the treatment of nuclear forces may seem rather questionable. However, it seems to us doubtful whether the effect of nucleon recoil can be adequately taken into account in the extended-source meson theory. Moreover, as we have already pointed out, the perturbation theory even with strong coupling can be applied to derive the nuclear potential outside the phenomenological core, while there does not seem to be any theoretical justification at all for applying the perturbation theory to the pion-nucleon scattering. Therefore, it seems to us more reasonable to separate the problem of nuclear forces from the problem of pion-nucleon scattering, and test the validity of the present nuclear potential (both with as well as without the contribution of the ρ^0 meson) by applying it to the interaction of two nucleons at nonrelativistic energies.

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Some Analytic Properties of the Vertex Function*

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The absorptive part of the vertex function $F[k^2, p^2, (k-p)^2]$ is an analytic function of the mass variables k^2 and p^2. On the basis of causality and the spectral conditions, the region of regularity $D(\sigma)$ of the absorptive part $A(k^2,p^2,\sigma^2)$ is obtained for fixed values of $\sigma \geqslant c$. The boundary of $D(\sigma)$ is calculated explicitly for the case $k^2=p^2$, which is of interest in connection with form factors. By the use of examples based upon perturbation theory, it is shown that this boundary is characteristic for the physical assumptions that have been made. The intersection D of all domains $D(\sigma)$ for $\sigma \geqslant c$ is the region for which F is an analytic function of all three variables, with $(k-p)^2$ in the cut plane and (k^2,p^2) in D. The relation of these general results to the composite structure of particles is discussed.

A simple, direct representation for the vertex function F is used in order to find limits for the region in the $(k-p)^2$ plane where singularities are allowed by the axioms. For real $k^2=p^2=z$, it is shown that the singularities are restricted to a *finite* region, and the static cut $(k-p)^2 \geqslant c^2$, provided z is below the onset of the corresponding cut in the z plane.

I. INTRODUCTION

IN an earlier article[1] we have shown that in local field theories the electromagnetic form factors of particles can have singularities which are a consequence of the structure of these particles as composite systems. These "structure singularities" are related to the quantum-mechanical tunnel effect. They appear in the physical sheet of the complex z_3 plane [$z_3=(k-p)^2=$momentum-transfer variable] only if the particle in question can be considered as a loosely bound system of its constituents such that the binding energy, B, does not exceed a certain limit. In the case of two constituents with masses m and m_3, we have the limitation

$$B<m+m_3-(m^2+m_3^2)^{\frac{1}{2}}. \tag{1.1}$$

The restriction (1.1) can be obtained by the use of examples from perturbation theory, but it is actually more general. It also appears, in a somewhat different form, if one derives analytic properties of the vertex function on the basis of Lorentz invariance, causality, and the spectral conditions.[2] In fact, it was in this context that limitations corresponding to Eq. (1.1) were first obtained.[3]

In this paper we discuss some further analytic properties of the vertex function which can be obtained from the axioms mentioned above. In the first two sections, we explore the "cut-plane" representation of the vertex function. This representation has been introduced in a previous reference.[2] It defines the domain D in the space of the complex variables $z_1=k^2$ and $z_2=p^2$ for which $F(z_1z_2z_3)$ is an analytic function for $(z_1,z_2)\epsilon D$, and z_3 in the whole z_3 plane except for the static cut $x_3\geqslant c^2$, $y_3=0$. The region D is the intersection of all $D(\sigma)$ for $\sigma\geqslant c$, where $D(\sigma)$ denotes the domain of analyticity of the absorptive part

$$A(z_1z_2\sigma^2)=\lim_{\epsilon\to 0+}(1/2i)[F(z_1z_2\sigma^2+i\epsilon) - F(z_1z_2\sigma^2-i\epsilon)] \tag{1.2}$$

as a function of z_1 and z_2. The mass $c\geqslant 0$ is given by the spectral conditions. For the case $z_1=z_2=z$, which is of interest in connection with form factors, we compute the boundary of $D(\sigma)$ in the complex z plane. Then we show, using examples based upon perturbation theory, that the boundary is characteristic for the axioms that

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[1] R. Oehme, Nuovo cimento **13**, 778 (1959). This paper will be referred to as II; it contains further references.

[2] R. Oehme, Phys. Rev. **111**, 1430 (1958). This paper will be referred to as I.

[3] H. J. Bremermann, R. Oehme, and J. G. Taylor, Phys. Rev. **109**, 2178 (1958).

we have used. This means that the region $D(\sigma)$ can only be enlarged by introducing additional, new assumptions into the problem.

In Sec. 3, we use a "direct" representation of the vertex function in order to obtain a limitation for the region S_3 in the z_3 plane where complex singularities are allowed by the axioms. In the derivation of the direct representation, only a fraction of the consequences of causality and spectrum are used, and hence the resulting region of analyticity is not characteristic for these physical principles. However, the representation is sufficient to show that S_3 is finite if z is real and below the onset of the static cut in the z plane.

Let us add here a few remarks concerning the limitation (1.1). Physically we require that the masses m and m_3 are restricted to the experimental masses of *existing* particles having the right quantum numbers and interactions to form the composite system with mass $M_0=m+m_3-B$, the form factor of which we are considering. However, in the general approach we use very little information about the low-mass states in the Hilbert space corresponding to the strongly interacting particles. The spectral conditions give only lower limits; they require for instance

$$m+m_3\geqslant a,\quad 2m\geqslant c. \tag{1.3}$$

Often it is possible to find *un*physical masses m and m_3 which satisfy the conditions (1.3) and lead to a binding energy B such that the inequality (1.1) holds. Then there appear structure singularities in the z_3 plane that are unphysical, because they describe the composite structure of the M_0 particle due to the probability distribution of the physically nonexistent m and m_3 particles with respect to the center of mass of the bound system. As an example, take the form factor of the nucleon, where $a=M+m_\pi$, $c=2m_\pi$, and $M_0=M$. For the physical choice, $m=m_\pi$, $m_3=M$, the inequality (1.1) is not fulfilled, whereas the unphysical possibility $m=m_3=\frac{1}{2}(M+m_\pi)$ gives

$$B=m_\pi<[(\sqrt{2}-1)/\sqrt{2}](M+m_\pi).$$

Physically, and according to the examples from perturbation theory,[1] we expect that the composite structure singularities appear always on the real axis for real values of the mass variable z. On the other hand, we know that in many practical cases the axioms, i.e., Lorentz-invariance, causality, and spectrum, do not exclude the appearance of complex singularities in a finite region of the z_3 plane.[4] It is not known at present, to what extent the additional assumptions, which are necessary to eliminate the *unphysical* structure singularities, will also restrict these complex singularities.

II. THE CUT-PLANE REPRESENTATION

It may be instructive to sketch briefly the main steps in the derivation of the cut-plane representation for the vertex function.[2] As usual, since we are interested in the analytic properties, it is sufficient to consider only real scalar fields. Let us introduce three such fields $\phi_A(x)$, $\phi_B(x)$ and $\phi_C(x)$. We denote the corresponding current operators by

$$A(x)=(\Box-m_A^2)\phi_A(x),\ \text{etc.}$$

The vertex function is then given as the Fourier transforms of the vacuum expectation value of a retarded or advanced product. We choose to write it in the form

$$G(k_1,k_2)=\int\int d^4x_1d^4x_2\, e^{ik_1x_1+ik_2x_2}\tilde{G}(x,y), \tag{2.1}$$

where

$$\begin{aligned}\tilde{G}(x,y)&=\left\langle 0\left|\frac{\delta^2C(0)}{\delta\phi_B(x_2)\delta\phi_A(x_1)}\right|0\right\rangle\\ &=-\theta(-x_1)\langle 0|\theta(-x_2)[[C(0),B(x_2)],A(x_1)]\\ &\qquad+\theta(x_1-x_2)[C(0),[A(x_1),B(x_2)]]|0\rangle. \end{aligned}\tag{2.2}$$

The current operators $A(x)$, $B(x)$, and $C(x)$ satisfy causality requirements of the form

$$\delta A(x_1)/\delta\phi_B(x_2)=-i\theta(x_1-x_2)[A(x_1),B(x_2)]=0 \tag{2.3}$$

unless we have

$$(x_1-x_2)^2\geqslant 0\quad\text{and}\quad (x_{10}-x_{20})\geqslant 0.$$

By means of an asymptotic condition, the function $G(k_1,k_2)$ is, of course, directly related to the matrix element

$$\langle p|C(0)|k\rangle,\quad p=k_1,\quad k=-k_2,$$

of the current operator $C(0)$ between the one-particle states corresponding to the fields ϕ_A and ϕ_B, respectively.

As a consequence of the causality condition (2.3), the integrand in Eq. (2.1) has support only if x_1 and x_2 both lie inside or on the past light cone; that is, we have

$$\tilde{G}(x,y)=0,\quad\text{unless}\quad x_{10}\leqslant-|\mathbf{x}_1|,\quad x_{20}\leqslant-|\mathbf{x}_2|.$$

These support properties imply that $G(k_1,k_2)$ is the boundary value of an analytic function in the components of the four vectors k_1 and k_2; it is regular in the tube domain

$$\operatorname{Im}k_{10}>|\operatorname{Im}\mathbf{k}_1|,\quad \operatorname{Im}k_{20}>|\operatorname{Im}\mathbf{k}_2|. \tag{2.4}$$

The invariance of $G(k_1,k_2)$ under orthochronous Lorentz transformations and the analyticity in the tube (2.4) are sufficient to assure that the analytic function depends only upon the inner products[5]

$$z_1=k_1^2,\quad z_2=k_2^2,\quad\text{and}\quad z_3=(k_1+k_2)^2. \tag{2.5}$$

It is regular in the domain $\mathfrak{M}$ over which the inner

[4] R. Jost, Helv. Phys. Acta **31**, 263 (1958).

[5] D. Hall and A. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 5 (1957).

products vary if the vectors vary over the tube (2.4). The region $\mathfrak{M}$ is a domain in the space of three complex variables which is bounded by pieces of analytic hypersurfaces.[6] Here we are only interested in the property of $\mathfrak{M}$ to contain the whole cut z_3 plane, provided z_1 and z_2 are real and negative. For such values of z_1 and z_2 we may then write[7]

$$F(z_1z_2z_3)=\frac{1}{\pi}\int_0^\infty d\sigma^2\frac{A(z_1z_2\sigma^2)}{\sigma^2-z_3}, \tag{2.6}$$

assuming that $F(z_1z_2z_3)$ is sufficiently bounded for $z_3\to\infty$. In general a finite number of subtractions may be required. The absorptive part $A(z_1z_2\sigma^2)$ is given by Eq. (1.2); it can be directly expressed as a Fourier transform of vacuum expectation values. Using Eqs. (2.1), (2.2), and (2.6), we find

$$A[k_1^2, k_2^2, (k_1+k_2)^2]=\frac{i}{2}\int\int d^4x_1d^4x_2\, e^{ik_1x_1+ik_2x_2}$$
$$\times\langle 0|\theta(-x_2)[[C(0),B(x_2)],A(x_1)]$$
$$+\theta(x_1-x_2)[C(0),[A(x_1),B(x_2)]]|0\rangle. \tag{2.7}$$

The representation (2.7) will enable us to show that, for fixed σ^2, the absorptive part $A(z_1z_2\sigma^2)$ is an analytic function of z_1 and z_2, which is regular in a certain domain $D(\sigma)$.

So far we have used only the causality conditions (2.3), but we shall need now also the spectral conditions, which may be expressed in the form

$$\begin{aligned}\langle 0|A|n\rangle&=0 \quad\text{unless}\quad p_n^2\geqslant a^2,\\ \langle 0|B|n\rangle&=0 \quad\text{unless}\quad p_n^2\geqslant b^2,\\ \langle 0|C|n\rangle&=0 \quad\text{unless}\quad p_n^2\geqslant c^2.\end{aligned} \tag{2.8}$$

Here $|n\rangle$ denotes a state with positive total energy p_{n0} and total momentum $\mathbf{p}_n$. Using the spectral conditions, we can simplify the expression (2.7). The first term in this representation may be decomposed with respect to a complete set of intermediate states $|n\rangle$ such that there always appears a factor $\langle n|A(x_1)|0\rangle$ or its complex conjugate. Hence we may use Eq. (2.8) to show that the first term in Eq. (2.7) vanishes for $k_1^2<a^2$. In the region of interest, i.e., for z_1 and z_2 real and negative, we may now write

$$A[k_1^2, k_2^2, (k_1+k_2)^2]$$
$$=\frac{i}{2}\int\int d^4xd^4y\exp[\tfrac{1}{2}i(k_1-k_2)\cdot y+i(k_1+k_2)\cdot x]$$
$$\times\theta(y)\langle 0|[C(-x),[A(\tfrac{1}{2}y),B(-\tfrac{1}{2}y)]]|0\rangle. \tag{2.9}$$

It is convenient to choose a Lorentz frame such that $k_1+k_2=(\sigma,0)$, $\sigma>0$. We introduce in Eq. (2.9) a sum over intermediate states in order to bring it into the form

$$(1/2i)\sum_{|n\rangle}\int d^4y\exp[\tfrac{1}{2}i(k_1-k_2)\cdot y]\theta(y)$$
$$\times\langle 0|[A(\tfrac{1}{2}y),B(-\tfrac{1}{2}y)]|n\rangle$$
$$\times\langle n|C(0)|0\rangle(2\pi)^4\delta(k_1+k_2-p_n). \tag{2.10}$$

Because of $\sigma>0$, the first term in the commutator (2.8) gives no contribution. Let us write $q=\frac{1}{2}(k_1-k_2)$ and consider the functions

$$f_{r,a}(q)=\mp(i/2)\int d^4y\, e^{iq\cdot y}\theta(\pm y)$$
$$\times\langle 0|[A(\tfrac{1}{2}y),B(-\tfrac{1}{2}y)]|n\rangle, \tag{2.11}$$

which are Fourier transforms of retarded and advanced functions, respectively. As a consequence of the spectral conditions, we find

$$f_r(q)-f_a(q)=0$$

unless we have

$$\begin{aligned}&(\tfrac{1}{2}p_n+q)^2\geqslant a^2,\quad \tfrac{1}{2}p_{n0}+q_0\geqslant 0,\\ &\text{and}\\ &(\tfrac{1}{2}p_n-q)^2\geqslant b^2,\quad \tfrac{1}{2}p_{n0}-q_0\geqslant 0.\end{aligned} \tag{2.4}$$

In Eq. (2.11) we take $p_n=k_1+k_2=(\sigma,0)$; then we have $f_r=f_a$ for all q that satisfy

$$\tfrac{1}{2}\sigma-(b^2+\mathbf{q}^2)^{\frac{1}{2}}<q_0<(a^2+\mathbf{q}^2)^{\frac{1}{2}}-\tfrac{1}{2}\sigma. \tag{2.12}$$

The properties of the functions f_r and f_a, as expressed in Eqs. (2.11) and (2.12), are sufficient to prove, by the usual chain of arguments, that both functions are boundary values of an analytic function $f(q)$.[3] This function is regular in the envelope of holomorphy of the domain $W\bigcup N(S)$, where we have

$$W=\{q:|\mathrm{Im}\, q_0|>|\mathrm{Im}\,\mathbf{q}|\}, \tag{2.13}$$

and $N(S)$ is a complex neighborhood of the region in real space defined by Eq. (2.12). Disregarding possible convergence factors and polynominals, the function $f(q)$ may be represented in the form[8]

$$f(q)=\int d^4u\int_{\kappa_0(u)}^\infty d\kappa\frac{\Phi(\kappa,u;\sigma^2)}{\kappa^2-(q-u)^2}, \tag{2.14}$$

where Φ vanishes except for

$$|u_0|+|\mathbf{u}|\leqslant\tfrac{1}{2}\sigma,\quad \kappa\geqslant\kappa_0 \tag{2.15}$$

with

$$\kappa_0=\max\{0, a-[(\tfrac{1}{2}\sigma+u_0)^2-\mathbf{u}^2]^{\frac{1}{2}}, b-[(\tfrac{1}{2}\sigma-u_0)^2-\mathbf{u}^2]^{\frac{1}{2}}\}.$$

[6] G. Källèn and A. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Skrifter 1, No. 6 (1958).
[7] The relations (2.6) and (2.7) can also be obtained by the use of the method described in Sec. 2 of reference 3.

[8] F. J. Dyson, Phys. Rev. 110, 1460 (1958); R. Jost and H. Lehmann, Nuovo cimento 5, 1598 (1957); L. Gårding and A. Wightman (private communication).

The envelope $E[W\bigcup N(S)]$ is then given by the set of points q for which the denominator in Eq. (2.14) does *not* vanish for *any* set of parameters satisfying the conditions (2.15). Note that $f(q)$ depends upon the internal variables of the state $|n\rangle$ with $p_n=(\sigma,0)$.

We may now insert the representation (2.14) into Eq. (2.10) and introduce, as a new weight function, the result of the summation over all states $|n\rangle$ with $p_n=(\sigma,0)$. Then we obtain a representation for $A(z_1z_2\sigma^2)$ which can be written in the form

$$A(k_1^2,k_2^2,\sigma^2)=\int d^4u\int_{\kappa_0(u)}^{\infty}d\kappa\frac{\Psi(\kappa^2,u;\sigma^2)}{\kappa^2-[\frac{1}{2}(k_1-k_2)-u]^2}. \quad (2.16)$$

The weight function Ψ has support only for u and κ satisfying Eqs. (2.15). In addition it follows from Eq. (2.10) and the spectral condition involving $C(x)$ that Ψ vanishes for $\sigma^2<c^2$. Although we have chosen a special Lorentz frame, we still have to make use of the invariance of A under space rotations. It follows that the weight Ψ can only depend upon the length of the vector $\mathbf{u}$. Therefore, taking q real, we can perform the angle integration in Eq. (2.16) and find a representation that depends only upon q_0 and $\mathbf{q}^2$. In order to avoid the appearance of a logarithm it is convenient to redefine the weight function and write the representation in the form

$$A(z_1z_2\sigma^2)=\int_0^{\frac{1}{2}\sigma}du\int_{u-\frac{1}{2}\sigma}^{\frac{1}{2}\sigma-u}du_0\times\int_{\kappa_0}^{\infty}d\kappa\frac{\bar{\chi}(\kappa,u_0,u;\sigma^2)}{[\kappa^2-(q_0-u_0)^2+\mathbf{q}^2+\mathbf{u}^2]^2-4\mathbf{u}^2\mathbf{q}^2}. \quad (2.17)$$

Here q_0 and $\mathbf{q}^2$ may now be taken complex. They are given in terms of the covariant variables z_1, z_2, and σ^2 by

$$q_0=(z_1-z_2)/2\sigma,\quad \mathbf{q}^2=(\sigma^2/4)-[(z_1+z_2)/2]+[(z_1-z_2)/2\sigma]^2. \quad (2.18)$$

Using the definition

$$\lambda(z_1z_2z_3)=z_1^2+z_2^2+z_3^2-2z_1z_2-2z_1z_3-2z_2z_3, \quad (2.19)$$

we finally obtain the cut-plane representation

$$F(z_1z_2z_3)=\frac{1}{\pi}\int_{c^2}^{\infty}d\sigma^2\frac{A(z_1z_2\sigma^2)}{\sigma^2-z_3},$$

where

$$A(z_1z_2\sigma^2)=\int_0^1d\xi\int_{\xi-1}^{1-\xi}d\eta\int_{\kappa_0}^{\infty}d\kappa\frac{\chi(\kappa,\xi,\eta;\sigma^2)}{[2\kappa^2+\frac{1}{2}(1+\xi^2-\eta^2)\sigma^2-(z_1+z_2)+\eta(z_1-z_2)]^2-\xi^2\lambda(z_1z_2\sigma^2)}, \quad (2.20)$$

and

$$\kappa_0=\max\{0,\ a-\tfrac{1}{2}\sigma[(1+\eta)^2-\xi^2]^{\frac{1}{2}},\ b-\tfrac{1}{2}\sigma[(1-\eta)^2-\xi^2]^{\frac{1}{2}}\}. \quad (2.21)$$

In the language of function theory, the simple angle integration which leads from Eq. (2.16) to Eqs. (2.17) or (2.20) corresponds to an enlargement of the envelope $E[W\bigcup N(S)]$. This gain is due to the fact that we have restricted our functions to the narrower class of rotation-invariant functions depending only upon q_0 and $\mathbf{q}^2$.

III. THE DOMAIN OF REGULARITY

In the space of two complex variables, the domain $D(\sigma)$ is the set of all points z_1, z_2 for which the denominator in Eq. (2.20) cannot vanish for any allowed set of parameters. The intersection of $D(\sigma)$ with the real space has already been discussed in I. Here we will generalize these results and compute the boundaries of the domains $D(\sigma)$ for certain cases of interest. As far as form factors are concerned we are mainly interested in the function $F(z_1z_2z_3)$ for $z_1=z_2\equiv z$ and $a=b$. In this case $D(\sigma)$ degenerates to a region in the complex z plane. For every $\sigma\geqslant c$ we can characterize the *complement* of $D(\sigma)$ by the set of all points which may be represented in the form

$$z=x+iy=\kappa^2+g^2\pm i\kappa(\sigma^2-4g^2)^{\frac{1}{2}}, \quad (3.1)$$

with

$$0\leqslant g\leqslant\tfrac{1}{2}\sigma \quad\text{and}\quad \kappa\geqslant\max\{0,a-g\}.$$

Then we find that the boundary can be described by pieces of the three curves

$$\begin{aligned}y=y_{\max}(x,\sigma)&\equiv\{[x+a(2x-a^2)^{\frac{1}{2}}]\times[\tfrac{1}{2}\sigma^2-x+a(2x-a^2)^{\frac{1}{2}}]\}^{\frac{1}{2}},\\ y=y_{\min}(x,\sigma)&\equiv\{[x-a(2x-a^2)^{\frac{1}{2}}]\times[\tfrac{1}{2}\sigma^2-x-a(2x-a^2)^{\frac{1}{2}}]\}^{\frac{1}{2}},\\ y&=\sigma x^{\frac{1}{2}}.\end{aligned} \quad (3.2)$$

For $\sigma=0$ we have the cut plane $x\geqslant a^2$, $y=0$, but as soon as σ becomes finite the cut is embedded into a singular region. In the interval $0\leqslant\sigma\leqslant a$, the domain $D(\sigma)$ is then given by

$$|y|>y_{\max}(x,\sigma)\quad\text{if}\quad \tfrac{1}{2}\sigma^2-a\sigma+a^2\leqslant x\leqslant a^2,$$

and

$$|y|>\sigma x^{\frac{1}{2}}\quad\text{if}\quad x\geqslant a^2. \quad (3.3)$$

For $a\leqslant\sigma\leqslant 2a$, we find

$$|y|<y_{\min}(x,\sigma)\quad\text{or}\quad |y|>y_{\max}(x,\sigma)\quad\text{if}\quad \tfrac{1}{2}a^2\leqslant x\leqslant\tfrac{1}{2}\sigma^2-a\sigma+a^2,$$

$$|y|>y_{\max}(x,\sigma)\quad\text{if}\quad \tfrac{1}{2}\sigma^2-a\sigma+a^2\leqslant x\leqslant a^2, \quad (3.4)$$

and

$$|y|>\sigma x^{\frac{1}{2}}\quad\text{if}\quad x\geqslant a^2.$$

Finally we obtain for $\sigma \geqslant a$

$$|y| < y_{\min}(x,\sigma) \quad \text{or} \quad |y| > y_{\max}(x,\sigma) \quad \text{for} \quad a^2/2 \leqslant x \leqslant a^2, \tag{3.5}$$

and

$$|y| > \sigma x^{\frac{1}{2}} \quad \text{for} \quad x \geqslant a^2.$$

In order to give a qualitative picture of the domains $D(\sigma)$ for various values of σ, we have plotted the boundary in Fig. 1 for several characteristic cases.

Of special interest is the intersection of the regions $D(\sigma)$ for all $\sigma \geqslant c$. Let us define

$$D(a,c) = \bigcap_{\sigma \geqslant c} D(\sigma).$$

Then the vertex function $F(zzz_3)$ is an analytic function of the two complex variables z and z_3, which is regular for $z \epsilon D(a,c)$ and all values of z_3 except those on the static cut $x_3 \geqslant c^2$, $y_3 = 0$. For spectral conditions with $c \leqslant a$, the domain $D(a,c)$ consists of all points z which lie to the *left* of the line $x = \frac{1}{2}a^2$, $-\infty \leqslant y \leqslant +\infty$. But for $c > a$, the region $D(a,c)$ contains, *in addition* to the points with $x < \frac{1}{2}a^2$, the set

$$|y| < y_{\min}(x,c), \quad \tfrac{1}{2}a^2 \leqslant x \leqslant \frac{c^2}{2} + a^2 - ac.$$

Here the quantity $y_{\min}(x,c)$ is the function given in Eq. (3.2) with a replaced by c. Using Fig. 1, one can easily visualize the shape of $D(a,c)$ as the intersection of all $D(\sigma)$ for $\sigma \geqslant c$.

We may now ask whether or not the regions $D(\sigma)$ and (or) $D(a,c)$ are characteristic for the assumptions we have made, namely Lorentz invariance, causality, and the spectral conditions. Here we call a domain characteristic if it is not possible to continue analytically beyond its boundary without introducing additional, new assumptions into the theory. From the method used in Sec. 2 for the derivation of the cut-plane representation, it is not evident that the domains $D(\sigma)$ are characteristic. However, we shall show in the following that it is possible to find functions that satisfy all the conditions we have imposed and that have singularities at points on the boundary of $D(\sigma)$. In this way we can account for all boundary points of the domains $D(\sigma)$ and thus prove that they are characteristic.

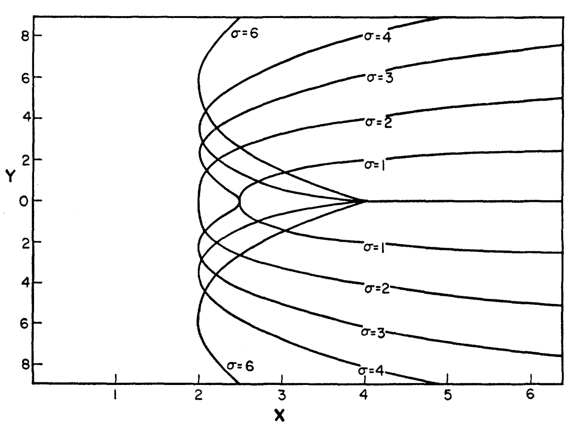

FIG. 1. Boundaries of the domains $D(\sigma)$ for various values of σ and $z \equiv z_1 = z_2$, $a = b = 2$. The region of analyticity lies always to the left of the corresponding curve. The region $D(a,c)$ is the intersection of all domains $D(\sigma)$ for $\sigma \geqslant c$.

Perturbation theory is a rich source for the construction of functions satisfying our axioms. We use here especially the function

$$F_p(z,z_3) = F_p(zzz_3, mmm_3),$$

which has been discussed extensively in I and II. In II we have shown that, for all $z = x + iy$ with $x < m^2 + m_3^2$, the function F_p may be represented in the form

$$F_p(z,z_3) = \int_{4m^2}^{\infty} d\sigma^2 \frac{A_p(z,\sigma^2)}{\sigma^2 - z_3}, \tag{3.6}$$

where the absorptive part $A_p(z,\sigma^2)$ is given by

$$A_p(z,\sigma^2) = \frac{1}{m_3^2\sigma^2 + [z - (m+m_3)^2][z - (m-m_3)^2]} \times \frac{2(z - m_3^2 + m^2) - \sigma^2}{[\sigma^2(\sigma^2 - 4m^2)]^{\frac{1}{2}}}. \tag{3.7}$$

By construction, the function $F_p(z,z_3)$ satisfies all our axioms including the spectral conditions, provided we require

$$m + m_3 \geqslant a \quad \text{and} \quad 2m \geqslant c. \tag{3.8}$$

We may also generalize $F_p(z,z_3)$ somewhat by choosing the lower limit of the integral in Eq. (3.6) to be an independent constant $c_0 \geqslant c$. Then we consider $A_p(z,\sigma^2)$ only for $\sigma \geqslant c_0$, and the spectral conditions are satisfied for $m + m_3 \geqslant a$ even though the mass $2m$ in Eq. (3.7) may be smaller than c.

For a given $\sigma \geqslant c$, the absorptive part $A_p(z,\sigma^2)$ has a pair of poles at

$$z = x + iy = m^2 + m_3^2 \pm i m_3[\sigma^2 - 4m^2]^{\frac{1}{2}}. \tag{3.9}$$

Let us choose the "masses" m and m_3 such that we have

$$m + m_3 = a$$

and

$$m^2 + m_3^2 = x,$$

or

$$2m = a \pm (2x - a^2)^{\frac{1}{2}},$$

and

$$2m_3 = a \mp (2x - a^2)^{\frac{1}{2}}, \tag{3.10}$$

for $x \geqslant \frac{1}{2}a^2$. Using the upper sign in Eqs. (3.10), we see from Eq. (3.9) that the position of the pair of poles is given by x and $y = \pm y_{\min}(x,\sigma)$. On the other hand, the lower sign gives $y = \pm y_{\max}(x,\sigma)$. Here the quantities $y_{\max}(x,\sigma)$ and $y_{\min}(x,\sigma)$ are just the functions that have been defined in Eqs. (3.2). We can now account for

the complete boundary of $D(\sigma)$ for any given $\sigma \geqslant c$. For every pair of points on the boundary, we use function (3.7) for different values of the parameters m and m_3:

(a) The points on the curve $y = \pm y_{\min}(x,\sigma)$ are obtained by the use of the upper sign in Eqs. (3.10). This curve is applicable in the interval $\frac{1}{2}a^2 \leqslant x \leqslant \frac{1}{2}\sigma^2 - a\sigma + a^2$ for $a < \sigma \leqslant 2a$ and in $\frac{1}{2}a^2 \leqslant x \leqslant a^2$ for $\sigma \geqslant 2a$. In both cases we have $\sigma \geqslant 2m \geqslant a$.

(b) We obtain $y = \pm y_{\max}(x,\sigma)$ using the lower sign in Eqs. (3.10). For $\sigma \geqslant a$ the curve is applicable in the range $\frac{1}{2}a^2 \leqslant x \leqslant a^2$ and for $\sigma < a$ in the interval $\frac{1}{2}\sigma^2 - a\sigma + a^2 \leqslant x \leqslant a^2$. We have always $a \geqslant 2m \geqslant 0$ besides the conditions $\sigma \geqslant c$.

(c) Points on the curve $y = \pm \sigma x^{\frac{1}{2}}$ for $x > a^2$ can be obtained with our example if we use the parameters $m = 0$, $m_3 = x^{\frac{1}{2}}$.

In the previous considerations we have used perturbation theory only as a mathematical tool for the construction of functions which satisfy the axioms. However, as we have seen in I and especially in II, it can also be used as a guide for the understanding of the relation of certain singularities to physical properties of particles. It is the boundary of the domain $D(a,c)$ that is of primary interest in this connection. For real, positive values of the mass variable $z = x$ and real, negative values of the momentum-transfer variable $z_3 = x_3$, the vertex function $F(z,z_3) \equiv F(zzz_3)$ may be interpreted as an electromagnetic form factor of a stable particle with mass $x^{\frac{1}{2}}$. For $z \epsilon D(a,c)$, this form factor is an analytic function in the z_3 plane except for the static cut $x_3 \geqslant c$, $y_3 = 0$, which is related to absorptive processes. For the examples in II a condition corresponding to $z \epsilon D(a,c)$ guarantees the absence of structure singularities in the physical sheet of the z_3 plane. But for real values of the mass variable, which are above the boundary point of $D(a,c)$ on the positive real axis, the structure singularities are in the physical sheet; often they determine the slope of the distribution in coordinate space. The formal connection between the real boundary point of $D(a,c)$ in perturbation theory and in representation (2.20) has been discussed in I. In view of the results obtained in II, we can now describe these limitations in a physical language. Let us use an example. In perturbation theory, the nucleon form factors have only the usual absorptive singularities (static cuts) in the physical sheet of the z_3 plane[9,2]; they describe the pion cloud. Because of the conservation of nucleon number, it is *not* possible to consider the nucleon as a loosely bound system of two other particles with masses m and m_3 such that the binding energy $B = m + m_3 - M$ is smaller than $m + m_3 - (m^2 + m_3^2)^{\frac{1}{2}}$. However, if we use only the axioms employed in Sec. 2, the spectral conditions do not specify the type of particles appearing in the intermediate states. All they do is set a lower bound for the total mass of a given state. Let us take the isotopic scalar part of the electromagnetic nucleon form factor. Then we have $a = M + m_\pi$, $c = 3m_\pi$, and the boundary curve of the domain $D(a,c)$ is given by the line $x = \frac{1}{2}(M + m_\pi)^2$, $-\infty \leqslant y \leqslant +\infty$. The point $z = M^2$ corresponding to the mass shell lies outside of $D(a,c)$, and consequently there will be certain structure singularities in the physical sheet of the z_3 plane. For example, the spectral conditions do not prevent us from considering the nucleon as a bound state of a boson with mass $m = \frac{3}{2}m_\pi$ and a baryon with $m_3 = M - \frac{1}{2}m_\pi$ such that $m + m_3 = M + m_\pi$ and $2m = 3m_\pi$. This system is bound loosely enough for the slope of the m-particle distributions to reach outside the pion cloud. The maximal range of the probability distribution of the m particle is given by[2]

$$r_0 = \frac{2M - m_\pi}{4m_\pi[(2M + m_\pi)(M - m_\pi)]^{\frac{1}{2}}} > \frac{1}{3m_\pi}.$$

Although the conservation of nucleon number has been used in order to obtain the spectral condition with $a = M + m_\pi$, the condition itself does not exhaust the information about the corresponding lowest intermediate state $|n\rangle$. We have not used the fact that this state consists of one physical nucleon and one pion. If we want to exhaust such information in the general approach, we are led to consider relations of the vertex function with Green's functions of higher order. It is reasonable to expect that a more complete analysis of this kind will eliminate the *unphysical* structure singularities. However, according to their definition in II, structure singularities appear always on the real axis, and we know, especially from Jost's example,[4] that they are not the only source of nonabsorptive singularities in the z_3 plane. In any case, as we have seen from the example given in I as well as in Sec. 1 of this paper, the limitation $x < \frac{1}{2}(M + m_\pi)^2$ can be understood in terms of unphysical structure singularities.[10]

IV. DIRECT REPRESENTATION

In the previous sections we have discussed the analytic properties of the vertex function $F(z_1z_2z_3)$ starting with the requirement of regularity in the cut z_3 plane. From Eq. (2.20) we can in general obtain a domain $D(a,b,c)$ such that $F(z_1z_2z_3)$ is an analytic function of three complex variables for $(z_1,z_2) \epsilon D(a,b,c)$ and z_3 in the cut plane. If (z_1,z_2) are not in $D(a,b,c)$, we expect additional singularities to appear somewhere in the z_3 plane. For this case it would be of interest to know the exact shape of the region of analyticity in the z_3 plane, especially for real, positive values $z_1 = x_1 < a^2$

[9] Y. Nambu, Nuovo cimento **6**, 1064 (1957); and **9**, 610 (1958). K. Symanzik, Progr. Theoret. Phys. (Kyoto) **20**, 690 (1958); R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. **111**, 1187 (1958).

[10] In the case of the pion-nucleon vertex and the nucleon-nucleon scattering amplitude, the situation is completely analogous to the one described here.

and $z_2=x_2<b^2$. The problem of finding the region that is characteristic for the axioms is essentially equivalent to that of computing the complete envelope of holomorphy of the primitive domain obtained from Lorentz invariance, causality, and spectrum. We shall not undertake this rather involved task.[11] Instead we derive a simple representation, which is not the best possible, but which is sufficient to show that, for positive a, b, and c, the points where singularities may occur are restricted to a finite region in the z_3 plane and the static cut $x_3\geqslant c^2$, $y_3=0$.

Let us write the vertex function $\langle k|C(0)|p\rangle$ in the form

$$G(k,-p)=-i\int d^4x\, e^{ik\cdot x}\theta(-x)\langle 0|[C(0),A(x)]|p\rangle,$$

where $|p\rangle$ is a one-particle state corresponding to the field ϕ_B. We introduce the related advanced function and write, using translation invariance,

$$G_{r,a}(k,-p)=\pm i\int d^4x\, e^{iq\cdot x}\theta(\pm x)\times\langle 0|[A(\tfrac{1}{2}x),C(-\tfrac{1}{2}x)]|p\rangle, \quad (4.1)$$

where $q=k-\frac{1}{2}p$. We choose a Lorentz frame such that $p=(2t,0)$. As a consequence of the spectral conditions (2.8), we find then that we have $G_r-G_a=0$ for all $q\epsilon S'$ where S' is the region

$$t-(c^2+\mathbf{q}^2)^{\frac{1}{2}}<q_0<(a^2+\mathbf{q}^2)^{\frac{1}{2}}-t. \quad (4.2)$$

By arguments that are completely analogous to those used in Sec. II, it follows that G_r and G_a are boundary values of an analytic function $G(q,t)$ which may be represented in the form[8]

$$G(q,t)=\int d^4u\int_{\kappa_1(u)}^{\infty}d\kappa\,\frac{\phi(\kappa,u;t)}{\kappa^2-(q-u)^2}. \quad (4.3)$$

The weight function ϕ vanishes unless we have

$$|u_0|+|\mathbf{u}|\leqslant t,$$

and

$$\kappa\geqslant\kappa_1=\max\{0,\,a-[(t+u_0)^2-\mathbf{u}^2]^{\frac{1}{2}},\; c-[(t-u_0)^2-\mathbf{u}^2]^{\frac{1}{2}}\}.$$

Because of rotation invariance, ϕ depends only upon the amount of the vector $\mathbf{u}$. We can perform the redundant angle integration, and, using

$$q_0=\frac{z_1-z_3}{4t}\quad\text{and}\quad \mathbf{q}^2=t^2-\frac{z_1+z_3}{2}+\left(\frac{z_1-z_3}{4t}\right)^2,$$

we obtain the following representation for the vertex function $F(z_14t^2z_3)$:

$$F(z_14t^2z_3)=\int_0^1 d\xi\int_{\xi-1}^{1-\xi}d\eta\int_{\kappa_1}^{\infty}d\kappa\,\frac{\rho(\kappa,\xi,\eta;t)}{[2\kappa^2+2(1+\xi^2-\eta^2)t^2-(z_1+z_3)+\eta(z_1-z_3)]^2-\xi^2\lambda(z_14t^2z_3)}. \quad (4.4)$$

The function $\lambda(z_1z_2z_3)$ has been defined in Eq. (2.19), and κ_1 is given by

$$\kappa_1=\max\{0,\,a-t[(1+\eta)^2-\xi^2]^{\frac{1}{2}},\; c-t[(1-\eta)^2-\xi^2]^{\frac{1}{2}}\}. \quad (4.5)$$

Note that in Eq. (4.4) the quantity $x_2=4t^2$ is a real parameter, $2t$ being the mass of the one-particle state $|p\rangle$. The spectral condition (2.8) of the B field does not appear explicitly in the formula (4.4), but we have always $b>2t$ because the field $\phi_B(x)$ describes a stable particle. The region of analyticity obtained from the representation (4.4) is the envelope of holomorphy of the primitive domain $W\bigcup N(S')$ with respect to the class of rotation-invariant functions depending upon q_0 and $\mathbf{q}$. Here W is the tube domain (2.13) resulting mainly from the causality condition, and $N(S')$ is a suitable complex neighborhood of the real region S' given in Eq. (4.2). The regularity of F for $(z_1,z_3)\epsilon N(S')$ is a consequence of causality and the spectral conditions for the A and the C field. We would like to stress that for the derivation of Eq. (4.4) we have not used all of the implications of causality and spectrum, and hence we cannot expect that the resulting region of analyticity is characteristic as far as these physical requirements are concerned. However, disregarding possible subtractions, the representation (4.4) is the most general function that satisfies the mathematical conditions contained in Eqs. (4.1) and (4.2).

Let us now discuss the region of analyticity in the z_3 plane for real values of the mass variable z_1. Since we are mainly interested in form factors, we shall take $a=b$ and $x_1=x_2\equiv 4t^2$, where $2t<a$. In addition to the static cut $x_3\geqslant c^2$, $y_3=0$, we have then singularities at those points of the z_3 plane that can be represented in the form $z_3=x_3+iy_3$, where x_3 and y_3 are real, and given by

$$x_3=2t^2\frac{(1+\eta)[\lambda^2-(1-\eta)^2+\xi^2]-4\xi^2}{(1+\eta)^2-\xi^2}, \quad (4.6)$$

and

$$y_3=\pm\frac{2t^2\xi}{(1+\eta)^2-\xi^2}(\{\lambda^2-2+[(1+\eta)^2-\xi^2]^{\frac{1}{2}}\}\times\{2-[(1+\eta)^2-\xi^2]^{\frac{1}{2}}-\lambda^2\})^{\frac{1}{2}},$$

where $\lambda\geqslant\kappa_1/t$, $0\leqslant\xi\leqslant 1$ and $|\eta|\leqslant 1-\xi$. We consider first the special case $a=c$, in order to get an idea about the shape of the region defined by Eqs. (4.6). The

[11] Compare reference 6, where the envelope has been computed for the case $a=b=c=0$. For arbitrary spectral conditions, it is somewhat more difficult to guess the boundary of the envelope, which is more or less a prerequisite for the present method of proof. We hope that the results of this paper may be of some help in this connection.

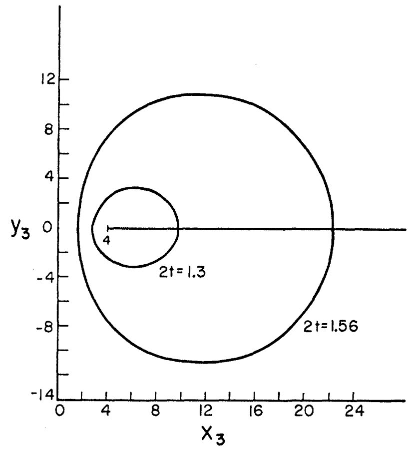

FIG. 2. Region of analyticity of the vertex function in the z_3 plane. The parameters are $a=b=c=2$, $x_1=x_2=4t^2$. For a given value of the mass variable, $2t$, the function is regular outside the corresponding closed curve and with the exception of the static cut.

domain of *analyticity* is as follows:

(a) for $0\leqslant 2t\leqslant \frac{1}{2}a$, we have the whole z_3 plane except for the static cut $x_3\geqslant a^2$, $y_3=0$;

(b) in the range $\frac{1}{2}a\leqslant 2t\leqslant \frac{1}{4}a[(17)^{\frac{1}{2}}-1]$, the region

$$\begin{gathered}|y_3|\leqslant y_1(x_3),\\ 2a(a-2t)\leqslant x_3\leqslant 8t^2[1+2t/(a-2t)]\end{gathered}\tag{4.7}$$

is excepted in addition to the static cut, and

(c) for $a/4[(17)^{\frac{1}{2}}-1]\leqslant 2t<a$, the cut and the region

$$\begin{gathered}|y_3|\leqslant y_1(x_3),\\ 2a(a-2t)\leqslant x_3\leqslant 8t^2[1+2t/(a-2t)];\\ y_2(x_3)\leqslant |y_3|\leqslant y_1(x_3),\\ 8t^2[1-a^2/4(a^2-4t^2)]\leqslant x_3\leqslant 2a(a-2t)\end{gathered}\tag{4.8}$$

are excepted. The functions $y_1(x_3)$ and $y_2(x_3)$ are given by

$$y_{1,2}(x_3)=H[x_3,h_{\mp}(x_3)],\tag{4.9}$$

where we have

$$h_{\mp}(x_3)=\frac{1}{8t^2-x_3}\left\{2at\mp\left[2(a^2-4t^2)\times\left(x_3-8t^2+\frac{2a^2t^2}{a^2-4t^2}\right)\right]^{\frac{1}{2}}\right\}\tag{4.10}$$

and

$$H(x_3,h)=[1-h^2]^{\frac{1}{2}}[(8t^2/h)^2-(x_3-8t^2)^2]^{\frac{1}{2}}.\tag{4.10a}$$

In Figs. 2 and 3 we have plotted some of the regions (4.7) and (4.8). As long as $2t$ is less than a, the complex singular region is finite; especially, we have always analyticity in a strip along the negative real axis. But as $2t$ approaches a, the point $x_3=\alpha$ in Fig. 3 moves to $-\infty$, the point $x_3=\beta$ to zero, and $x_3=\gamma$ to $+\infty$. At $2t=a$, the mass variable $x_1=2t$ coincides with the lowest branch point $x_1=a$, $y_1=0$ in the z_1 plane, and the singular region (4.8) covers the whole z_3 plane.

We have mentioned earlier that the boundaries of the regions (4.7) and (4.8) cannot be expected to be characteristic for the physical assumptions we have made. That this is actually the case may be seen by comparison with the cut-plane representation. Take for instance the vertex function $F(z_1z_2z_3)$ for $z_1=z_2=x<a^2$ and assume $a=b=c$. According to the cut-plane representation the real points $z_3=x_3<2a(a-x^{\frac{1}{2}})$, $x_3<a^2$ are inside the region of analyticity. On the other hand, we know from Sec. 2 that $F(xxz_3)$ is analytic for $x_3<a^2$ provided $x\leqslant\frac{1}{2}a^2$; for $\frac{1}{4}a^2<x<\frac{1}{2}a^2$ we have $2a(a-x^{\frac{1}{2}})<a^2$.

In view of the applications, we are especially interested in the region of analyticity in the z_3 plane for the case $c<a$, $x_1=x_2=4t^2$. Again the direct representation (4.4) shows that we have regularity except for the static cut $x_3\geqslant c^2$, $y_3=0$ and a complex region around the lower end of this cut. It can be seen from Eqs. (4.5) and (4.6) that the points where singularities are allowed are restricted to a finite region as long as $2t$ is less than a. We shall not describe here the shape of the singular region for $c<a$, because its boundary is not characteristic. We give only the points where the boundary of the complex singular domain intersects the real axis. The left-hand point has already been mentioned in an earlier publication.[12] It corresponds to $x_3=\alpha$ in Fig. 3 and is given by

$$x_3=\frac{4tc(a-2t)}{2t+a-c}.\tag{4.11}$$

For the right-hand point, which corresponds to $x_3=\gamma$ in Fig. 3, we find the following expressions: for

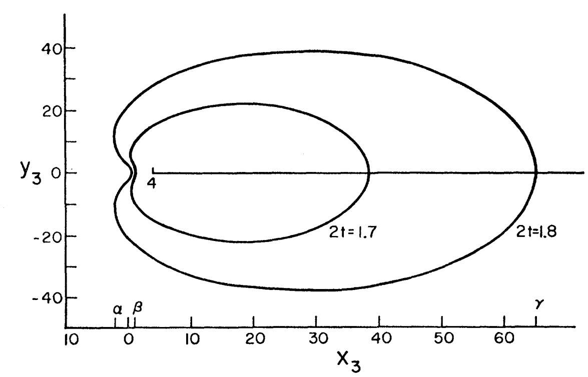

FIG. 3. Region of analyticity of the vertex function in the z_3 plane. Parameters are the same as Fig. 2, but for larger values of the mass variable, $2t$. For $2t\to a=2$ we have $\alpha\to-\infty$, $\beta\to 0_+$ and $\gamma\to+\infty$.

[12] R. Oehme and J. G. Taylor, Phys. Rev. **113**, 371 (1959); see below Eq. (3.10).

$a \geqslant c > 2a/3$ we obtain

$$x_3^{(\gamma)} = 8t^2\left(1+\frac{2t}{a-2t}+\frac{(a-c)(c-2t)}{2t(a-2t)}\right), \quad (4.12)$$

provided we have $\frac{1}{2}c \leqslant 2t \leqslant 2c-a$, and

$$x_3^{(\gamma)} = 8t^2\{1+[2t/(a-2t)]+(a-2t)/8t\} \quad (4.13)$$

for $2c-a \leqslant 2t < a$. For cases where we have $0 \leqslant c \leqslant 2a/3$, we find Eq. (4.13) provided $\frac{1}{3}a \leqslant 2t < a$. In the special case of the electromagnetic form factors for the nucleon, we have for the isotopic vector part $a = M+m_\pi$, $c = 2m_\pi$, and $2t = M$, which gives

$$\begin{aligned} x_3^{(\beta)} &= 2m_\pi^2 2M/(2M-m_\pi), \\ x_3^{(\gamma)} &= (M/2m_\pi)(2M+m_\pi). \end{aligned} \quad (4.14)$$

The isotopic scalar part requires $c = 3m_\pi$ and leads to

$$\begin{aligned} x_3^{(\beta)} &= 3m_\pi^2 M/(M-m_\pi),^{12} \\ x_3^{(\gamma)} &= (M/2m_\pi)(2M+m_\pi). \end{aligned} \quad (4.15)$$

In problems related to the question of consistency of quantum electrodynamics, it is sometimes useful to know some analytic properties of the electron-photon vertex function. From the direct representation, we can say only the following: if one is willing to introduce a small, auxiliary photon mass $\lambda > 0$ such that we have $x = m_e^2$, $a = b = m_e + \lambda$, $c = 3\lambda$, then the singularities in the z_3 plane are restricted to a finite region and the static cut $x_3 \geqslant (3\lambda)^2$, $y_3 = 0$. The real boundary points of the region with complex singularities are given by Eqs. (4.15) with M replaced by m_e, and m_π by λ. Note that for $\lambda \to 0$ the mass variable $x_1 = x_2 = m_e^2$ coincides with the static cut $x \geqslant a^2 = \lim_{\lambda\to 0}(m_e+\lambda)^2$, $y = 0$, and the singular region covers the whole z_3 plane.

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Kinematics of General Scattering Processes and the Mandelstam Representation

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The kinematics of an arbitrary process involving two incoming and two outgoing particles is studied in terms of the invariants used in Mandelstam's representation, treating the three processes described by the same Green's function simultaneously. It is shown that the physical regions for these processes are bounded by a cubic curve in the plane of the two independent invariants. The unitarity conditions are discussed in the approximation of neglecting intermediate states of more than two particles. The formula for the spectral functions of the double dispersion relation is obtained explicitly in terms of the invariants chosen.

1. INTRODUCTION

MANDELSTAM[1] has recently proposed a representation of the scattering amplitude for meson-nucleon scattering, which is obtained from a plausible assumption about its behavior as an analytic function of two variables, the energy and momentum transfer. He has also been able to show,[2] for a more general process, that the representation is satisfied by the lower orders of the perturbation series, and that this series can actually be constructed from the representation and the unitarity relations,[3] in a two-particle approximation. In this paper we shall discuss certain aspects, mainly kinematical, of the extension of this representation to a general process. We consider together the three processes

$$\text{I}: \quad 1+2 \to 3+4,$$
$$\text{II}: \quad 1+\bar{3} \to \bar{2}+4,$$
$$\text{III}: \quad 1+\bar{4} \to \bar{2}+3.$$

The complications of spin and isotopic spin will be ignored, and all the particles will be assumed to be stable.

In Sec. 2 we shall find the physical regions for the three scattering processes in terms of the three invariants r, s, t, whose sum is equal to the sum of squared masses of the four particles. These invariants may be regarded as homogeneous coordinates in a plane, and the physical regions are then bounded by a cubic curve in this plane. The curve has three branches corresponding to the physical regions for the three scattering processes, and also a closed branch within the rst-triangle. The interior of this closed curve would correspond to the physical region for the decay process

$$\text{IV}: \quad 1 \to \bar{2}+3+4$$

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).
[3] S. Mandelstam, Phys. Rev. **115**, 1752 (1959).