Stopping of π-Minus and K-Minus Mesons on Hydrogen in Nuclear Emulsion*

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We show that the experimentally observed fact that the number of K^- mesons stopping on hydrogen in nuclear emulsion is significantly larger than the number of π^- is consistent with other known properties of these mesons. The reason is chiefly that the zero energy π^--p cross section is anomalously low.

1. INTRODUCTION

EXPERIMENTAL information on the stopping of π^- and K^- mesons in nuclear emulsion reveals a very curious state of affairs. It has been known for some time that the number of π^- mesons stopped on hydrogen was almost undetectible.[1] (These results would appear to correspond to less than 0.03% stopping on hydrogen in emulsion.) This apparent contradiction to the Z law of Fermi and Teller[2] which predicts that the number of mesons stopping on an atom is roughly proportional to the number of atomic electrons has been explained[3] by observing that when a meson is captured by hydrogen, the single atomic electron is ejected and so the mesic hydrogen-atom cannot be de-excited by the rapid Fermi-Teller process of electron ejection. "It can reach the mesic K shell only via optical transitions. Since the probability for optical transitions is quite small,[4] the neutral, mesic hydrogen-atom will generally diffuse around and collide with other atoms before the mesic K shell in hydrogen is reached. During the course of these collisions, there is an excellent chance for the π^- meson to be transferred from hydrogen to a heavier atom. Hence it is not surprising that the Z law for the capture of negative mesons in mixtures is contradicted by an experiment in which hydrogen is one of the elements."[3] In light of this explanation, it seems very curious that a significantly larger number of K^- mesons should stop on hydrogen under similar conditions.[5] It is the purpose of this study to explain this curious phenomenon. We shall show that it is in fact in accord with the known properties of the π^- and K^- mesons.

Our approach is quite straightforward. We simply follow the history of the meson from the time it is moving slowly through the emulsion to the time of its absorption by nuclear interaction. An exact prediction of the fraction of mesons captured on hydrogen would be very difficult because of the complex processes which occur. We shall therefore content ourselves with crude estimates, which will retain only the qualitative features of the results. We shall neglect almost entirely the molecular structure of the emulsion and treat it as a gas of monoatomic particles, and we shall employ first-order perturbation theory throughout. We shall also make the Born-Oppenheimer approximation that the mass of the meson may be neglected with respect to that of the proton. These approximations, while admittedly crude, are not so crude as to obscure the basic physical explanation.

We find that while atomic processes tend to militate against the K^- meson relative to the π^-, the π^- cross section is so much smaller than that of the K^- that many more K^- mesons can be expected to be retained on hydrogen than π^-.

2. INITIAL CAPTURE PHASE

Let us consider the capture of negatively charged mesons into bound states in the Coulomb fields of the various atoms of a nuclear emulsion. The emulsion is composed of gelatin and silver halide crystals. The problem therefore separates into two parts[6]: the particles which stop in gelatin and those which stop in the silver halide crystals. We shall concern ourselves with the former part of the problem. The competition which we must consider is that between hydrogen and the carbon, oxygen, nitrogen group.

We imagine the initial capture process to proceed by the meson being captured by either (i) exciting an electron or (ii) by direct Coulomb capture emitting a gamma ray. At the low energies we are considering, the direct nuclear-capture cross sections will be many orders of magnitude smaller[7] than the capture cross section into Coulomb states, We expect the direct nuclear-capture to be important only "in flight." We must calculate the ratio of the initial capture probabilities for hydrogen and for the carbon, oxygen, nitrogen group, and the distribution of the captured meson in the various hydrogenlike bound-states about the proton. Direct Coulomb capture will be seen to be relatively unimportant compared to the electron-ejection mechanism.

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[1] W. K. H. Panofsky, R. L. Aamodt, and J. Hadley, Phys. Rev. **81**, 565 (1951). See also, W. Spry, Phys. Rev. **95**, 1295 (1954), footnote 8, who obtains a somewhat larger value.

[2] E. Fermi and E. Teller, Phys. Rev. **72**, 399 (1947).

[3] R. E. Marshak, *Meson Physics* (McGraw-Hill Book Company, Inc., New York, 1952).

[4] J. A. Wheeler, Revs. Modern Phys. **21**, 133 (1949).

[5] W. H. Barkas, J. N. Dyer, P. E. Giles, H. H. Heckman, C. J. Mason, N. A. Nickols, and F. M. Smith, Phys. Rev. **112**, 622 (1958).

[6] Page 171, reference 3 above, and also, W. F. Fry, Nuovo cimento **10**, 490 (1953).

[7] One can easily extrapolate the results quoted in reference 3 above and see that at "zero" energy the "total interaction" cross section is quite small compared to the cross section for the processes we are considering.

The method of calculation for electron ejection will be first order perturbation theory, using the exact Coulomb-wave functions. We also make the Born-Oppenheimer approximation that the proton "remains fixed" throughout the interaction. As we have mentioned, while these approximations are definitely approximations, they are not so poor that they will obscure the basic physical results.

3. DIRECT COULOMB CAPTURE

We will now calculate the cross section for direct Coulomb capture into the ground state of a mesic atom. According to Schiff,[8] Section 50, the transition probability in the dipole approximation (here satisfied) is given by

$$w=(4e^2\omega^3/3\hbar c)\,|\,(\mathbf{r})_{n'n}|^2, \tag{3.1}$$

where $(\mathbf{r})_{n'n}$ is the matrix element of the coordinate of the meson in the unperturbed system. From this formula we see that it will be the low-lying states of the system which will be most important because of the factor of ω^3.

As the ground state of the mesic atom lies so far below that of an electronic atom, the meson ground state is essentially a pure hydrogenlike ground state. In large nuclei the finite extension will be important but as direct Coulomb capture is neglectible, we will not consider it.

Approximating the free-state wave function by $L^{-\frac{3}{2}}\exp(i\mathbf{k}\cdot\mathbf{r})$, we may evaluate (3.1) and obtain

$$w=(4e^2\omega^3)[16\pi a_0^2(mZ)^{-5}(a_0/L)^3/\hbar c^3]\times[144K^2(1+K^2)^{-6}], \tag{3.2}$$

where

$$\omega=Z^2e^2m(1+K^2)/(2a_0\hbar),\quad K=a_0k/mZ. \tag{3.3}$$

The cross section is given by

$$\sigma=m_ea_0L^3w/Z\hbar K. \tag{3.4}$$

Thus

$$\sigma/\pi a_0^2=(552K)/[m^2(c\hbar/e^2)^3(1+K^2)^3]. \tag{3.5}$$

The maximum is for $K=(5)^{-\frac{1}{2}}$, we have, for the largest case, that of π^-,

$$\sigma_{\max}/\pi a_0^2=1.34\times10^{-8}. \tag{3.6}$$

This maximum corresponds, for hydrogen, to an energy of the incident π^- meson of about 650 ev. As we shall see, this cross section is quite small compared to those for electron ejection.

4. ELECTRON EXCITATION

We shall treat the case where the electron is completely ejected from a hydrogen atom. Our unperturbed Hamiltonian is (let the subscript 1 denote the meson quantities, and 2 denote electron quantities),

$$H_0=(p_1^2/2m_1)+(p_2^2/2m_2)-(e^2/r_1)-(e^2/r_2), \tag{4.1}$$

and the perturbation is

$$H'=e^2/(|\mathbf{r}_1-\mathbf{r}_2|). \tag{4.2}$$

The masses are, of course, the customary reduced masses. The reason for breaking the total Hamiltonian up in this way is so that there will be eigenstates of H_0 which will closely approximate the bound states of the total system. Further as the meson and electron variables are completely separated in H_0, there can never occur a capture in the system described by it.

The transition rate, w, is then, according to first order perturbation theory,

$$w=(2\pi/\hbar)|H_{km}'|^2\rho(k), \tag{4.3}$$

where $\rho(k)$ is the density of final states. We wish to calculate the probability of the capture of a meson into a hydrogenlike state described by n, l, m (principle quantum number, total angular momentum, z component of angular momentum) from a free state which is the appropriate Coulomb modification of a plane wave, by hydrogen in its ground state.

Our initial conditions are[8–10]

$$\text{electron wave function}=(\pi a_0^3)^{-\frac{1}{2}}\exp(-r_2/a_0), \tag{4.4}$$

$$\text{meson wave function}\ =(L)^{-\frac{3}{2}}C(\mathbf{k}_1,\mathbf{r}_1), \tag{4.5}$$

where

$$C(\mathbf{k}_1,\mathbf{r}_1)=\sum_{l=0}^{\infty}(2l+1)i^lC_l(k_1,r_1)P_l(\cos\theta), \tag{4.6}$$

with

$$\cos\theta=(\mathbf{k}_1\cdot\mathbf{r}_1)/(k_1r_1),$$

and

$$C_l(k_1,r_1)=\frac{\exp(\frac{1}{2}\pi t_1)|\Gamma(l+1+it_1)|(2k_1r_1)^l}{(2l+1)!}\sum_{j=0}^{\infty}\alpha_j(k_1r_1)^j, \tag{4.7}$$

where

$$\alpha_j=-(2t_1\alpha_{j-1}+\alpha_{j-2})/[j^2+(2l+1)j],\quad \alpha_0=1,\quad \alpha_1=-[t_1(l+1)]^{-1},\quad t_1=(m_1e^2)/(\hbar^2k_1). \tag{4.8}$$

Our final conditions are:

$$\text{meson wave function}=\psi_{nlm}(r_1)=R_{nl}(r_1)\Theta_{lm}(\theta_1)\Phi_m(\varphi_1), \tag{4.9}$$

where, if we define

$$\rho_1=(2mr_1)/(na_0),\quad m=m_1/m_2, \tag{4.10}$$

[8] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, New York, 1949).

[9] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953).

[10] G. Breit, *Tables of Coulomb Wave Functions*, U. S. National Bureau of Standards, Applied Mathematics Series–17 (U. S. Government Printing Office, Washington, D. C., 1952), Vol. I.

then

$$R_{nl}(r_1)=-\left((2m/na_0)^3\frac{(n-l-1)!}{2n[(n+l)!]^3}\right)^{\frac{1}{2}}\times\rho_1{}^l L_{n+l}{}^{2l+1}(\rho_1)e^{-\frac{1}{2}\rho_1}, \quad (4.11)$$

$$\Theta_{lm}(\theta_1)=\left(\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}\right)^{\frac{1}{2}}P_l{}^{|m|}(\cos\theta_1), \quad (4.12)$$

$$\Phi_m(\varphi_1)=(2\pi)^{-\frac{1}{2}}\exp(im\varphi_1), \quad (4.13)$$

where the $L_b{}^a$ and $P_b{}^a$ are the associated Laguerre and Legendre polynomials, respectively. The electron wave function is

$$(L)^{-\frac{3}{2}}C(\mathbf{k}_2,\mathbf{r}_2), \quad (4.14)$$

which is the same as $C(\mathbf{k}_1,\mathbf{r}_1)$ except that the subscript 2 instead of 1 appears throughout the definition. We determine k_2 by conservation of energy as

$$k_2=[(m/n^2)-1+(a_0k_1)^2/m]^{\frac{1}{2}}/a_0. \quad (4.15)$$

We may now write down the square of the matrix element of the transition-causing perturbation. It is

$$|H'_{nlm;k_1}|^2=[e^4/(\pi a_0{}^3L^6)]\left|\int\cdots\int C(\mathbf{k}_1,\mathbf{r}_1)\exp(-r_2/a_0)\times(|\mathbf{r}_1-\mathbf{r}_2|)^{-1}C^*(\mathbf{k}_2,\mathbf{r}_2)\psi_{nlm}{}^*(r_1)d^3r_1d^3r_2\right|^2. \quad (4.16)$$

Now, using the well-known expansion of the Coulomb potential in spherical harmonics,[9] we get

$$(|\mathbf{r}_1-\mathbf{r}_2|)^{-1}=4\pi/r_1\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}(r_2/r_1)^{-\frac{1}{2}\pm(\lambda+\frac{1}{2})}\times\Theta_{\lambda\mu}{}^*(\theta_1)\Theta_{\lambda\mu}(\theta_2)\Phi_\mu{}^*(\varphi_1)\Phi(\varphi_2)/(2\lambda+1), \quad (4.17)$$

where the upper sign is used when $r_2<r_1$ and the lower sign otherwise. Let us re-express (4.14) by the addition theorem for Legendre polynomials[9] as

$$C^*(\mathbf{k}_2,\mathbf{r}_2)=(4\pi)\sum_{\nu=0}^{\infty}\sum_{\epsilon=-\nu}^{\nu}(-i)^\nu C_\nu(k_2,r_2)\Theta_{\nu\epsilon}{}^*(\theta_2)\times\Theta_{\nu\epsilon}(\theta_2')\Phi_\epsilon{}^*(\varphi_2)\Phi_\epsilon(\varphi_2'), \quad (4.18)$$

where (θ_2',φ_2') is the direction of $\mathbf{k}_2$. If we multiply (4.17) by (4.18) we obtain,

$$(16\pi^2/r_1)\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\sum_{\nu=0}^{\infty}\sum_{\epsilon=-\nu}^{\nu}(r_2/r_1)^{-\frac{1}{2}\pm(\lambda+\frac{1}{2})}(-i)^\nu\times C_\nu(k_2,r_2)(2\lambda+1)^{-1}\Theta_{\lambda\mu}{}^*(\theta_1)\Theta_{\lambda\mu}(\theta_2)\Phi_\mu{}^*(\varphi_1)\Phi_\mu(\varphi_2)\times\Theta_{\nu\epsilon}{}^*(\theta_2)\Theta_{\nu\epsilon}(\theta_2')\Phi_\epsilon{}^*(\varphi_2)\Phi_\epsilon(\varphi_2'). \quad (4.19)$$

When we integrate over all solid angles in electron space, we get, by orthonormality

$$=(16\pi^2/r_1)\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}(r_2/r_1)^{-\frac{1}{2}\pm(\lambda+\frac{1}{2})}C_\lambda(k_2,r_2)(-i)^\lambda\times(2\lambda+1)^{-1}\Theta_{\lambda\mu}{}^*(\theta_1)\Theta_{\lambda\mu}(\theta_2')\Phi_\mu{}^*(\varphi_1)\Phi_\mu(\varphi_2'). \quad (4.20)$$

Since (4.20) appears as an absolute square, we may square and then average over all final directions of $\mathbf{k}_2$. This operation yields

$$[(64\pi^2e^4)/(a_0{}^3L^6)]\times\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\left|\int_0^\infty r_2{}^2dr_2\int d^3r_1\,C(\mathbf{k}_1,\mathbf{r}_1)R_{nl}(r_1)\times\Theta_{lm}{}^*(\theta_1)\Phi_m(\varphi_1)\Theta_{\lambda\mu}{}^*(\theta_1)\Phi_\mu{}^*(\varphi_1)(2\lambda+1)^{-1}(r_1)^{-1}\times C_\lambda(k_2,r_2)\exp(-r_2/a_0)(r_2/r_1)^{-\frac{1}{2}\pm(\lambda+\frac{1}{2})}\right|^2. \quad (4.21)$$

The cross terms vanish by the orthogonality relations.

If we now re-express (4.6) by the addition theorem for Legendre polynomials we have

$$C(\mathbf{k}_1,\mathbf{r}_1)=(4\pi)\sum_{\nu=0}^{\infty}\sum_{\epsilon=-\nu}^{\nu}i^\nu C_\nu(k_1,r_1)\Theta_{\nu\epsilon}(\theta_1)\Theta_{\nu\epsilon}{}^*(\theta_1')\times\Phi_\epsilon(\varphi_1)\Phi_\epsilon{}^*(\varphi_1'). \quad (4.22)$$

Making this substitution and averaging over all initial meson directions we get

$$[(256\pi^3e^4)/(a_0{}^3L^6)]\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\sum_{\nu=0}^{\infty}\sum_{\epsilon=-\nu}^{\nu}\left|\int_0^\infty r_2{}^2dr_2\times\int_0^\infty r_1{}^2dr_1\,C_\nu(k_1,r_1)R_{nl}(r_1)(r_1)^{-1}\times(r_2/r_1)^{-\frac{1}{2}\pm(\lambda+\frac{1}{2})}C_\lambda(k_2,r_2)\exp(-r_2/a_0)\right|^2\times\left|\int d\Omega_1\,\Theta_{lm}{}^*(\theta_1)\Phi_m{}^*(\varphi_1)\Theta_{\lambda\mu}{}^*(\theta_1)\Phi_\mu{}^*(\varphi_1)\times(2\lambda+1)^{-1}\Theta_{\nu\epsilon}(\theta_1)\Phi_\epsilon(\varphi_1)\right|^2. \quad (4.23)$$

We may evaluate the angular part of (4.23) for, if we sum over μ and ϵ, we have, by the addition theorem for Legendre polynomials,

$$\int\int d\Omega_1d\Omega_1'\,\Theta_{lm}(\theta_1)\Phi_m(\varphi_1)\Theta_{lm}{}^*(\theta_1')\Phi_m{}^*(\varphi_1')\times[(2\lambda+1)/4\pi]P_\lambda(\cos\vartheta)[(2\nu+1)/4\pi]P_\nu(\cos\vartheta), \quad (4.24)$$

where (θ_1',φ_1') now denotes the direction of $\mathbf{r}_1'$ and ϑ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_1'$. According to Whittaker and Watson,[11]

$$P_\lambda(\cos\vartheta)P_\nu(\cos\vartheta)=\sum_{r=0}^{\min(\lambda,\nu)}[(A_{\lambda-r}A_rA_{\nu-r})/A_{\lambda+\nu+r}]\times[(2\lambda+2\nu-4r+1)/(2\lambda+2\nu-2r+1)]\times P_{\lambda+\nu-2r}(\cos\vartheta), \quad (4.25)$$

[11] E. T. Whittaker and G. M. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1927), 4th ed., p. 331, Ex. 11.

where

$$A_m = 1\cdot 3\cdot 5\cdots(2m-1)/m!,\quad A_0=1. \tag{4.26}$$

By the addition theorem for Legendre polynomials we expand (4.25) as

$$\begin{aligned}(4\pi)\sum_{r=0}^{\min(\lambda,\nu)}\sum_{\mu=-\lambda-\nu+2r}^{\lambda+\nu-2r}&[(A_{\lambda-r}A_rA_{\nu-r})/A_{\lambda+\nu-r}]\\ &\times(2\lambda+2\nu-2r+1)^{-1}\Theta_{\lambda+\nu-2r,\mu}(\theta_1)\\ &\times\Theta_{\lambda+\nu-2r,\mu}^*(\theta_1')\Phi_\mu(\varphi_1)\Phi_\mu^*(\varphi_1').\end{aligned} \tag{4.27}$$

If we now integrate over all directions of $\mathbf{r}_1$ and $\mathbf{r}_1'$, we get, by orthonormality for (4.23)

$$\begin{aligned}&(64\pi^2e^4/a_0^3L^6)\\ &\times\sum_{\lambda=0}^{\infty}\sum_{\substack{\nu=|\lambda-l|\\ \text{steps of two}}}^{\lambda+l}(2\nu+1)(2\lambda+1)^{-1}(\lambda+\nu+l+1)^{-1}\\ &\times[(A_{\frac{1}{2}(\lambda-\nu+l)}A_{\frac{1}{2}(\lambda+\nu-l)}A_{\frac{1}{2}(\nu-\lambda+l)})/(A_{\frac{1}{2}(\lambda+\nu+l)})]\\ &\times\left|\int_0^\infty r_2^2dr_2\int_0^\infty r_1^2dr_1\,R_{nl}(r_1)C_\nu(k_1,r_1)(r_1)^{-1}\right.\\ &\qquad\left.\times(r_2/r_1)^{-\frac{1}{2}\pm(\lambda+\frac{1}{2})}C_\lambda(k_2,r_2)\exp(-r_2/a_0)\right|^2.\end{aligned} \tag{4.28}$$

The density of final states will be (see, for instance, Schiff[8]).

$$\rho(k)dE=(L/2\pi)^3 4\pi k_2 m(2l+1)\hbar^{-2}dE, \tag{4.29}$$

which, on substitution for k_2 becomes

$$\begin{aligned}\rho(k)=4\pi(2l+1)&(2\pi)^{-3}(a_0/e^2)(L/a_0)^3\\ &\times[(m/n^2)-1+(a_0k_1)^2/m]^{\frac{1}{2}}.\end{aligned} \tag{4.30}$$

To obtain the partial Coulomb capture cross sections we make use of the relation

$$\sigma=L^3w/v=[(2\pi L^3)/v\hbar]|H'|^2\rho(k). \tag{4.31}$$

Thus

$$\begin{aligned}&\frac{\sigma(n,l;k_1)}{\pi a_0^2}\\ &=64m(k_2/k_1)\sum_{\lambda=0}^{\infty}\sum_{\substack{\lambda=|\lambda-l|\\ \text{steps of two}}}^{\lambda+l}\frac{(2\nu+1)(2l+1)}{(2\lambda+1)(\lambda+\nu+l+1)}\\ &\times[(A_{\frac{1}{2}(\lambda-\nu+l)}A_{\frac{1}{2}(\lambda+\nu-l)}A_{\frac{1}{2}(\nu-\lambda+l)})/(A_{\frac{1}{2}(\lambda+\nu+l)})]a_0^{-7}\\ &\times\left[\int_0^\infty dr_1\,r_1^{1-\lambda}C_\nu(k_1,r_1)R_{nl}(r_1)\right.\\ &\times\int_0^{r_1}dr_2\,r_2^{2+\lambda}C_\lambda(k_2,r_2)\exp(-r_2/a_0)\\ &+\int_0^\infty dr_1\,r_1^{2+\lambda}C_\nu(k_1,r_1)R_{nl}(r_1)\\ &\qquad\left.\times\int_{r_1}^\infty dr_2\,r_2^{1-\lambda}C_\lambda(k_2,r_2)\exp(-r_2/a_0)\right]^2.\end{aligned} \tag{4.32}$$

We have coded Eq. (4.32) for the IBM 704 and calculated numerically values of $\sigma(n,l;k_1)$. The details of this calculation are described in Appendix A. We find that our results for $\sigma(n)=\sum_{l=0}^{n-1}\sigma(n,l)$ may be represented within $\pm 2\%$ for both k_1 and $n\lesssim\sqrt{m}$ by

$$\begin{aligned}\sigma(n,m,k_1)\approx An^5&[1-\exp(-Bn)]\\ &\times\exp\{-[(C+Dn^2)^{\frac{1}{2}}-C^{\frac{1}{2}}]\},\end{aligned} \tag{4.33}$$

where

$$\begin{aligned}A&=[26.8+(138/\sqrt{m})-268/m]/(k_1^2m^3),\\ B&=0.123+2.60/\sqrt{m},\\ C&=40.6(k_1^2/m)/[1+4.44(k_1^2/m)],\\ D&=\{[(0.615)^2+27.1(k_1^2/m)]^{\frac{1}{2}}-0.615\}/\sqrt{m}.\end{aligned} \tag{4.34}$$

It is to be noted that because of the k_1^{-2} factor, most of the captures will be of thermal mesons and the relative distribution of mesic atoms will be essentially that of (4.33) with $k_1=0$. The n^5 factor insures that most of the mesons will be captured in the highest n state allowed by energy conservation ($n\leq\sqrt{m}$). (We compared this formula in the range $9\leq m\leq 630$.)

Figures 1 and 2 show the distributions for π^- and K^- mesic atoms in the different (n,l) states. Figure 3 shows the probability of finding a meson between r and $r+dr$

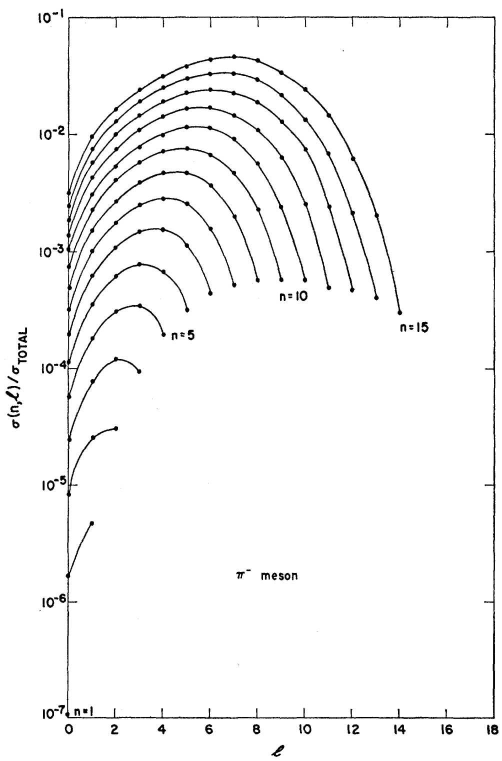

FIG. 1. Partial Coulomb capture cross sections as fractions of the total cross section for π^- mesons on hydrogen at nearly zero energy. The lines represent constant principle quantum number, n.

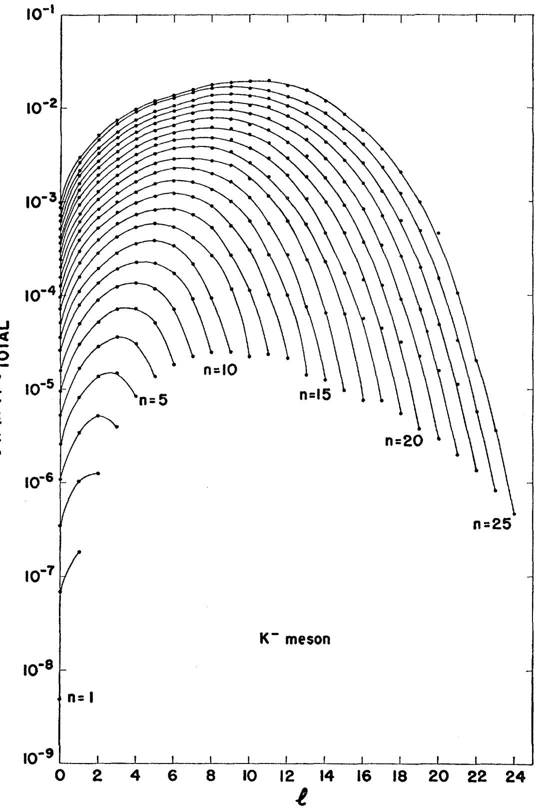

FIG. 2. Partial Coulomb capture cross sections as fractions of the total cross section for K^- mesons on hydrogen at nearly zero energy. The lines represent constant principle quantum number, n. Values for both n and l large are somewhat uncertain due to the calculational difficulty.

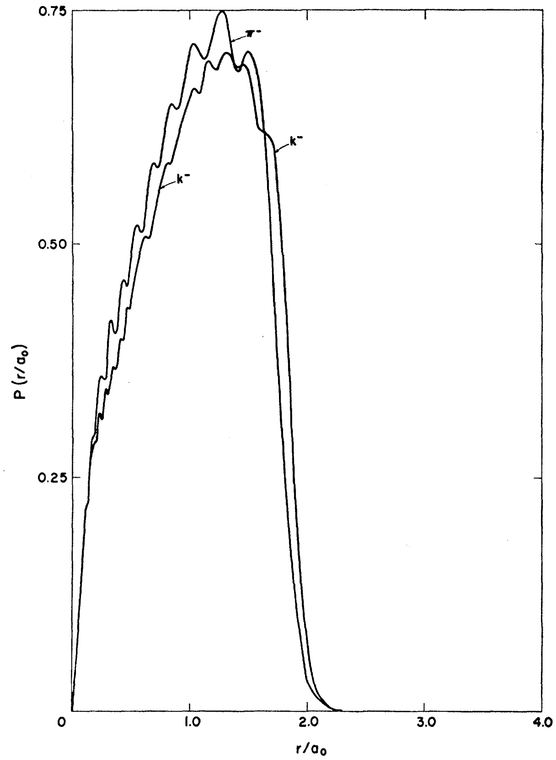

FIG. 3. Probability distributions for captured π^- and K^- mesons on hydrogen at nearly zero energy.

in an ensemble of mesic atoms. It is a plot of

$$P(r)=r^2 \sum_{n=1}^{[\sqrt{m}]} \sum_{l=0}^{n-1} \frac{\sigma(n,l)}{\sigma_{\text{total}}} [R_{nl}(r)]^2. \tag{4.35}$$

One can see in this plot that both the π^- and the K^- mesons are captured at approximately the same radial distances. It is easy to show that the fractions of mesons captured between E and $E+dE$ is also approximately the same for π^- and K^-. This result follows because $\sigma(n)$ is approximately proportional to n to some power.

Figure 4 shows a comparison for large values of k_1 of (4.33) and our numerical results for the cross sections for a meson of reduced mass 100. A factor of the form $[1+E\exp(Fn)]$ would appear to correct (4.33) properly, but the computational difficulties connected with large k, n, and m make it difficult to determine such a correction reliably.

5. CAPTURE PROCESS IN THE CARBON, OXYGEN, NITROGEN GROUP

For our purposes, a crude estimate of the capture process will suffice. We shall simply adopt the estimate of Fermi and Teller.[2] (See also, Burbridge and deBrode,[12] Ferretti,[13] Rosenberg,[14] and Wightman.[15]) They find that the total capture cross section is roughly proportional to Z, the number of atomic electrons. After the meson is captured, it descends rapidly by electron ejection, and in the last stages by radiative transition, to a low-lying state where it is absorbed by means of strong interactions with the nucleus. All this transpires in a time of the order of 10^{-13} seconds, which is short compared to the lifetime of the π^- or K^- and with respect to a close collision time in the emulsion.

6. MESIC HYDROGEN ATOM PHASE

Shortly after (about 10^{-13} second) a meson has been captured, if it lands on a carbon, oxygen, or nitrogen atom, it will be absorbed by a nuclear reaction. On the other hand if it lands on hydrogen, it will have ejected the single electron and be unable to descend to a lower state by the Fermi-Teller process. It cannot decay by gamma emission, as this process is too slow.[4]

The transition rate for direct nuclear capture[16] from

[12] G. R. Burbridge and A. H. deBrode, Phys. Rev. **89**, 189 (1953).
[13] B. Ferretti, Nuovo cimento **5**, 325 (1948).
[14] R. L. Rosenberg, Phil. Mag. **40**, 759 (1949).
[15] A. S. Wightman, Phys. Rev. **77**, 521 (1953).
[16] H. A. Bethe and F. deHoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), Vol. II, p. 101.

S states is for the π^-

$$W_\pi \approx (7.0\times 10^{14}/n^3)\ \mathrm{sec}^{-1}, \tag{6.1}$$

including both of the relatively slow reactions

$$\begin{aligned} \pi^- + P &\rightarrow N+\gamma, \\ \pi^- + P &\rightarrow N+\pi^0. \end{aligned} \tag{6.2}$$

The transition rate for direct nuclear capture[17] from S states is, for the K^-

$$W_K \approx 4.7\times 10^{17}/n^3\ \mathrm{sec}^{-1}. \tag{6.3}$$

Day, Snow, and Sucher[18] point out that the Stark effect is strong enough to produce very good mixing between the n^2 different eigenstates with principle quantum number n. In the solid state the mesic atom will be almost continuously subject to fields which produce Stark transitions at a significant rate. Adopting the estimates of Day, Snow, and Sucher we see that almost all the mesons captured will be subject to Stark transitions at rates of 10^{15} to 4×10^{16} sec^{-1}. These rates are larger than the depletion rates of most states because of the direct capture ($n\geq 4$ for K^-, $n\geq 2$ for π^-), radiative transitions and collisional transfer or de-excitation. Thus we will assume that Stark "pumping" maintains an equidistribution between the n^2 eigenstates with principle quantum number n. As there is only one S state per n, the depletion rate due to direct capture may be found by dividing (6.1) and (6.3) by n^2.

We assume that the transfer rate to other atoms, W_T, is the same in all states. It is probably higher for larger n but as its effect is chiefly to transfer the mesons out of the higher n states and allow direct capture from the lower n states, this assumption should not seriously affect the conclusions. We also assume the de-excitation probability rate, W_D, for $n\rightarrow n-1$ is a constant. The results are not very sensitive to W_D so long as $W_D \lesssim W_T$. Under these assumptions, if $Y_n(t)$ denotes the population of the n'th level at time t after capture and $W_{DC}(n)$ the direct capture rate from the n'th level, then

$$\dot{Y}_n(t) = -W_{nj}Y_j(t), \tag{6.4}$$

where

$$W_{11}=W_T+W_{DC}(1),\quad W_{jj}=W_T+W_D+W_{DC}(j)\quad j\neq 1,$$
$$W_{j,j+1}=-W_D. \tag{6.5}$$

The total loss caused by nuclear capture will be

$$\int_0^\infty \sum_{n=1}^{[\sqrt{m}]} W_{DC}(n)Y_n(t)dt = \sum_{n=1}^{[\sqrt{m}]}\sum_{j=1}^{[\sqrt{m}]} W_{DC}(n)[W^{-1}]_{nj}Y_j(0). \tag{6.6}$$

Equation (6.6) is easy to evluate as W_{nj} is a triangular matrix. Barkas *et al.*[5] find about 0.43% K^- mesons stopping on hydrogen in nuclear emulsion out of 4700 observed stoppings. Estimating roughly from the Z law, we find about 65% stop initially on the heavy elements, 30.5% on carbon nitrogen, or oxygen and 4.5% on hydrogen. Hence a little less than 10% of those originally captured on hydrogen are retained. We find that this percentage may be approximately reproduced with either

$$W_T=1.3\times 10^{12}\ \mathrm{sec}^{-1},\quad W_D=0, \tag{6.7a}$$

or

$$W_T=W_D=1.7\times 10^{12}\ \mathrm{sec}^{-1}. \tag{6.7b}$$

These rates are in reasonable agreement with estimates for close collision rates in emulsion analogous to those given by Day, Snow, and Sucher[18] for liquid hydrogen. If we assume that these rates are the same for π^- mesons then we get 0.25% from (6.7a) and 0.64% from (6.7b). These results imply that the number of π^- observed stopping on hydrogen should be of the order of 2×10^{-4}.

That so many fewer π^- undergo nuclear capture on hydrogen than K^- is caused chiefly by the anomalously low π^- nuclear interaction cross section at zero energy, but, due to the Coulomb capture distribution, the number of π^- is larger than one might suppose from a simple comparison of the direct capture rates.

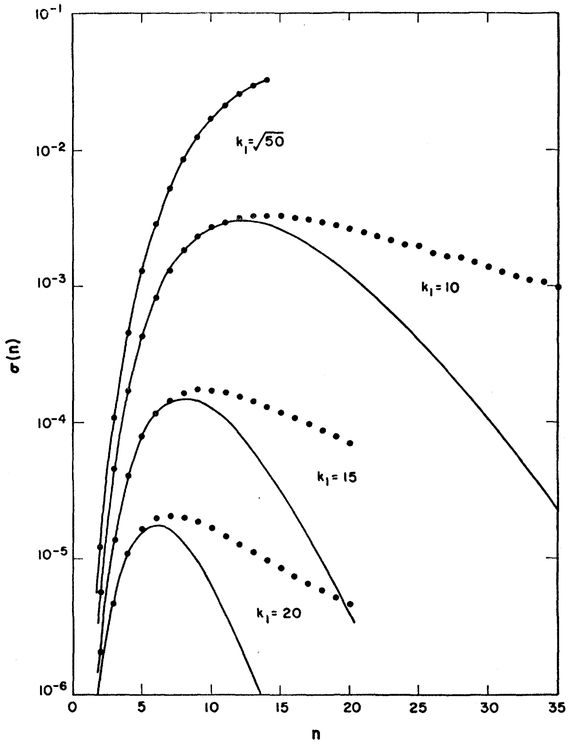

FIG. 4. Comparison of Eq. (4.33) with Eq. (4.32) for large k_1 for a meson of reduced mass 100. The lines represent Eq. (4.33), and the dots Eq. (4.32).

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[17] G. Frye, Phys. Rev. **113**, 688 (1959).
[18] T. B. Day, G. A. Snow, and J. Sucher, Phys. Rev. Letters **3**, 61 (1959).

discussions of it with him. Mrs. Julie Wagoner kindly checked some of our calculations before we had them programmed for numerical computation and we have enjoyed a number of discussions of them with her. We wish to thank Benny J. Hill for programming this problem for numerical computation on the IBM 704.

APPENDIX A

In order to evaluate (4.32) we first calculated the various wave functions. The free state electron wave functions were calculated by integrating the differential equation[10] for $[C_\lambda(k_2,r_2)/r_2^\lambda]$ to the edge of the angular momentum barrier and then the differential equation[10] for $[C_\lambda(k_2r_2)]$ to the maximum value of r. This integration was performed for the two largest values of λ occurring and the rest of the wave functions were calculated by the standard recursion relations.[10] A similar procedure was followed for the free state meson wave functions, except inside the angular momentum barrier $[C_\nu(k_1,r_1)/r_1^\nu]$ was calculated by means of a "double precision" power series because of the difficulty in integrating the differential equation for it. For the bound state meson wave functions we calculated the two highest l values for each value of n directly from (4.11) and calculated the rest from the easily derivable recursion relation

$$R_{n,l-1}(r_1)=-\frac{l}{l+1}\left(\frac{n^2-(l+1)^2}{n^2-l^2}\right)^{\frac{1}{2}}R_{n,l+1}(r_1) -\frac{n(2l+1)}{l+1}\left(1-\frac{l(l+1)}{mr_1}\right)\frac{R_{n,l}(r_1)}{(n^2-l^2)^{\frac{1}{2}}}, \quad \text{(A-1)}$$

where r_1 is in units of the electron Bohr radius and m in units of the electron mass. The indefinite electron integrals were formed by Simpson's rule integration. In order to accommodate the different rate of variation of the meson wave functions at different values of r we used 300 integration steps and varied the step size by a factor of 64 between r small and r large.

In order to prevent the use of an inordinately large number of integration steps for large r when k_1 is large we have generalized Simpson's rule for a weight function which is not a constant, following Filon.[19] It is easy to show by expanding $f(x)$ in a power series that to Simpson's rule accuracy,

$$\begin{aligned}\int_a^b f(x)w(x)dx\approx\{&\alpha(b,\Delta)f(b)-\alpha(a,\Delta)f(a)\\&+\tfrac{1}{2}[\beta(a,\Delta)f(a)+\beta(b,\Delta)f(b)]\\&+\sum_{r=1}^{m}f(a+(2r-1)\Delta)\gamma(a+(2r-1)\Delta,\Delta)\\&+\sum_{r=1}^{m-1}f(a+2r\Delta)\beta(a+2r\Delta,\Delta)\}\Delta, \quad \text{(A-2)}\end{aligned}$$

[19] L. N. G. Filon, Proc. Roy. Soc. (Edinburgh) **49**, 38 (1928–1929).

where

$$a+2m\Delta=b. \quad \text{(A-3)}$$

and we define

$$\alpha(x,\Delta)=\frac{1}{4}\int_{-1}^{+1}(\theta^2+\theta)\times[w(x-\Delta(1-\theta))-w(x+\Delta(1-\theta))]d\theta, \quad \text{(A-4)}$$

$$\beta(x,\Delta)=\frac{1}{2}\int_{-1}^{+1}(\theta^2+\theta)\times[w(x-\Delta(1-\theta))+w(x+\Delta(1-\theta))]d\theta, \quad \text{(A-5)}$$

$$\gamma(x,\Delta)=\int_{-1}^{+1}(1-\theta^2)w(x+\Delta\theta)d\theta. \quad \text{(A-6)}$$

It will be noted that for $w=1$, we obtain Simpson's rule. This generalization of Simpson's rule is useful when w varies rapidly with respect to f. We used it for r larger than about two Bohr radii where $R_{nl}(r)$ vary relatively slowly compared to $C_\nu(k_1,r)$ for k_1 large. We approximated (A-4–6) by a Simpson's rule integration and obtained

$$\alpha(x,\Delta)=+\tfrac{1}{8}[w(x+\tfrac{3}{4}\Delta)-w(x-\tfrac{3}{4}\Delta)]+\tfrac{3}{8}[w(x-\tfrac{1}{4}\Delta)-w(x+\tfrac{1}{4}\Delta)], \quad \text{(A-7)}$$

$$\beta(x,\Delta)=-\tfrac{1}{4}[w(x-\tfrac{3}{4}\Delta)+w(x+\tfrac{3}{4}\Delta)]+\tfrac{3}{4}[w(x-\tfrac{1}{4}\Delta)+w(x+\tfrac{1}{4}\Delta)]+w(x), \quad \text{(A-8)}$$

$$\gamma(x,\Delta)=\tfrac{3}{2}[w(x-\tfrac{1}{4}\Delta)+w(x+\tfrac{1}{4}\Delta)]+w(x). \quad \text{(A-9)}$$

In order to check the accuracy of our calculations we compared the results of numerical integration with the following analytically done integrals

$$\int_0^\infty dr_2\, r_2^{\lambda+2}C_\lambda(k_2,r_2)e^{-r_2/a_0}=\frac{2\lambda|\Gamma(\lambda+1+i/k_2)|(2k_2)^\lambda}{(a_0^{-2}+k_2^2)^{\lambda+2}}\times\exp\{[\pi-4\tan^{-1}(a_0k_2)]/(2k_2a_0)\}. \quad \text{(A-10)}$$

and for $n+\lambda-\nu=1$

$$\begin{aligned}\int_0^\infty dr_1\, r_1^{2+\lambda}R_{n,n-1}(r_1)C_\nu(k_1,r_1)&=-\frac{2m^{\frac{3}{2}}|\Gamma(\nu+1+im/k_1)|(2k_1)^\nu(2m/na_0)^{n-1}}{n^2[(2n-1)!]^{\frac{1}{2}}}\\&\times\left[\frac{(2m/a_0)[1-(\nu+1)/n]}{[(m/na_0)^2+k_1^2]^{\nu+2}}\times\exp\left\{\frac{m}{2k_1a_0}\left[\pi-4\tan^{-1}\left(\frac{na_0k_1}{m}\right)\right]\right\}\right]. \quad \text{(A-11)}\end{aligned}$$

We found the accuracy to be sufficiently good to lead us to believe that the cross sections herein reported are good to at least a few percent.

We found that it was only necessary to carry the summation over λ to a maximum value of 3 to get almost all the cross sections good to about 0.1%.