad+|a({}^3P_1)|^2\}+\{\tfrac{1}{2}|a({}^1D)|^2-\sqrt{2}\mathrm{Re}[a({}^1S)^*a({}^1D)]\}\\
&\quad\times3\cos^2\theta-P\sin\theta\cos\varphi\{\sqrt{2}\,\mathrm{Im}[a({}^1S)^*a({}^3P_1)]\\
&\quad+\mathrm{Im}[a({}^1D)^*a({}^3P_1)]\}).\end{aligned}\tag{3}$$

Only a moderate degree of success has hitherto met attempts to obtain these numbers from theory. Lichtenberg,[7] treating the nuclear interactions by means of phenomenological nucleon-nucleon potentials and using the p-wave interaction for the production of the meson, calculated the p-wave production amplitudes [$a({}^1S_0)$ and $a({}^1D_2)$], obtaining $a({}^1S_0)\approx0$ and, when rescattering of the pion in a $\frac{3}{2}$, $\frac{3}{2}$ state off the second nucleon was included,[8] a magnitude of $2.3\eta^{\frac{3}{2}}$ for $a({}^1D_2)$.[9] Geffen,[10] using a similar method, was able to fit the angular distribution and excitation function including the s-wave production by introducing an interaction linear in $\varphi(x)$ to produce the s-wave mesons. Rescattering of the meson was not included. The p- and s-wave coupling constants were treated as independent parameters; agreement with the experimental differential cross section was obtained with a value for the p-wave constant greater than the one obtained from scattering and photoproduction (i.e., $f^2\approx0.08$), and an s-wave constant smaller by a factor of order μ/M as would be

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[1] A. H. Rosenfeld, Phys. Rev. **96**, 130 (1954).

[2] F. S. Crawford and M. L. Stevenson, Phys. Rev. **97**, 1305 (1955).

[3] R. D. Tripp, Phys. Rev. **102**, 862 (1956).

[4] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

[5] M. Gell-Mann and K. M. Watson, Ann. Rev. Nuclear Sci. **4**, 219 (1954).

[6] F. Mandl and T. Regge, Phys. Rev. **99**, 1478 (1955).

[7] D. B. Lichtenberg, Phys. Rev. **100**, 303 (1955).

[8] D. B. Lichtenberg, Phys. Rev. **105**, 1084 (1957). See also B. Durney, Proc. Phys. Soc. London **71**, 654 (1958).

[9] This was calculated from Lichtenberg's graphs for an energy of 340 Mev, as used in the present calculation. Lichtenberg omitted charge-exchange scattering in the rescattering process, to simulate higher order effects which presumably suppress π^0 production, at least on the energy shell [D. Lichtenberg (private communication)]. If the charge-exchange scattering is included, as in the present calculation, Lichtenberg's $a({}^1D)$ would be $2.7\eta^{\frac{3}{2}}$.

[10] D. A. Geffen, Phys. Rev. **99**, 1534 (1955).

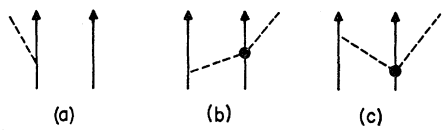

FIG. 1. Types of processes considered in this work. The matrix elements are taken between phonomenological nuclear wave functions.

expected. However, the relative magnitudes of $a(^1D_2)$ and $a(^1S_0)$ and the relative phases obtained from the nuclear potential used by Geffen do not agree with the relations (2).

We have found it possible, by extending the method used by these authors, to fit the experimental data except for the magnitude of the s-wave production amplitude $a(^3P_1)$; our ignorance of the proper functional form of the s-wave pion scattering matrix off the energy shell[11] prevents us from making a definite calculation for the magnitude of $a(^3P_1)$. No undetermined parameters are used with which to fit the meson production; we take $f^2=0.08$ and cutoff the momentum integrals at a momentum of 6μ.[12] The "Galilean-invariant" s-wave interaction proves to give too little s-wave production; s-wave rescattering of the pion off the second nucleon can give a contribution of sufficient magnitude. Rescattering in the small p-wave states is included as well as the static p-wave production term and the $\frac{3}{2}$, $\frac{3}{2}$ rescattering. Two-meson diagrams in which the "rescattering" occurs before the linear interaction [Fig. 1, diagram (c)] are included. The D state of the deuteron wave function (obtained from the Gartenhaus potential[13]) proves to be important. Substantially the correct relative phases are ensured by using the Gammel-Thaler potential[14] to obtain the initial-state functions. The calculation was performed for a laboratory energy of 340 Mev.

II. THEORETICAL FORMULATION

We shall first derive a formal expression for the transition matrix element $(1|T|0)$ relevant to our problem,[15] where $|0)$ denotes a state with two free nucleons, and $|1)$ with two free nucleons and one meson. The nuclear interaction is assumed to be contained in T. The initial-state interaction is separated out by writing

$$(1|T|0)=(1|T|0)'[1+(1/a)(0|T|0)], \tag{4}$$

where $a=E-H_0+i\epsilon$, E is the total energy and H_0 the free-particle Hamiltonian, and the prime on $(1|T|0)'$ means that all processes containing intermediate states with no mesons are excluded. $(0|T|0)$ is the nucleon-nucleon scattering transition matrix element, so that

$$(1|T|0)=(1|T|\psi_i^{(+)})', \tag{5}$$

where

$$|\psi_i^{(+)})=|0)+(1/a)|0)(0|T|0), \tag{6}$$

is the initial outgoing-wave state including the nuclear interaction. If we make the approximation that the interaction of the produced meson with the nucleons occurs and is completed before the final-state nuclear interaction, we may write

$$\begin{aligned}(1|T|\psi_i^{(+)})' &= [1+(1|T|1)^N(1/a)](1|T|\psi_i^{(+)})'' \\ &= (\psi_f,\pi|T|\psi_i^{(+)})'',\end{aligned} \tag{7}$$

where in $(1|T|1)^N=(0|T|0)$, the real meson does not interact, and $(1|T|\psi_i^{(+)})''$ means that all processes containing intermediate states with no virtual mesons are excluded. $|\psi_f,\pi)$ is the final state including the nucleon-nucleon interaction but with a free meson. Making the further approximation that only one scattering-type interaction of the meson off the second nucleon occurs (i.e., no multiple rescattering),[16] we obtain finally

$$\begin{aligned}(1|T|0)=(\psi_f,\pi|H'|\psi_i^{(+)})&+(\psi_f,\pi|T_{\pi N}|1)'' \\ &\times(1/a)(1|H'|\psi_i^{(+)}) \\ &+(\psi_f,\pi|H'|2)(1/a)(2|T_{\pi N}|\psi_i^{(+)})'',\end{aligned} \tag{8}$$

corresponding to the diagrams in Fig. 1 taken between the exact nuclear wave functions, which we derive from phenomenological nuclear potentials.

The interaction Hamiltonian used in this work is

$$\begin{aligned}H'=(4\pi)^{\frac{1}{2}}(f/\mu)i\boldsymbol{\sigma}\cdot\nabla\tau\cdot\varphi(x)&+(4\pi)^{\frac{1}{2}}(f/\mu)(i/2M) \\ \times\{\boldsymbol{\sigma}\cdot\mathbf{p},\,\tau\cdot\pi(x)\}&+\lambda_0 4\pi(f/\mu)^2\varphi^2(x) \\ &+\lambda 4\pi(f/\mu)^2\tau\cdot\varphi(x)\times\pi(x),\end{aligned} \tag{9}$$

μ is the meson, and M the nucleon mass; $\varphi(x)$ is the meson field and $\pi(x)=\dot{\varphi}(x)$ the conjugate field; $\{a,b\}=ab+ba$; $\mathbf{p}$ is the relative momentum operator; λ_0 and λ are constant parameters, defined below; and τ, φ, and π are written as vectors in isotopic spin space. This form for the Hamiltonian can be obtained as an approximation to the pseudoscalar interaction by a Foldy-Dyson transformation.[17] The first term is the usual p-wave interaction, the second the corresponding s-wave recoil term ("Galilean invariant" interaction), and the last two are the pair theory and isotopic-spin dependent quadratic s-wave interactions, respectively. Direct production [diagram (a)] involves the first

[11] See in particular C. J. Goebel, Phys. Rev. **101**, 468 (1956). It may be noted that the most recent Russian data [as reported by B. Pontecorvo, Ninth Annual International Conference on High-Energy Physics, Kiev, 1959 (unpublished)] approach the energy dependence of the Born approximation expression (17) more closely than the previous data indicated; in particular, α, appears to increase linearly with momentum up to above 300 Mev and the magnitude of α_3 to increase more rapidly than linearly.

[12] See footnote 20.

[13] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).

[14] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).

[15] We follow the method used by F. Zachariasen [Phys. Rev. **101**, 371 (1956)] in treating the photodisintegration of the deuteron.

[16] The effects of rescattering off the initial nucleon are presumed to be contained in the coupling constant.

[17] F. Dyson, Phys. Rev. **73**, 929 (1948); L. Foldy and S. Wouthuysen, Phys. Rev. **78**, 29 (1950); L. Foldy, Phys. Rev. **84**, 169 (1951); J. M. Berger, L. L. Foldy, and R. K. Osborn, Phys. Rev. **87**, 1061 (1952).

two terms only; the s-wave rescattering is due almost exclusively to the last two. λ_0, λ were fitted to the zero energy s-wave scattering data[18] in Born approximation, yielding[19]

$$\lambda_0=0.104\mu,\quad \lambda=0.58. \tag{10}$$

The relative-coordinate nuclear wave functions have the form

$$\psi_{in}^{(+)}(r)=\left(\frac{\sqrt{2}}{pr}\right)\{[u_0e^{i\delta_0}-5u_2\tfrac{1}{2}(3(\hat{p}\cdot\hat{r})^2-1)e^{i\delta_2}]\chi_s +i\tfrac{3}{2}u_1\mathbf{S}\cdot\hat{r}\mathbf{S}\cdot\hat{p}e^{i\delta_1}\chi\}\varphi_1, \tag{11}$$

$$\psi_f(r)=\frac{1}{(4\pi r_D)^{\frac{1}{2}}}\left(\frac{1}{r}\right)\left[u+\frac{1}{2\sqrt{2}}S_{12}w\right]\chi\varphi_s$$

p is the initial relative momentum, $\chi_s(\chi)$ the singlet (triplet) spin function, $\varphi_s(\varphi_1)$ the corresponding isospin function, and $\mathbf{S}=\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$. The diproton radial wave functions were obtained numerically from the Gammel-Thaler potential[14] and normalized to

$$u_l(r)\underset{r\rightarrow\infty}{\longrightarrow}\sin[pr-(l\pi/2)+\delta_l], \tag{12}$$

while the Gartenhaus deuteron wave functions[13] are normalized according to the relation

$$\int[u^2+w^2]dr=r_D=4.315\times10^{-13}\text{ cm}. \tag{13}$$

In obtaining the matrix elements of T [Eq. (6)] integrals proportional to q^2 and higher powers were neglected, and only the $j_0(qr/2)$ component of the retardation factor from the meson wave function was included. In general, this approximation is not necessarily any better than the replacement of j_0 by 1, but in the one important integral in which it makes a large difference, the direct p-wave production term in $a(^1S)$ [term (A_1)], inclusion of the factor j_0 gives the exact expression (except for a transition to the deuteron D state) and is preferred. The resulting expressions are

$$(1|T|0)=-4\pi\frac{f}{\mu}\frac{1}{(2\omega_q r_D)^{\frac{1}{2}}}\frac{2}{p}\Bigg\{\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\mathbf{q}\sum_{i=1}^{11}A_ie^{i\delta_0} -\frac{1}{p^2}\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot(\mathbf{q}-3\mathbf{pp}\cdot\mathbf{q})\sum_{i=1}^{13}B_ie^{i\delta_2}+\frac{\omega_q}{M}\frac{1}{p}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{p}\sum_{i=1}^{6}C_ie^{i\delta_1}\Bigg\}, \tag{14}$$

$$A_1=\int dr\,uj_0u_0,$$

$$A_2=-\frac{\omega_q}{6M}\left[\int dr\,u'rj_0u_0-\sqrt{2}\int dr\,w'rj_0u_0\right],$$

$$A_3=\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}\frac{k}{q}\sqrt{2}\int dr\,wj_2(kr)j_0u_0T_{33,13}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$A_4=\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{qM}\frac{1}{\sqrt{2}}\int dr\,w'j_1(kr)j_0u_0T_{33,13}^{\pi},$$

$$A_5=\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}\frac{k}{q}\int dr\,uj_0(kr)j_0u_0T_{31,11}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$A_6=-\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{qM}\int dr\,u'j_1(kr)j_0u_0T_{31,11}^{\pi},$$

$$A_7=\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{M}\frac{1}{2}\int dr\,uj_0(kr)j_0u_0T_{3,1}^{\pi},$$

$$A_8=\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}k\frac{1}{6}\int dr\,urj_1(kr)j_0u_0T_{3,1}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$A_9=-\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}k\frac{1}{3\sqrt{2}}\int dr\,wrj_1(kr)j_0u_0T_{3,1}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$A_{10}=\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{M}\frac{1}{6}\int dr\,u'rj_0(kr)j_0u_0T_{3,1}^{\pi},$$

[18] *1958 Annual International Conference on High-Energy Physics at* CERN, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958), Sec. II.

[19] A similar calculation was included in A. Klein, Phys. Rev. **99**, 998 (1955).

$$A_{11}=-\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{M}\frac{1}{3\sqrt{2}}\int dr\, w'rj_0(kr)j_0u_0T_{3,1}{}^{\pi},$$

$$B_1=(1/\sqrt{2})\int dr\, wj_0u_2,$$

$$B_2=(\omega_q/6M)\left[\int dr\, u'rj_0u_2-\sqrt{2}\int dr\, w'rj_0u_2\right],$$

$$B_3=\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}\frac{k}{q}\int dr\, uj_2(kr)j_0u_2T_{33,13}{}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$B_4=-\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}\frac{k}{q}\frac{1}{\sqrt{2}}\int dr\, wj_2(kr)j_0u_2T_{33,13}{}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$B_5=\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{qM}\frac{1}{2}\int dr\, u'j_1(kr)j_0u_2T_{33,13}{}^{\pi},$$

$$B_6=-\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{qM}\frac{1}{2\sqrt{2}}\int dr\, w'j_1(kr)j_0u_2T_{33,13}{}^{\pi},$$

$$B_7=\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}\frac{k}{q}\frac{1}{\sqrt{2}}\int dr\, wj_0(kr)j_0u_2T_{13,11}{}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$B_8=-\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{qM}\frac{1}{\sqrt{2}}\int dr\, w'j_1(kr)j_0u_2T_{13,11}{}^{\pi},$$

$$B_9=\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{M}\frac{1}{2\sqrt{2}}\int dr\, wj_0(kr)j_0u_2T_{3,1}{}^{\pi},$$

$$B_{10}=-\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}k\frac{1}{6}\int dr\, urj_0(kr)j_0u_2T_{3,1}{}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$B_{11}=\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_q}{\omega_k}\right)^{\frac{1}{2}}k\frac{\sqrt{2}}{6}\int dr\, wrj_1(kr)j_0u_2T_{3,1}{}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$B_{12}=-\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{M}\frac{1}{6}\int dr\, u'rj_0(kr)j_0u_2T_{3,1}{}^{\pi},$$

$$B_{13}=\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{\frac{1}{2}}\frac{1}{M}\frac{\sqrt{2}}{6}\int dr\, w'rj_0(kr)j_0u_2T_{3,1}{}^{\pi},$$

$$C_1=\int dr\, u'j_0u_1,$$

$$C_2=(1/\sqrt{2})\int dr\, w'j_0u_1,$$

$$C_3=\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_k}{\omega_q}\right)^{\frac{1}{2}}\int dr\, u'j_0(kr)j_0u_1T_{3,1}{}^{\pi},$$

$$C_4=\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{-\frac{1}{2}}Mk\int dr\, uj_1(kr)j_0u_1T_{3,1}{}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

$$C_5=\int\frac{k^2dk}{2\pi^2}\left(\frac{\omega_k}{\omega_q}\right)^{\frac{1}{2}}\frac{1}{\sqrt{2}}\int dr\, w'j_0(kr)j_0u_1T_{3,1}{}^{\pi},$$

$$C_6=\int\frac{k^2dk}{2\pi^2}(\omega_q\omega_k)^{-\frac{1}{2}}Mk\frac{1}{\sqrt{2}}\int dr\, wj_1(kr)j_0u_1T_{3,1}{}^{\pi}\left(1-\frac{\omega_k}{2M}\right),$$

where

$$u'=r(d/dr)(u/r),\quad w'=r[(d/dr)(w/r)+3w/r^2],$$

the spherical Bessel function j_0 with its argument suppressed is $j_0(qr/2)$, and

$$T_{a,b}{}^{\pi}=-(\tfrac{4}{3}t_a-\tfrac{1}{3}t_b')\left[\frac{1}{\omega_k-\omega_q/2}\pm\frac{1}{\omega_k+\omega_q/2}\right],\quad (15)$$

t_a being the appropriate meson-nucleon scattering matrix. The $\pm$ sign in (15) is determined by the functional form of the scattering matrix t, as discussed below. The kinetic energy of the fast nucleon, $p^2/2M\approx\omega_q/2$, has been included in the intermediate state energy. The terms (A_1), (B_1) in (14) are direct p-wave production terms; (A_2), (B_2), (C_1), and (C_2) are direct production by the Galilean-invariant s-wave interaction, giving rise in the case of (A_2) and (B_2) to p-wave production relative to the center of mass. The terms with the factor $\omega_k/2M$ represent rescattering of a meson produced by the s-wave interaction at the other nucleon. Lichtenberg[7,8] calculated (A_1), (B_1), and the (b) type term of (B_2) using the static denominator $(\omega_q-\omega_k+i\epsilon)$, without the retardation factor $j_0(qr/2)$ and without the $I=\frac{1}{2}$, $J=\frac{3}{2}$ rescattering or the rescattering term in $\omega_k/2M$.

The p-wave rescattering matrices t_a' are obtained by using Gammel's Fredholm solution approximation[20] to the scattering integral equation:

$$t_a'(q,k)=v_a(q,k)+\int\frac{l^2dl}{2\pi^2}v_a(q,l)\frac{1}{E-\omega_l+i\epsilon}t_a'(l,k)$$
$$=v_a(q,k)\Delta_a(q,k),$$
$$v_a(q,k)=\frac{\lambda_a}{3}4\pi\left(\frac{f}{\mu}\right)^2\frac{qk}{(4\omega_q\omega_k)^{\frac{1}{2}}}\frac{1}{E-\omega_q-\omega_k}$$
$$\lambda_a=\begin{cases}4 & a=33 \quad (I=\tfrac{3}{2},J=\tfrac{3}{2})\\ -2 & a=13,31\\ 1 & a=11\end{cases}\quad (16)$$
$$\Delta_a(q,k)=\frac{1+\lambda_a\Delta_I(\omega_k)-\lambda_a\Delta_F}{1-\lambda_a\Delta_F},$$
$$\Delta_I(\omega_k)=\tfrac{1}{3}4\pi\left(\frac{f}{\mu}\right)^2\frac{\omega_k}{2}\int\frac{l^2dl}{2\pi^2}\frac{l^2}{\omega_l^2(\omega_l-\mu)(\omega_l+\omega_k-\mu)},$$
$$\Delta_F=\tfrac{1}{3}4\pi\left(\frac{f}{\mu}\right)^2\frac{1}{2}\int\frac{l^2dl}{2\pi^2}\frac{l^2}{\omega_l(2\omega_l-\mu)(\omega_l-\mu)}.$$

[20] J. L. Gammel, Phys. Rev. **95**, 209 (1954). The cutoff used here was $l_{\max}=7\mu$, in order to fit the pion scattering resonance with $f^2=0.08$. See Fig. 1 of Gammel's article.

We assume $\lambda_{11}=+1$ (not -8) because the uncrossed term is omitted in calculating t_{11}'. t_{11}' contributes very little to the meson production. For the s-wave rescattering matrices the Born approximation using the quadratic terms in (9) [with the coefficients (10)] was used,[21] giving

$$t_a=t_a'=\frac{1}{(\omega_q\omega_k)^{\frac{1}{2}}}4\pi\left(\frac{f}{\mu}\right)^2[\lambda_0+\Gamma_a\lambda(\omega_q+\omega_k)]\quad (17)$$
$$\Gamma_a=-1\quad a=1\quad (I=\tfrac{1}{2})$$
$$=\tfrac{1}{2}\quad a=3\quad ,$$

t_a in (17) is fitted to the zero energy s-wave scattering data. Less phenomenological treatments of the s-wave pion scattering in the Hamiltonian formalism have been unable to fit these data.[11] Therefore, we have also used an alternative approach, using simple, symmetric scattering matrices giving a linear dependence of phase shift on momentum:

$$t_a{}^L=-\frac{2\pi}{(\omega_q\omega_k)^{\frac{1}{2}}\mu}\frac{\alpha_a}{\eta}$$
$$=-\frac{2\pi}{(\omega_q\omega_k)^{\frac{1}{2}}\mu}\times\begin{cases}0.173 & a=1\\ -0.110 & a=3\end{cases}.\quad (18)$$

The factor k (ω_k) which appears linearly in the direct p- (or s-) wave production interaction changes sign when the virtual meson is absorbed [diagram (c)] instead of being created [diagram (b)]. The similar factor k in the p-wave rescattering (16), as well as the $\lambda\omega_k$ term in the s-wave rescattering (17), also changes sign, so that for these terms the diagrams (b) and (c) interfere constructively [+ sign in (15)]. The remaining terms in λ_0 and $\lambda\omega_q$ in (17) do not change sign, so that for these terms the negative sign is appropriate in (15). For consistency with crossing symmetry,[22] the isotopic-spin dependent part of the scattering matrices defined in (18) for physical values of ω must be odd in $\omega=\omega_q=\omega_k$. If we assume that (18) is an approximation to the functional form

$$t_a{}^L(q,k)=\frac{1}{(\omega_q\omega_k)^{\frac{1}{2}}}4\pi\left(\frac{f}{\mu}\right)^2$$
$$\times\{\lambda_0+\Gamma_a(\lambda/2f(\mu))[f(\omega_q)+f(\omega_k)]\},\quad (19)$$

it follows that $f(-\omega)=-f(\omega)$, so that as a rough approximation for diagram (c) we take

$$t_a{}^L=\frac{1}{(\omega_q\omega_k)^{\frac{1}{2}}}4\pi\left(\frac{f}{\mu}\right)^2\lambda_0\approx 0.$$

[21] The Born approximation using the crossed diagram with the Galilean-invariant interaction is much smaller than the contribution of the quadratic interactions.

[22] See for example C. J. Goebel, Phys. Rev. **101**, 468 (1956), p. 478.

III. NUMERICAL RESULTS AND CONCLUSIONS

The integrals in (12) were evaluated on the IBM 650 computer at the University of Rochester. The matrix elements in (14) must be multiplied by the factor $\{16\pi(2\pi/v)[\rho_E/(2\pi)^3]\}^{\frac{1}{2}}$ to obtain the amplitudes a. If s-wave rescattering is not included at all, we obtain[23]

$$\begin{aligned} a(^1S) &= 0.76\eta^{\frac{3}{2}}e^{2.65i}\ (\mathrm{mb}^{\frac{1}{2}}), \\ a(^1D) &= 1.70\eta^{\frac{3}{2}}, \\ a(^3P_1) &= 0.13\eta^{\frac{1}{2}}e^{0.77i}. \end{aligned} \tag{20}$$

If the fitted Born approximation (17) is used for the s-wave rescattering,

$$\begin{aligned} a(^1S) &= 0.62\eta^{\frac{3}{2}}e^{2.65i}, \\ a(^1D) &= 1.75\eta^{\frac{3}{2}}, \\ a(^3P_1) &= 1.18\eta^{\frac{1}{2}}e^{0.77i}, \end{aligned} \tag{21}$$

while with the linear approximation (19)

$$\begin{aligned} a(^1S) &= 0.70\eta^{\frac{3}{2}}e^{2.65i}, \\ a(^1D) &= 1.73\eta^{\frac{3}{2}}, \\ a(^3P_1) &= 0.51\eta^{\frac{1}{2}}e^{0.77i}. \end{aligned} \tag{22}$$

TABLE I. Partial coefficients of the appropriate power of η appearing in the moduli of the amplitudes a, contributed by each partial wave term. The units are $\mathrm{mb}^{\frac{1}{2}}$, as in the text.

Direct and p-scattering processes			S-scattering processes: Fitted born approx	S-scattering processes: Linear approx
A_1	0.223	A_7	−0.042	−0.026
A_2	0.014	A_8	−0.159	−0.054
A_3	0.384	A_9	0.065	0.021
A_4	0.266	A_{10}	−0.007	−0.002
A_5	−0.134	A_{11}	~0	~0
A_6	0.010			
B_1	1.042	B_9	~0	~0
B_2	−0.190	B_{10}	0.111	0.061
B_3	1.308	B_{11}	−0.063	−0.035
B_4	−0.346	B_{12}	~0	~0
B_5	−0.014	B_{13}	~0	~0
B_6	−0.099			
B_7	~0	C_3	−0.081	0.000
B_8	~0	C_4	−0.911	−0.459
C_1	−0.285	C_5	−0.069	−0.058
C_2	0.415	C_6	−0.247	−0.120

[23] The phases relative to $a(^1D)$ are listed, as in (2).

The contributions of the individual terms are given in Table I. Aside from the approximations made in the derivation of (8), other sources of error are the (partial) omission of terms in higher powers of q, which, if considered, would affect the larger integrals by less than 15% [except for the term (A_1) discussed above] and would introduce more transitions of comparably small size; the wide spacing of points in the momentum integrals (generally less than 5% error) and their cutoff dependence; the experimental uncertainty in such parameters as $f^2=0.08$; and the approximations made in the meson rescattering matrices, which are, of course, especially crude for the s-wave rescattering. It will be seen that the p-wave amplitudes are in fairly good agreement with the experimental data (2). Concerning $a(^1D)$ reference to Table I shows that the direct production and the $\frac{3}{2}$, $\frac{3}{2}$ rescattering to the deuteron S state [(B_1) and (substantially) (B_3)], which were also calculated by Lichtenberg, contribute the main portion (modulus of 2.34). The $\frac{3}{2}$, $\frac{3}{2}$ rescattering (to the deuteron D state) must be considered to obtain a sufficiently large $a(^1S)$. The small p-wave and s-wave rescattering contributions are also not negligible, since the terms which could be expected to be largest [(A_1), in which the integral is unexpectedly small, and $\frac{3}{2}$, $\frac{3}{2}$ rescattering to the deuteron S state, which is forbidden] do not contribute much.

The Galilean-invariant interaction is too small to give the experimental s-wave production amplitude $a(^3P_1)$, in spite of the moderately high nucleon recoil momenta involved; this occurs because of a cancellation between the transitions to the deuteron S-state (C_1) and D-state (C_2). When the s-wave rescattering is included, our ignorance of the correct functional form for this virtual process makes it impossible to conclude more than that the rescattering can be large enough to yield the experimental value for $a(^3P_1)$.

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