Pion Theory of Nuclear Forces with Nucleon Recoil*

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The nuclear potential between two nucleons with nonrelativistic velocities in their center-of-mass system is calculated by using the relativistic pion theory and taking fully into account the effect of the nucleon recoil. The resulting potential completely disagrees with the Klein potential but differs from the Lévy potential to a lesser extent. It is shown that an expansion of the contribution of the nucleon recoil in powers of the ratio of the pion and nucleon masses leads to erroneous results.

1. INTRODUCTION

It has often been suggested[1] that perturbation theory might yield a fairly reliable result for the nuclear potential in all regions except in the immediate neighborhood of the nucleons. Therefore, a semiphenomenological theory of nuclear forces, in which one uses a phenomenological potential at short distances and the pion theoretical potential at larger distances, has been developed by Lévy,[2] Klein,[3] Gartenhaus,[4] and others.[5] The second-order nuclear potential due to the pion field is easy to derive, and the result is well known. However, an exact derivation of the nuclear potential in fourth and higher orders is extremely complicated, and in all the calculations so far either the nucleon recoil has been completely ignored or it has not been adequately taken into consideration.

In this paper we shall reinvestigate the problem of the fourth-order nuclear potential by using the relativistic pion theory with pseudoscalar coupling. We shall assume that the nucleons are moving with nonrelativistic velocities in their center-of-mass system, but within this approximation we shall completely take into account the effect of the nucleon recoil. We shall also compare our results with those of Lévy and Klein to clarify the role of nucleon recoil in nuclear forces. The most interesting conclusion of the present paper is that *the widely used practice of expanding the contribution of the nucleon recoil in powers of the ratio of the pion and nucleon masses is mathematically incorrect.*

2. RELATION BETWEEN POTENTIAL AND SCATTERING OPERATOR

We shall first discuss briefly how we can obtain in general a Hermitian two-particle potential from the scattering operator.

The scattering operator for a system of interacting fields is usually expressed as[6]

$$S=1+\sum_{n=1}^{\infty} S_n, \tag{1}$$

where S_n is related to the interaction energy density $H(x)$ in the interaction representation as

$$S_n=\frac{1}{n!}\left(\frac{-i}{c\hbar}\right)^n\int dx'\int dx''\cdots \times\int dx^{(n)}\,P[H(x'),H(x''),\cdots,H(x^{(n)})]. \tag{2}$$

However, as we have already pointed out,[7] it is in some respects more advantageous to put

$$S=\frac{1-\frac{1}{2}iK}{1+\frac{1}{2}iK}, \tag{3}$$

and then expand K as

$$K=\sum_{n=1}^{\infty} K_n. \tag{4}$$

It is easy to express the K_n in terms of the S_n, and we find

$$K_n=iS_n-(i/2)\sum_{n_1+n_2=n} S_{n_1}S_{n_2} +i(-1/2)^{p-1}\sum_{n_1+n_2+\cdots+n_p=n} S_{n_1}S_{n_2}\cdots S_{n_p}+\cdots, \tag{5}$$

where

$$\sum_{n_1+n_2+\cdots+n_p=n},$$

denotes summation over all possible positive integral values of n_1, n_2, $\cdots$, n_p such that $n_1+n_2+\cdots+n_p=n$.

When $S_n=0$ for odd values of n, it follows from (5) that

$$K_2=iS_2, \tag{6}$$

$$K_4=\tfrac{1}{2}i(S_4-S_4^*), \tag{7}$$

where the latter relation shows that K_4 is equal to the

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[1] See particularly M. Taketani, S. Nakamura, and M. Sasaki, Progr. Theoret. Phys. (Kyoto) **6**, 581 (1951).
[2] M. M. Lévy, Phys. Rev. **88**, 725 (1952).
[3] A. Klein, Phys. Rev. **90**, 1101 (1953).
[4] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).
[5] For a survey of the considerable theoretical work on this subject see G. Breit and M. H. Hull, Am. J. Phys. **21**, 184 (1953); H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), Vol. II; and Suppl. Progr. Theoret. Phys. (Kyoto), No. 3 (1956).
[6] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).
[7] S. N. Gupta, Proc. Cambridge Phil. Soc. **47**, 454 (1951).

Hermitian part of iS_4. However, in general

$$K_n \neq \tfrac{1}{2}i(S_n - S_n^*).$$

We shall call the simple expansion, given by (1), the direct expansion of the scattering operator. On the other hand, the expansion, given by (3) and (4), will be called the unitary expansion, because this expansion is exactly unitary in any approximation.[7] We also define an *effective interaction energy density* H_{eff}, which is such that the contribution of H_{eff} to the unitary expansion of the scattering operator in the first approximation is equal to the total scattering operator. It is then evident from (5) and (2) that H_{eff} is related to K as

$$K = (1/c\hbar)\int dx\, H_{\mathrm{eff}}(x). \tag{8}$$

Since S is unitary, it follows from (3) that K is Hermitian, and therefore H_{eff}, given by (8), is also Hermitian.[8]

We now consider the mutual scattering of two nucleons, whose propagation four-vectors are p and q in the initial state and p' and q' in the final state. We can express the effective interaction energy of the two nucleons as

$$\int H_{\mathrm{eff}}(x)dV = \int dV\, \psi^{*-}(\mathbf{p}')e^{-ip'x}\psi^{*-}(\mathbf{q}')e^{-iq'x} \times V(\mathbf{p}'-\mathbf{p})\psi^{+}(\mathbf{q})e^{iqx}\psi^{+}(\mathbf{p})e^{ipx}, \tag{9}$$

where $\psi^{+}(\mathbf{p})e^{ipx}$ denotes a Fourier component of the positive frequency part of the nucleon field operator, $\psi^{*-}(\mathbf{p})$ denotes the Hermitian conjugate of $\psi^{+}(\mathbf{p})$, and $V(\mathbf{p}'-\mathbf{p})$ is related to the two-nucleon potential $V(\mathbf{r})$ as

$$V(\mathbf{r}) = (2\pi)^{-3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}V(\mathbf{k}). \tag{10}$$

Substituting (9) in (8), we get

$$K = (2\pi)^4(1/c\hbar)\delta(p-p'+q-q')\psi^{*-}(\mathbf{p}')\psi^{*-}(\mathbf{q}') \times V(\mathbf{p}'-\mathbf{p})\psi^{+}(\mathbf{q})\psi^{+}(\mathbf{p}), \tag{11}$$

which gives us the required relationship between the scattering operator and the two-nucleon potential.

3. PION-THEORETICAL NUCLEAR POTENTIAL

Let us consider the mutual scattering of two nucleons, whose propagation four-vectors are p and q in the initial state and p' and q' in the final state, and let

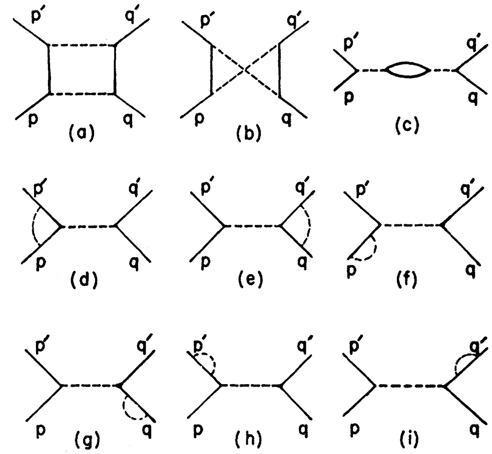

FIG. 1. Diagrams for the fourth-order pion theoretical nuclear potential.

$$k_\mu = p_\mu' - p_\mu = -(q_\mu' - q_\mu). \tag{12}$$

We then have in the center-of-mass system of the two nucleons

$$\mathbf{p} = -\mathbf{q},\quad \mathbf{p}' = -\mathbf{q}',\quad p_0 = q_0 = p_0' = q_0',$$
$$\mathbf{k} = \mathbf{p}' - \mathbf{p} = -(\mathbf{q}' - \mathbf{q}),\quad k_0 = 0. \tag{13}$$

The derivation of the second-order pion theoretical nuclear potential in the center-of-mass system of the nucleons is quite straightforward, and it is known to be

$$V_2 = \frac{1}{12}\left(\frac{g^2}{4\pi c\hbar}\right)\lambda c\hbar\left(\frac{\lambda}{\kappa}\right)^2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{e^{-x}}{x} \times\left[\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)} + \left(1+\frac{3}{x}+\frac{3}{x^2}\right)S^{(1,2)}\right], \tag{14}$$

where λ and κ are related to the pion mass μ and the nucleon mass M as $\lambda = \mu c/\hbar$ and $\kappa = Mc/\hbar$, $x = \lambda r$, $S^{(1,2)}$ is the tensor operator, and other symbols have the usual meaning.

To obtain the fourth-order nuclear potential we have to consider the diagrams shown in Fig. 1. After renormalization, the contributions of these diagrams to the scattering operator are found to be

$$S_4(a) = (i\pi^2 g^4/c^2\hbar^2)\delta(p-p'+q-q')[\bar{\psi}^{-}(\mathbf{q}')\gamma_\mu\tau_i\tau_j\psi^{+}(\mathbf{q})][\bar{\psi}^{-}(\mathbf{p}')\tau_i\tau_j\gamma_\nu\psi^{+}(\mathbf{p})]$$
$$\times\int_0^1 du\int_0^u dv\int_0^v dw\left(\frac{\tfrac{1}{2}\delta_{\mu\nu}}{[\kappa^2(v-2w)^2+\lambda^2(1-v)+k^2(u-v-u^2+uv)+(p-q)^2(w^2-vw)]}\right.$$
$$\left.+\frac{[vq_\mu - w(p_\mu+q_\mu)][vq_\nu - w(p_\nu+q_\nu)]}{[\kappa^2(v-2w)^2+\lambda^2(1-v)+k^2(u-v-u^2+uv)+(p-q)^2(w^2-vw)]^2}\right), \tag{15}$$

[8] It should be noted that our effective interaction energy density (8) is different from that in reference 6, which is not Hermitian in general.

$$S_4(b)=(-i\pi^2g^4/c^2\hbar^2)\delta(p-p'+q-q')[\bar{\psi}^-(\mathbf{q}')\gamma_\mu\tau_i\tau_j\psi^+(\mathbf{q})][\bar{\psi}^-(\mathbf{p}')\gamma_\nu\tau_j\tau_i\psi^+(\mathbf{p})]$$
$$\times\int_0^1 du\int_0^u dv\int_0^v dw\left(\frac{\frac{1}{2}\delta_{\mu\nu}}{[\kappa^2v^2+\lambda^2(1-v)+k^2(u-v-u^2+uv-vw+w^2)+(p-q)^2(vw-w^2)]}\right.$$
$$\left.+\frac{[vq_\mu+w(p_\mu-q_\mu)][vq_\nu+w(p_\nu-q_\nu)]}{[\kappa^2v^2+\lambda^2(1-v)+k^2(u-v-u^2+uv-vw+w^2)+(p-q)^2(vw-w^2)]^2}\right),\quad(16)$$

$$S_4(c)=(-8i\pi^2g^4/c^2\hbar^2)\delta(p-p'+q-q')[\bar{\psi}^-(\mathbf{p}')\gamma_5\tau_i\psi^+(\mathbf{p})][\bar{\psi}^-(\mathbf{q}')\gamma_5\tau_i\psi^+(\mathbf{q})]\frac{1}{(k^2+\lambda^2)^2}\int_0^1 du\left\{3(k^2+\lambda^2)(u-u^2)\right.$$
$$\left.\times\ln\left(1+\frac{(u-u^2)(k^2+\lambda^2)}{\kappa^2+\lambda^2(u^2-u)}\right)+(\kappa^2+3u^2\lambda^2-3u\lambda^2)\left[\ln\left(1+\frac{(u-u^2)(k^2+\lambda^2)}{\kappa^2+\lambda^2(u^2-u)}\right)-\frac{(u-u^2)(k^2+\lambda^2)}{\kappa^2+\lambda^2(u^2-u)}\right]\right\},\quad(17)$$

$$S_4(d)=S_4(e)=(i\pi^2g^4/c^2\hbar^2)\delta(p-p'+q-q')[\bar{\psi}^-(\mathbf{q}')\gamma_5\tau_i\psi^+(\mathbf{q})][\bar{\psi}^-(\mathbf{p}')\gamma_5\tau_i\psi^+(\mathbf{p})]\frac{1}{(k^2+\lambda^2)}$$
$$\times\int_0^1 du\int_0^u dv\left[-2\ln\left(1-\frac{v^2k^2+2uvpk}{u^2\kappa^2+\lambda^2(1-u)}\right)+\frac{\lambda^2(1-u)}{u^2\kappa^2+\lambda^2(1-u)-v^2k^2-2uvpk}-\frac{\lambda^2(1-u)}{u^2\kappa^2+\lambda^2(1-u)}\right],\quad(18)$$

$$S_4(f)=S_4(g)=S_4(h)=S_4(i)=0.\quad(19)$$

We have evaluated all divergent integrals by the method of auxiliary fields,[9] which is mathematically unambiguous as well as free from difficulties of physical interpretation.

So far we have not made any approximations. However, to simplify our calculations we shall now assume that the two nucleons have nonrelativistic velocities in the initial and the final states, so that we can treat $\mathbf{p}^2$ and $\mathbf{k}^2$ as small quantities compared with κ^2. It is then easy to see that the leading terms in $S_4(a)$ and $S_4(b)$ correspond to the values $\mu=\nu=4$ of the indices μ and ν in (15) and (16), and that $S_4(c)$ and $S_4(d)$ are negligible compared with $S_4(a)$ and $S_4(b)$. Thus, retaining only the leading terms and carrying out some simplifications, we obtain for the total contribution of the fourth-order diagrams

$$S_4=(i\pi^2g^4/c^2\hbar^2)\delta(p-p'+q-q')\{[\psi^{*-}(\mathbf{q}')\tau_i\tau_j\psi^+(\mathbf{q})][\psi^{*-}(\mathbf{p}')\tau_i\tau_j\psi^+(\mathbf{p})]I_a$$
$$+[\psi^{*-}(\mathbf{q}')\tau_i\tau_j\psi^+(\mathbf{q})][\psi^{*-}(\mathbf{p}')\tau_j\tau_i\psi^+(\mathbf{p})]I_b\},\quad(20)$$

where

$$I_a=\int_0^1 du\int_0^u dv\,v\int_0^1 dw\left\{\frac{\frac{1}{2}}{[\kappa^2v^2(2w-1)^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)]}-\frac{v^2(2w-1)^2\kappa^2}{[\kappa^2v^2(2w-1)^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)]^2}\right\},\quad(21)$$

$$I_b=-\int_0^1 du\int_0^u dv\,v\int_0^1 dw\left(\frac{\frac{1}{2}}{[\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)]}-\frac{v^2\kappa^2}{[\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)]^2}\right).\quad(22)$$

When we carry out the integrations over w, we obtain

$$I_a=\int_0^1 du\int_0^u dv\,v\frac{\frac{1}{2}}{[\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)]},\quad(23)$$

$$I_b=\int_0^1 du\int_0^u dv\,v\left(\frac{\frac{1}{2}}{[\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)]}-\frac{\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)}{[\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)]^2}\right).\quad(24)$$

By using (7) and (11), we get from (20)

$$V_4(\mathbf{k})=-(g^4/16\pi^2c\hbar)[\tau_i^{(2)}\tau_j^{(2)}\tau_i^{(1)}\tau_j^{(1)}I_a+\tau_i^{(2)}\tau_j^{(2)}\tau_j^{(1)}\tau_i^{(1)}I_b]$$
$$=-(g^4/16\pi^2c\hbar)[3(I_a+I_b)+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}(I_b-I_a)].\quad(25)$$

[9] S. N. Gupta, Proc. Phys. Soc. (London) **A66**, 129 (1953).

Substituting (23) and (24) in (25), and carrying out integration over the $\mathbf{k}$ space, we find that the fourth-order nuclear potential as a function of $x=\lambda r$ is given by

$$V_4=(2\pi)^{-3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}V_4(\mathbf{k})$$
$$=-(g^2/4\pi c\hbar)^2\lambda c\hbar[P(x)-\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}Q(x)], \quad (26)$$

where

$$P(x)=\frac{3\kappa^2}{8\pi\lambda^2}\int_0^1 du\int_0^u dv\frac{v^3}{(u-v)^2(1-u)^2}\frac{1}{f}e^{-fx}, \quad (27)$$

$$Q(x)=\frac{1}{2\pi}\int_0^1 du\int_0^u dv\frac{v}{(u-v)(1-u)}\frac{1}{x}e^{-fx}-\tfrac{2}{3}P(x), \quad (28)$$

with

$$f=\left[\frac{(\kappa^2/\lambda^2)v^2+(1-v)}{(u-v)(1-u)}\right]^{\frac{1}{2}}. \quad (29)$$

We have evaluated the integrals $P(x)$ and $Q(x)$ numerically for various values of x by taking $\lambda/\kappa=0.15$, and the results obtained are given in Table I.

The static potential (26) has been obtained by treating $\mathbf{p}^2$ and $\mathbf{k}^2$ as small compared with κ^2 and retaining only the leading terms in S_4. If we also retain some smaller terms in S_4, we obtain a spin-orbit interaction, which is however too small to be of any great significance for nonrelativistic velocities of the nucleons.[10]

TABLE I. Results obtained by numerical integrations of the quantities $P(x)$ and $Q(x)$ for various values of the parameter x.

x	$P(x)$	$Q(x)$
0.1	0.86632	0.37890
0.2	0.18116	0.04868
0.3	0.06264	0.01216
0.4	0.02711	0.00405
0.5	0.01331	0.00161
0.6	0.00708	0.00073
0.7	0.00399	0.00037
0.8	0.00234	0.00021
0.9	0.00142	0.00013
1.0	0.00089	0.00008
1.1	0.00057	0.00005
1.2	0.00037	0.00004
1.3	0.00024	0.00003
1.4	0.00016	0.00002
1.5	0.00011	0.00001
1.6	0.00008	0.00001
1.7	0.00005	0.00001
1.8	0.00004	0.00000
1.9	0.00003	0.00000
2.0	0.00002	0.00000

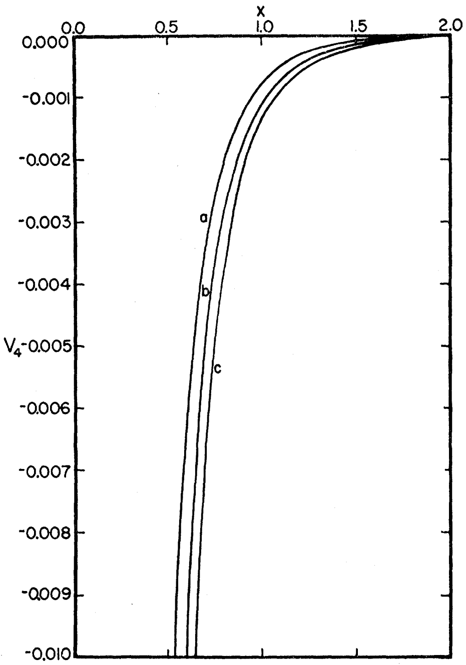

FIG. 2. Fourth-order pion-theoretical nuclear potential V_4 in units of $(g^2/4\pi c\hbar)^2\lambda c\hbar$ as a function of the distance $x=\lambda r$. The curves a and b describe the nuclear potential with recoil in different isotopic spin states as explained in the text, while the curve c describes the nuclear potential without recoil.

We shall not derive higher-order contributions to the pion-theoretical nuclear potential, which presumably do not give rise to any appreciable effect outside the phenomenological core.

4. REMARKS ON THE ROLE OF NUCLEON RECOIL IN PION-THEORETICAL NUCLEAR POTENTIAL

The evaluation of the nuclear potential can be considerably simplified, if we make the further assumption that $\lambda/\kappa\ll 1$. This approximation can also be looked upon as neglect of the recoil of the nucleons, because it is equivalent to assuming that the nucleons have an infinitely large mass compared with that of mesons.

The fourth-order pion-theoretical nuclear potential with complete neglect of nucleon recoil was first obtained by Lévy,[11] and was found to be

$$V_4(\text{Lévy})=-(g^2/4\pi c\hbar)^2\lambda c\hbar(3\lambda^2/2\pi\kappa^2)K_1(2x)/x^2. \quad (30)$$

Later, Klein modified the above result by including partially the effect of the nucleon recoil, and thus he obtained[3]

$$V_4(\text{Klein})=-(g^2/4\pi c\hbar)^2\lambda c\hbar(3\lambda^2/2\pi\kappa^2)(1/x^2)$$
$$\times[K_1(2x)-(\pi\lambda/2\kappa)(1+1/x)^2e^{-2x}]. \quad (31)$$

It should be observed that (30) and (31) do not involve

[10] For a discussion of the spin-orbit interaction in the fourth-order pion-theoretical nuclear potential, see G. Breit, Phys. Rev. **111**, 652 (1958); S. Otsuki, Progr. Theoret. Phys. (Kyoto) **20**, 171 (1958); N. Tzoar, R. Raphael, and A. Klein, Phys. Rev. Letters **2**, 433 (1959) and **3**, 145 (1959); S. Okubo and S. Sato, Progr. Theoret. Phys. (Kyoto) **21**, 383 (1959).

[11] We have dropped a spurious term in Lévy's result, as given in reference 2.

the isotopic spin operators, while our fourth-order nuclear potential (26) involves these operators.

The fourth-order pion-theoretical nuclear potential in units of $(g^2/4\pi c\hbar)^2\lambda c\hbar$ as a function of the distance $x=\lambda r$ is shown in Fig. 2. The curve a describes the potential (26) in the singlet even or the triplet odd states, the curve b describes the same potential in the triplet even or the singlet odd states, and the curve c describes the potential (30). We find that our potential (26) has the same general features as the Lévy potential (30), but the two potentials differ considerably numerically. For instance, at a distance of one pion Compton wavelength the magnitude of our potential is less than that of the Lévy potential by 40% in the singlet even or the triplet odd states and by 20% in the triplet even or the singlet odd states. It is even more significant that our potential (26) is in complete disagreement with the Klein potential (31), because the second term within the square brackets in (31) completely changes the general features of the Lévy potential.

We shall now discuss why the neglect of the nucleon recoil in nuclear forces is not very reliable from the theoretical point of view. Replacing the variable of integration v in (23) and (24) by

$$z=\left[\frac{(\kappa^2/\lambda^2)v^2+1-v}{1-v/u}\right]^{\frac{1}{2}}, \tag{32}$$

we can express (23) as

$$I_a=\frac{\lambda^2}{2\kappa^2}\int_0^1 du\int_1^\infty dz\frac{z}{[\lambda^2z^2+\mathbf{k}^2u(1-u)]}\times[1+f(\lambda/\kappa)], \tag{33}$$

where

$$f(\lambda/\kappa)=-(\lambda/\kappa)(z^2-u)[4u^2(z^2-1)+(\lambda/\kappa)^2(z^2-u)^2]^{-\frac{1}{2}}. \tag{34}$$

We can neglect the effect of the nucleon recoil by neglecting λ/κ compared with 1. Thus, (33) reduces to

$$I_a=\frac{\lambda^2}{2\kappa^2}\int_0^1 du\int_1^\infty zdz\frac{1}{[\lambda^2z^2+\mathbf{k}^2u(1-u)]}, \tag{35}$$

and similarly we can reduce (24) to

$$I_b=\frac{\lambda^2}{2\kappa^2}\int_0^1 du\int_1^\infty zdz\left(\frac{1}{[\lambda^2z^2+\mathbf{k}^2u(1-u)]}-\frac{2\lambda^2+2\mathbf{k}^2u(1-u)}{[\lambda^2z^2+\mathbf{k}^2u(1-u)]^2}\right). \tag{36}$$

After some elementary integrations, we obtain from (35) and (36)

$$(2\pi)^{-3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}I_a=(2\pi)^{-3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}I_b=(\lambda/4\pi\kappa^2r^2)K_1(2\lambda r), \tag{37}$$

which, on using (25), gives us the Lévy potential.

However, in the above derivation we have neglected $f(\lambda/\kappa)$ compared with 1, while actually for large values of z the quantity $f(\lambda/\kappa)$ is independent of λ/κ and has the value -1. This shows that the Lévy approximation cannot be justified mathematically even if we treat λ/κ as small compared with 1.

Further, the Klein approximation is equivalent to retaining only the terms proportional to λ/κ in (34), which reduces (34) to

$$f(\lambda/\kappa)\approx-(\lambda/\kappa)(z^2-u)[4u^2(z^2-1)]^{-\frac{1}{2}}. \tag{38}$$

But, for large values of z, (34) behaves as -1 while (38) acquires a large value, which is proportional to z. This shows that the Klein approximation is even more objectionable than the Lévy approximation.

5. DISCUSSION

We shall now briefly remark on higher order pion-theoretical nuclear potentials and the role of heavier mesons in nuclear forces. Following the usual practice, we can assume some simple phenomenological potential at short distances and use the field-theoretical result beyond the phenomenological core. It is well known that the larger the mass of a meson, the smaller is the range of the nuclear force produced by it. Moreover, usually the perturbation theory leads to forces of shorter and shorter range in successive approximations. We shall, therefore, make the following plausible assumptions:

1. Only the second- and the fourth-order pion-theoretical potentials contribute to the effective nuclear force outside the phenomenological core.
2. Only the second-order potential due to the heavier ρ^0 meson[12,13] contributes to the effective nuclear force outside the phenomenological core.
3. The entire contribution of the still heavier K mesons is essentially contained within the phenomenological core.

It then follows that the effective field-theoretical nuclear potential outside the phenomenological core is obtained by adding the second-order contribution of the ρ^0 meson, given in an earlier paper,[13] to the pion-theoretical potentials (14) and (26).

Finally, it should be noted that the relativistic pion theory is unable to account for the observed pion-nucleon scattering, which can be explained much better by the Chew-Low-Wick extended-source meson theory. Therefore, the use of the relativistic meson theory in

[12] S. N. Gupta, Phys. Rev. 111, 1436, 1698 (1958).
[13] S. N. Gupta, Phys. Rev. Letters 2, 124 (1959).

the treatment of nuclear forces may seem rather questionable. However, it seems to us doubtful whether the effect of nucleon recoil can be adequately taken into account in the extended-source meson theory. Moreover, as we have already pointed out, the perturbation theory even with strong coupling can be applied to derive the nuclear potential outside the phenomenological core, while there does not seem to be any theoretical justification at all for applying the perturbation theory to the pion-nucleon scattering. Therefore, it seems to us more reasonable to separate the problem of nuclear forces from the problem of pion-nucleon scattering, and test the validity of the present nuclear potential (both with as well as without the contribution of the ρ^0 meson) by applying it to the interaction of two nucleons at nonrelativistic energies.

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Some Analytic Properties of the Vertex Function*

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The absorptive part of the vertex function $F[k^2, p^2, (k-p)^2]$ is an analytic function of the mass variables k^2 and p^2. On the basis of causality and the spectral conditions, the region of regularity $D(\sigma)$ of the absorptive part $A(k^2,p^2,\sigma^2)$ is obtained for fixed values of $\sigma \geqslant c$. The boundary of $D(\sigma)$ is calculated explicitly for the case $k^2=p^2$, which is of interest in connection with form factors. By the use of examples based upon perturbation theory, it is shown that this boundary is characteristic for the physical assumptions that have been made. The intersection D of all domains $D(\sigma)$ for $\sigma \geqslant c$ is the region for which F is an analytic function of all three variables, with $(k-p)^2$ in the cut plane and (k^2,p^2) in D. The relation of these general results to the composite structure of particles is discussed.

A simple, direct representation for the vertex function F is used in order to find limits for the region in the $(k-p)^2$ plane where singularities are allowed by the axioms. For real $k^2=p^2=z$, it is shown that the singularities are restricted to a *finite* region, and the static cut $(k-p)^2 \geqslant c^2$, provided z is below the onset of the corresponding cut in the z plane.

I. INTRODUCTION

In an earlier article[1] we have shown that in local field theories the electromagnetic form factors of particles can have singularities which are a consequence of the structure of these particles as composite systems. These "structure singularities" are related to the quantum-mechanical tunnel effect. They appear in the physical sheet of the complex z_3 plane [$z_3=(k-p)^2$ = momentum-transfer variable] only if the particle in question can be considered as a loosely bound system of its constituents such that the binding energy, B, does not exceed a certain limit. In the case of two constituents with masses m and m_3, we have the limitation

$$B < m + m_3 - (m^2 + m_3^2)^{\frac{1}{2}}. \qquad (1.1)$$

The restriction (1.1) can be obtained by the use of examples from perturbation theory, but it is actually more general. It also appears, in a somewhat different form, if one derives analytic properties of the vertex function on the basis of Lorentz invariance, causality, and the spectral conditions.[2] In fact, it was in this context that limitations corresponding to Eq. (1.1) were first obtained.[3]

In this paper we discuss some further analytic properties of the vertex function which can be obtained from the axioms mentioned above. In the first two sections, we explore the "cut-plane" representation of the vertex function. This representation has been introduced in a previous reference.[2] It defines the domain D in the space of the complex variables $z_1=k^2$ and $z_2=p^2$ for which $F(z_1z_2z_3)$ is an analytic function for $(z_1,z_2)\epsilon D$, and z_3 in the whole z_3 plane except for the static cut $x_3 \geqslant c^2$, $y_3=0$. The region D is the intersection of all $D(\sigma)$ for $\sigma \geqslant c$, where $D(\sigma)$ denotes the domain of analyticity of the absorptive part

$$A(z_1z_2\sigma^2) = \lim_{\epsilon\to 0+} (1/2i)[F(z_1z_2\sigma^2+i\epsilon) - F(z_1z_2\sigma^2-i\epsilon)] \qquad (1.2)$$

as a function of z_1 and z_2. The mass $c \geqslant 0$ is given by the spectral conditions. For the case $z_1=z_2=z$, which is of interest in connection with form factors, we compute the boundary of $D(\sigma)$ in the complex z plane. Then we show, using examples based upon perturbation theory, that the boundary is characteristic for the axioms that

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[1] R. Oehme, Nuovo cimento **13**, 778 (1959). This paper will be referred to as II; it contains further references.

[2] R. Oehme, Phys. Rev. **111**, 1430 (1958). This paper will be referred to as I.

[3] H. J. Bremermann, R. Oehme, and J. G. Taylor, Phys. Rev. **109**, 2178 (1958).