# Extension of the Blatt-Biedenharn Formalism for Resonance Reactions*

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The Blatt-Biedenharn formalism for the calculation of cross sections in resonance reactions is extended to provide for (1) empirical spin-orbit effects in the potential scattering, (2) the interference of resonances of unequal spin and/or parity, and (3) the energy spread of the incident particle. Explicit formulas are given for the total cross section and for the Legendre polynomial expansion coefficients in the differential cross section, for single channel elastic scattering, in the presence of all three effects. For a general multichannel reaction, explicit reaction and elastic scattering formulas, both differential and total, are derived only for the second situation.

## I. INTRODUCTION

THE formalism of Blatt and Biedenharn[1] (hereafter referred to as BB) is well known and widely used for the calculation of differential and total cross sections from interaction parameters, and conversely for the determination of nuclear properties, such as phase shifts and resonance parameters, from experimental cross-section data. In this formalism the cross sections are given as sums of terms involving the elements of scattering matrices $\mathcal{S}^{J\Pi}$.

The element $S^{J\Pi}(\alpha's'l';\alpha sl)$ of the scattering matrix $\mathcal{S}^{J\Pi}$ stands for the probability amplitude for a collision with total angular momentum $J$ and parity $\Pi$ from channel $\alpha sl$ into channel $\alpha's'l'$. The channel index $\alpha$ defines the type of incoming particle and the quantum state of the struck nucleus; the channel spin $s$ is the total spin angular momentum in the channel; $l$ is the incoming orbital angular momentum. The primed indices refer to the corresponding after-collision quantities. For an $N$ channel reaction there are $(\frac{1}{2}N)\times(N+1)$ independent elements of the unitary and symmetric matrix $\mathcal{S}^{J\Pi}$, $N$ of which are eigenphase shifts. The cross section formulas contain an $\mathcal{S}$ matrix for each combination of $J$ and $\Pi$ which enters into the reaction. *It is the function of a nuclear model to yield the elements of these scattering matrices; or failing that, at least to make statements which limit their complexity.*

A favorite and often satisfactory model is the assumption that the reaction proceeds by way of a single resonance level of the compound nucleus. To provide a specific (and important) example of the use of their formalism, BB have adapted the resonance theory of Wigner and Eisenbud[2] (hereafter referred to as WE) to obtain an explicit formula for the $\mathcal{S}$ matrix elements.

In many cases, however, in which a resonance structure is clearly observed in the cross sections, and where one resonance clearly predominates, the description in terms of hard sphere phase shifts and the parameters of a single level has turned out to be inadequate. To deal with such cases some experimenters have adopted the practice of treating the potential phase shifts as wholly empiricial quantities instead of as hard sphere parameters which depend only on the channel radius and energy. These potential phase shifts, which can be thought of as reflecting the effects of distant resonances, are then determined from experiment along with the parameters of the explicitly recognized single level.

It seems desirable to investigate the validity and the physical significance of this procedure in terms of the WE theory and to attempt to generalize the BB formulation to include the possibility of empirical potential parameters. This is done in Sec. II, in which we obtain an expression for the $\mathcal{S}$ matrix which is appropriate for substitution into the general BB cross section formalism. The elements of the $\mathcal{S}$ matrix depend on the parameters of a single level and on a set of empirical potential parameters which, unlike the hard sphere phase shifts or their usual *ad hoc* empirical generalizations, are characterized not only by the orbital angular momentum quantum number, but by the total angular momentum as well; they show spin-orbit coupling effects. They can be written in terms of (diagonal) phase shifts only for special cases.

Another modification of the single level assumption seems called for when several distinguishable resonances combine to give the observed cross section at a particular energy. If these levels belong to the same matrix, i.e., have both their spins and parities in common, the multilevel algebra becomes very complex[3]; but if it can be assumed that the resonance levels belong to different matrices, the generalization of the BB formalism is straightforward. In Sec. III we work out explicit formulas for the differential and total cross sections for *single channel elastic scattering*, which allow for *both* spin-orbit coupled empirical phase shifts and many levels of different spin and/or parity. The first of these

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[1] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. 24, 258 (1952).

[2] E. P. Wigner and L. Eisenbud, Phys. Rev. 72, 29 (1947).

[3] E. P. Wigner, Phys. Rev. 70, 606 (1946).

generalizations becomes more complicated, when applied to the multichannel case, and will not be done here[4]; the second, however, can be readily undertaken and is performed in Sec. IV.

Because of the energy spread of the particles, the actually observed cross sections, near a resonance, will differ from those given by BB and those so far mentioned in this paper. The precise correction will, of course, depend on the nature of the resolution function. While there is generally little difficulty in making the required correction for well isolated levels, no general formalism seems to exist for the case of overlapping resonances. We therefore devote Sec. V to a calculation of the cross sections for single level elastic scattering with spin-orbit phase shifts and many levels, taking account of the finite energy spread of the particles. We use a square resolution function and observe that the results will not be very different for any other reasonable distribution.[5]

## II. THE $\mathcal{S}$ MATRIX WITH SPIN-ORBIT POTENTIAL EFFECTS

We recall WE's expression for the scattering matrix in terms of the derivative matrix $\mathcal{R}$ [6]:

$$\mathcal{S}^J=\omega\frac{1+i\mathcal{B}\mathcal{R}^J\mathcal{B}+i\mathcal{C}}{1-i\mathcal{B}\mathcal{R}^J\mathcal{B}-i\mathcal{C}}\omega. \tag{1}$$

In this expression $\omega$, $\mathcal{B}$, and $\mathcal{C}$ are diagonal matrices; $\mathcal{B}$ and $\mathcal{C}$ are real, $\omega$ is of modulus 1, and all three can be readily calculated once the energy, channel radius, and non-nuclear interactions (centrifugal and Coulomb forces) are specified.[7]

The real and symmetric $\mathcal{R}$ matrix is wholly determined by the goings on in the interior region, i.e., by the nuclear interactions. It may be expressed in terms of the set of eigenstates of the nuclear system by

$$R^J(\alpha's'l';\alpha sl)=\sum_\lambda\frac{\gamma_\lambda{}^J(\alpha's'l')\gamma_\lambda{}^J(\alpha sl)}{E_\lambda{}^J-E}. \tag{2}$$

The $E_\lambda{}^J$ are the eigenvalues (resonance energies) of the system and $\gamma_\lambda{}^J(\alpha sl)$ is an energy independent coefficient which depends on the size of the $\lambda$th eigenfunction at the entrance to channel $\alpha sl$; it may be considered as the $\alpha sl$ component of a vector $\gamma_\lambda{}^J$.

The single level approximation consists of replacing the sum in Eq. (2) by a single term

$$R^J(\alpha's'l';\alpha sl)=\frac{\gamma_\lambda{}^J(\alpha's'l')\gamma_\lambda{}^J(\alpha sl)}{E_\lambda{}^J-E}. \tag{3}$$

This approximation leads to the Breit-Wigner formula and also to the more general single level formulas of BB. Significantly for our work, the derivation of the single level formula is done by WE in two steps; first the infinite sum in expression (2) is replaced by

$$R^J(\alpha's'l';\alpha sl)=\frac{\gamma_\lambda{}^J(\alpha's'l')\gamma_\lambda{}^J(\alpha sl)}{E_\lambda{}^J-E}+R_\infty{}^J(\alpha's'l';\alpha sl), \tag{4}$$

where $\mathcal{R}_\infty{}^J$ is an energy independent matrix, and only after an expression is obtained for the scattering matrix is terms of this "generalized one level formula" is the further assumption made that $\mathcal{R}_\infty{}^J$ is equal to zero.

*We make the basic assumption in this paper that not only is $\mathcal{R}_\infty{}^J$ not equal to zero but that it is to be replaced by a matrix $\mathcal{R}_p{}^J$ which varies slowly with energy:*

$$R^J(\alpha's'l';\alpha sl)=\frac{\gamma_\lambda{}^J(\alpha's'l')\gamma_\lambda{}^J(\alpha sl)}{E_\lambda{}^J-E}+R_p{}^J(\alpha's'l';\alpha sl). \tag{5}$$

Since, however, the formal work of WE does not make use of the fact that their $\mathcal{R}_\infty{}^J$ is independent of $E$, we can, in part, follow their procedure in arriving at an expression for the scattering matrix. This procedure introduces the symmetric and real matrix $\mathcal{C}'^J$, which in out case is given by

$$\mathcal{C}'^J=\mathcal{B}\mathcal{R}_p{}^J\mathcal{B}+\mathcal{C}; \tag{6}$$

the vectors $\mathbf{a}_\lambda{}^J$,[8]

$$\mathbf{a}_\lambda{}^J=(1-i\mathcal{C}'^J)^{-1}\mathcal{B}\gamma_\lambda{}^J; \tag{7}$$

the partial widths $\Gamma_\lambda{}^J(\alpha sl)$ and the total width $\Gamma_\lambda{}^J$,

$$\begin{aligned}\Gamma_\lambda{}^J(\alpha sl)&=2a_\lambda{}^{*J}(\alpha sl)a_\lambda{}^J(\alpha sl),\\ \Gamma_\lambda{}^J&=2(\mathbf{a}_\lambda{}^{*J},\mathbf{a}_\lambda{}^J)=\sum_{\alpha sl}\Gamma_\lambda{}^J(\alpha sl);\end{aligned} \tag{8}$$

and the level shift $\Delta_\lambda{}^J$,

$$\Delta_\lambda{}^J=(\mathbf{a}_\lambda{}^{*J};\mathcal{C}'^J a_\lambda{}^J). \tag{9}$$

In terms of these quantities the scattering matrix may then be written

$$\begin{aligned}S^J(\alpha's'l';\alpha sl)&\\ =\omega(\alpha's'l';\alpha's'l')\omega(\alpha sl;\alpha sl)\times&\left(\frac{1+iC'^J}{1-iC'^J}(\alpha's'l';\alpha sl)\right.\\ &\left.+\frac{2ia_\lambda{}^J(\alpha's'l')a_\lambda{}^J(\alpha sl)}{E_\lambda{}^J+\Delta_\lambda{}^J-E-\frac{1}{2}i\Gamma_\lambda{}^J}\right).\end{aligned} \tag{10}$$

[4] *Note added in proof.*—An expression for the $\mathcal{S}$ matrix for the general (multichannel) case has now been obtained. [H. Lustig, Nuclear Phys. (to be published).]

[5] After submission of this paper, the author was informed that results similar to those of Secs. III and IV are described by Willard, Biedenharn, Huber, and Baumgartner in the forthcoming book *Fast Neutron Physics*, edited by J. L. Fowler and J. B. Marion (Interscience Publishers, New York, 1959).

[6] Actually we write down not the WE $\mathfrak{U}^J$ matrix, but the BB matrix, by letting the elements of the matrix range over the three indices $\alpha$, $s$, $l$, instead of over the two indices $s$, $l$, and by letting the matrix element $\omega(\alpha sl;\alpha sl)$ equal $i^l$ times the WE element $\omega(sl;sl)$. The superscript for the parity, which serves no useful purpose at this stage, is left out.

[7] This notation differs from that used in the authoritative review paper of A. M. Lane and R. G. Thomas [Revs. Modern Phys. **30**, 257 (1958)]. We use the WE notation since it forms the basis of the BB formulation which, in turn, we consider particularly well suited for calculation and comparison with experiment.

[8] We use $\mathbf{a}_\lambda{}^J$ in place of the WE $\boldsymbol{\alpha}_\lambda{}^J$, in order to avoid confusion with our channel index.

Except for changes in notation and for the different meaning of the $\mathcal{C}'^J$ matrix, this expression is identical with formula (56) of WE. We now introduce the observed resonance energy $E_0{}^J=E_\lambda{}^J+\Delta_\lambda{}^J$, by neglecting the energy dependence of the $\Delta_\lambda{}^J$, drop the superfluous index $\lambda$, and introduce the square roots of the partial widths, $g^J(\alpha sl)$,

$$g^J(\alpha sl)=\pm[2a^{*J}(\alpha sl)a^J(\alpha sl)]^{\frac{1}{2}}=\pm[\Gamma^J(\alpha sl)]^{\frac{1}{2}}, \quad (11)$$

enabling us to write (10) as

$$\begin{aligned}S^J(\alpha's'l';\alpha sl)&=\omega(\alpha's'l';\alpha's'l')(a^J(\alpha's'l')/a^{*J}(\alpha's'l'))^{\frac{1}{2}}\\&\times\omega(\alpha sl;\alpha sl)(a^J(\alpha sl)/a^{*J}(\alpha sl))^{\frac{1}{2}}\\&\times\left[\frac{1+iC'^J}{1-iC'^J}(\alpha's'l';\alpha sl)\left(\frac{a^{*J}(\alpha's'l')}{a^J(\alpha's'l')}\right)^{\frac{1}{2}}\right.\\&\left.\times\left(\frac{a^{*J}(\alpha sl)}{a^J(\alpha sl)}\right)^{\frac{1}{2}}+\frac{ig^J(\alpha's'l')g^J(\alpha sl)}{E_0{}^J-E-\frac{1}{2}i\Gamma^J}\right]. \quad (12)\end{aligned}$$

Both the primed and unprimed factors in front of the bracket are complex functions of absolute value 1. We may, therefore, write

$$\begin{aligned}S^J(\alpha's'l';\alpha sl)&=\exp[i\eta^J(\alpha's'l')]\exp[i\eta^J(\alpha sl)]\\&\times\left[\frac{1+iC'^J}{1-iC'^J}(\alpha's'l';\alpha sl)\left(\frac{a^{*J}(\alpha's'l')}{a^J(\alpha's'l')}\right)^{\frac{1}{2}}\right.\\&\left.\times\left(\frac{a^{*J}(\alpha sl)}{a^J(\alpha sl)}\right)^{\frac{1}{2}}+\frac{ig^J(\alpha's'l')g^J(\alpha sl)}{E_0{}^J-E-\frac{1}{2}i\Gamma^J}\right], \quad (13)\end{aligned}$$

or, simply,

$$\begin{aligned}S^J(\alpha's'l';\alpha sl)&=\exp[i\eta^J(\alpha's'l')]\exp[i\eta^J(\alpha sl)]\\&\times\left(\mathcal{T}^J(\alpha's'l';\alpha sl)+\frac{ig^J(\alpha's'l')g^J(\alpha sl)}{E_0{}^J-E-\frac{1}{2}i\Gamma^J}\right), \quad (14)\end{aligned}$$

where $\mathcal{T}^J$ is a unitary, symmetric matrix. This expression for the scattering matrix element may now be substituted into the appropriate formula for the Legendre expansion coefficient $B_L$ [BB, formula (4.7)] and the differential and total cross sections may then be obtained. The elements of $\mathcal{T}^J$ as well as the $g^J$ are to be considered as empirical parameters which may, in principle, be found by comparison with enough experiments. In practice, this task will be facilitated by the knowledge that all the parameters vary slowly with energy as long as $\mathcal{R}_p{}^J$ does, and that the $g^J$ will show up only near the resonance energy. It may, for example, be possible to extrapolate the potential parameters calculated near one resonance with known resonance parameters to a neighboring resonance and then, in turn, to determine the parameters of that resonance.

Expressions (13) and (14) undergo a drastic simplification for single-channel reactions. Before proceeding to that situation it will be instructive to show how expression (13) goes over into BB's single level (but multi-channel) scattering matrix formula. Since $\mathcal{B}$, $\gamma^J$, and $\mathcal{C}'^J$ are real, expression (7) shows that

$$\mathbf{a}^J=(1-i\mathcal{C}'^J)^{-1}(1+i\mathcal{C}'^J)\mathbf{a}^{*J}. \quad (15)$$

If now $\mathcal{R}_p{}^J$ is set equal to zero, then $\mathcal{C}'^J$ is equal to the diagonal matrix $\mathcal{C}$ and (15) leads to

$$\frac{1+iC'^J}{1-iC'^J}(\alpha's'l';\alpha sl)=\frac{a^J(\alpha sl)}{a^{*J}(\alpha sl)}\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}, \quad (16)$$

whereupon (13) becomes

$$\begin{aligned}S^J(\alpha's'l';\alpha sl)&=e^{i\eta(\alpha's'l')}e^{i\eta(\alpha sl)}\\&\times\left(\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}+\frac{ig^J(\alpha's'l')g^J(\alpha sl)}{E_0{}^J-E-\frac{1}{2}i\Gamma^J}\right). \quad (17)\end{aligned}$$

The $\eta(\alpha sl)$ now depend only on $\omega$ and $\mathcal{C}$ and are independent of the total angular momentum $J$. Furthermore, if it is assumed that the channel radius is independent of the channel spin $s$ (although it may vary for different channels $\alpha$), the $\eta(\alpha sl)$ may be expressed in terms of the known, spin independent "hard sphere" phase shifts, plus appropriate Coulomb factors for charged particle reactions. If this is done, the BB single level scattering matrix element is obtained [their formula (5.6)].

Our more general expressions (13) and (14), unlike (17) contain quantities other than diagonal phase shifts and resonance parameters. This is because only $N$ of the $(\frac{1}{2}N)(N+1)$ independent elements of $\mathcal{S}^J$ are eigenphase shifts. For a *single channel reaction* (i.e., elastic scattering without change of channel spin or orbital angular momentum, for a given total angular momentum and parity) the one potential parameter is ipso facto a phase shift. Even though $\mathcal{C}'$ does not equal $\mathcal{C}$, equation (16) now holds *a fortiori*, and we obtain

$$\begin{aligned}S^J(\alpha sl;\alpha sl)&=\exp[2i\eta^J(\alpha sl)]\\&\times[1+i\Gamma^J/(E_0{}^J-E-\tfrac{1}{2}i\Gamma^J)]. \quad (18)\end{aligned}$$

Expression (18) looks similar to the single channel case of (17) or of BB. We should note, however, that our potential phase shifts $\eta^J(\alpha sl)$ are $J$ dependent; they are potential phase shifts in the presence of spin-orbit coupling. They vary slowly with energy and, unlike the equivalent parameters in the single level case, must be determined from experiment. Before this can be done, expressions for the observable quantities, viz., the cross sections, must be obtained. This is done in the following section. There is, or course, no formal reason why the matrix element for a single

channel reaction should be made to depend on three parameters $\eta$, $\Gamma$, and $E_0$. In fact we can write the single element of $\mathcal{S}^J$ as

$$S^J(\alpha sl;\alpha sl)=\exp[2i\delta^J(\alpha sl)], \tag{19}$$

where

$$\delta^J(\alpha sl)=\eta^J(\alpha sl)-\arctan[\tfrac{1}{2}\Gamma^J/(E-E_0{}^J)].$$

There is, however, a good practical reason for retaining the form (18), for only in this way can we take advantage of the different effects on the cross sections of the potential and of the resonance level parameters, respectively.

One further remark may be in order. The formulation of nuclear reaction theory in terms of the $\mathcal{R}$ matrix is completely general and is valid whether a true compound nucleus is formed or not. The single level approximation, however, introduces into the theory the Bohr assumption of a compound nucleus in which the incident particle "loses its memory" and in which the mode of decay is independent of the mode of formation; the single level postulate is a sufficient (but not necessary) condition for requiring the Bohr assumption to hold.[9] There is, by now, ample experimental evidence against the general validity of the Bohr assumption. Our formalism does not require the picture of a true compound nucleus; it is therefore valid for the treatment of direct interaction processes, and may be particularly suitable for those cases, in which part of the cross section may be ascribed to a direct interaction and the remainder to a single level of the compound nucleus.

## III. CROSS SECTIONS FOR SINGLE-CHANNEL REACTIONS

The general formulas for the cross sections in terms of the $\mathcal{S}$ matrices have been given by BB. The differential cross section $\sigma_{\alpha';\,\alpha}(\theta)$ must be derived from cross sections with definite incoming and outgoing channel spins by averaging over the possible values of $s$ and summing over the possible values of $s'$:

$$\sigma_{\alpha';\,\alpha}(\theta)=\sum_{s=|I-i|}^{I+i}\sum_{s'=|I'-i'|}^{I'+i'}\frac{2s+1}{(2I+1)(2i+1)}\sigma_{\alpha's';\,\alpha s}(\theta), \tag{20}$$

where $i$ and $I$ are the spins of the incoming particle and of the struck nucleus, and $i'$ and $I'$ the spins of the emerging particle and residual nucleus, respectively. The differential cross section for the $\alpha s\rightarrow\alpha's'$ collision can then be written in terms of the Legendre expansion coefficients $B_L$,

$$\sigma_{\alpha's';\,\alpha s}(\theta)=[\lambda\!\!\!^{-}_{\alpha}{}^2/(2s+1)]\sum_{L=0}^{\infty}B_L(\alpha's';\alpha s)\times P_L(\cos\theta), \tag{21}$$

where $\lambda\!\!\!^{-}_{\alpha}$ is the de Broglie wavelength (divided by $2\pi$) in the center-of-mass system and the $P_L(\cos\theta)$ are the usual Legendre polynomials. The expansion coefficients $B_L(\alpha's';\alpha s)$ are given by

$$\begin{aligned}B_L(\alpha's';\alpha s)=&\frac{(-1)^{s'-s}}{4}\sum_{J_1}\sum_{J_2}\sum_{l_1}\sum_{l_2}\sum_{l_1'}\sum_{l_2'}Z(l_1J_1l_2J_2,sL)\\&\times Z(l_1'J_1l_2'J_2,s'L)\,\text{R.P.}\{[\delta_{\alpha'\alpha}\delta_{s's}\delta_{l1'l1}\\&-S^{J_1\Pi_1}(\alpha's'l_1';\alpha sl)]^*\\&\times[\delta_{\alpha'\alpha}\delta_{s's}\delta_{l2'l2}-S^{J_2\Pi_2}(\alpha's'l_2';\alpha sl_2)]\}.\end{aligned} \tag{22}$$

The $Z$ (Racah) coefficients arise from having performed explicitly the sums over the magnetic quantum numbers[10]; they have been tabulated by Biedenharn.[11] The dummy indices 1 and 2 have their origin in the multiplication of amplitudes.[12] R.P. stands for the real part of the expression in brackets. The sums over angular momenta, except one, are restricted by vector addition and parity conservation rules[13]; we shall explicitly give these for our final expressions. The one unrestricted momentum runs, in principle, to infinity; in practice there is, for short range forces, a maximum effective value. The total cross section $\sigma_{\alpha';\,\alpha}$ is given by

$$\sigma_{\alpha';\alpha}=[4\pi\lambda\!\!\!^{-}_{\alpha}{}^2/(2I+1)(2i+1)]\textstyle\sum_s\sum_{s'}B_0(\alpha's';\alpha s). \tag{23}$$

For a single-channel reaction these formulas simplify considerably. Since there is only one value of $s=s'$, the differential cross section becomes

$$\sigma_{\alpha;\,\alpha}(\theta)=[\lambda\!\!\!^{-}_{\alpha}{}^2/(2I+1)(2i+1)]\textstyle\sum_L B_L(\alpha s;\alpha s)\times P_L(\cos\theta), \tag{24}$$

and the total cross section is

$$\sigma_{\alpha;\,\alpha}=[4\pi\lambda\!\!\!^{-}_{\alpha}{}^2/(2I+1)(2i+1)]B_0(\alpha s;\alpha s). \tag{25}$$

Since for a given value of $J$ and $\Pi$ only one $l$ is allowed, $B_L$ is given by[14]

$$\begin{aligned}B_L(\alpha s;\alpha s)=&(\tfrac{1}{4})\textstyle\sum_{J_1}Z^2(l_1J_1l_2J_2,sL)\\&\times\text{R.P.}\{[1-S^{J_1\Pi_1}(\alpha sl_1;\alpha sl_1)]^*\\&\times[1-S^{J_2\Pi_2}(\alpha sl_2;\alpha sl_2)]\}.\end{aligned} \tag{26}$$

[9] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. X.

[10] Biedenharn, Blatt, and Rose, Revs. Modern Phys. **24**, 249 (1952). These $Z$ coefficients differ from the $Z$ coefficients of Lane and Thomas by lacking a factor $i^{l_1-l_2-L}$.

[11] L. C. Biedenharn, Oak Ridge National Laboratory Reports ORNL-1098, 1952 and ORNL-1501, 1953 (unpublished).

[12] We shall consistently use numerical subscripts as dummies in multiplying amplitudes, and primes to distinguish the exit channel from the entrance channel. This notation differs from the practice of BB who use $l$ and $l'$ when only two indices are needed and have recourse to numerical subscripts when additional subscripts are required, irrespective of meaning.

[13] Our expression (22) differs from that given by BB in that we have restored the parity index to the scattering matrix. We need not include a formal summation over the parity index since this process is implicitly included in the summations over the $l$'s.

[14] One may, at first sight, wonder about the presence of sums over $l_1$ and $l_2$, considering that each $\mathcal{S}$ matrix now has only one element. We should remember, however, that for each value of $J$

It is into this formula that we must substitute our expression (18) for the matrix elements. The work of BB assumes that the entire reaction involves only a single level of the compound nucleus, i.e., that the $S$ matrix for one value of $J$ and $\Pi$ contains a parameter of a single resonance level as well as hard sphere parameters, and the $S$ matrices for the other values of $J$ and $\Pi$ depend on hard sphere parameters only. This picture may be generalized in two ways; (1) by explicitly admitting one resonance level into each $S^{J\Pi}$ matrix and (2) by making provision for additional resonances in a particular $S^{J\Pi}$ matrix. We have already carried out the latter generalization to the extent of including the $\mathcal{R}_p{}^J$ matrix; we shall now deal with the former in admitting an *arbitrary number of levels of different $J$ and/or $\Pi$*. These two generalizations are independent of one another; one may be made without the other. We shall proceed to work out the single channel case in the presence of both modifications. The reduction for the case of only one explicitly denumerated level will be immediately obvious. On the other hand, if we wish to introduce only the generalization to many levels of different $J$ and/or $\Pi$ and to restrict ourselves to the assumption that the reaction for a particular $J$ and $\Pi$ proceeds entirely by way of a single resonance state, i.e., that $\mathcal{R}_p{}^J$ equals zero for all $J$ and $\Pi$, there is no need to specialize to the single channel case. Results for the general multichannel case are obtained in Sec. IV.

Let there be $n$ denumerated levels of different $J$ and/or parity. The total angular momentum of the $k$th level will be denoted by $J^k$ and the orbital angular momentum (which is then determined by the parity of the level) by $l^k$. The width and resonance energy of this level will be abbreviated as $\Gamma_k$ and $E_k$. Dropping the indices $\alpha$ and $s$, expression (18) becomes

$$S^J(l)=\exp[2i\eta^J(l)]\times\left(1+\left[\frac{i\Gamma_k}{(E_k-E-\frac{1}{2}i\Gamma_k)}\right]\delta_{JJ^k}\delta_{ll^k}\right). \quad (27)$$

The calculation which ensues upon substitution of (27) into (26) is straightforward. We introduce the angle $\beta_k$,

$$\tan\beta_k=(E-E_k)/\tfrac{1}{2}\Gamma_k, \quad (28)$$

and make use of the rule[10]

$$Z(abcd,ef)=(-1)^fZ(cdab,ef). \quad (29)$$

We write the final result as

$$B_L(\alpha s;\alpha s)=H_L(\alpha s;\alpha s)+R_L(\alpha s;\alpha s)+I_L(\alpha s;\alpha s), \quad (30)$$

where, having insured for the purposes of practical computation that each separate term appears only once in a sum,

$$H_L(\alpha s;\alpha s)=\sum_{l=0}^{l_{\max}}\sum_{J=|l-s|}^{l+s}Z^2(lJlJ,sL)\sin^2\eta^J(l)+2\sum_{l_1=0}^{l_{\max}}\sum_{J_1=|l_1-s|}^{l_1+s}\Big\{\sum_{l_2=l_1+1}^{l_1+L}\sum_{J_2=|l_2-s|}^{l_2+s}Z^2(l_1J_1l_2J_2,sL)\times\sin\eta^{J_1}(l_1)\sin\eta^{J_2}(l_2)\cos[\eta^{J_1}(l_1)-\eta^{J_2}(l_2)]+\sum_{J_2=J_1+1}^{l_1+s}Z^2(l_1J_1l_1J_2,sL)\sin\eta^{J_1}(l_1)\times\cos[\eta^{J_1}(l_1)-\eta^{J_2}(l_2)]\Big\}, \quad (31)$$

$$R_L(\alpha s;\alpha s)=\sum_{k=1}^{n}Z^2(l^kJ^kl^kJ^k,sL)\cos^2\beta_k+2\sum_{k_1=1}^{n}\sum_{k_2=k_1+1}^{n}Z^2(l^{k_1}J^{k_1}l^{k_2}J^{k_2},sL)\cos\beta_{k1}\cos\beta_{k2}\times\cos[2\eta^{J^{k_1}}(l^{k_2})-2\eta^{J^{k_1}}(l^{k_1})+\beta_{k2}-\beta_{k1})], \quad (32)$$

and

$$I_L(\alpha s;\alpha s)=-2\sum_{k=1}^{n}\sum_{l=|l_k-L|}^{(l_k+L)\le l_{\max}}\sum_{J=|l-s|}^{l+s}Z^2(lJl^kJ^k,sL)\times\cos\beta_k\sin\eta^J(l)\sin[2\eta^{J^k}(l^k)-\eta^J(l)+\beta_k]. \quad (33)$$

The $H_L$, $R_L$, and $I_L$ terms may be thought of as potential scattering, resonance scattering, and interference terms, respectively; however, we again call attention to the fact that our phase shifts are not true potential phase shifts in the BB sense, and may include the effect of distant resonances. We have explicitly given the limits on the sums which arise from the laws of vector addition of momenta; the value of $l_{\max}$ is determined by the range of forces and energy of the incident particle.[15] In addition, the requirements[10]

there are two matrices (one for each value of the parity) and that we have chosen to absorb the summation over parity into the summation over orbital angular momentum. In this case the presence of the orbital angular momentum index allows us to dispense with the parity index altogether. We could, instead, explicitly sum over $\Pi_1$, and $\Pi_2$ and delete the sums over $l_1$ and $l_2$, but this would make for a clumsier notation in our results.

[15] In the presence of Coulomb forces the sum over $l$ converges so slowly in the phase shifts as to make our expression useless for practical computation, as it stands. The case of the scattering of charged particles may, however, be treated by explicit summation of the Coulomb amplitudes [see BB; also H. Lustig and J. M. Blatt, Phys. Rev. **100**, 777 (1955)].

that for each Racah coefficient $Z(abcd,ef)$,

$$f\leq b+d \quad \text{and} \quad a+c-f \text{ be even}, \tag{34}$$

reduce the number of terms as follows:

The first term of $H_L$ will be present only for even $L\leq 2J$; the terms involving $Z^2(l_1J_1l_2J_2,sL)$ will be present for even $L\leq J_1+J_2$ if $(-1)^{l_1}=(-1)^{l_2}$ and for odd $L\leq J_1+J_2$ otherwise; the terms with $Z^2(l_1J_1l_1J_2, sL)$ will remain only for even $L\leq J_1+J_2$. The first term of $R_L$ will occur only for even $L\leq 2J^k$, and the second term for even $L\leq J^{k_1}+J^{k_2}$ if $k_1$ and $k_2$ have opposite parity. The interference term will be present if $L$ is even for $(-1)^l=(-1)^{l_k}$, and if $L$ is odd for $(-1)^l=(-1)^{l_{k+1}}$; in any case $L\leq J+J^k$.

Since[10] $Z^2(abcd,e0)=\delta_{ac}\delta_{bd}(2b+1)$, the total cross section is given by

$$\sigma_{\alpha;\,\alpha}=[4\pi\lambda_\alpha^2/(2I+1)(2i+1)]\{\sum_{l=0}^{l_{\max}}\sum_{J=|l-s|}^{l+s}(2J+1)$$
$$\times\sin^2\eta^J(l)+\sum_{k=1}^{n}(2J^k+1)\cos^2\beta_k-2\sum_{k=1}^{n}(2J^k+1)$$
$$\times\cos\beta_k\sin\eta^{J^k}(l^k)\sin[\eta^{J^k}(l^k)+\beta_k]\}. \tag{35}$$

## IV. MULTICHANNEL RESULTS FOR MANY LEVELS OF DIFFERENT SPIN AND/OR PARITY

We now give results for multichannel reactions which involve an arbitrary number of resonance levels of the compound nucleus of different total angular momentum, parity, or both. The derivative matrix for a given value of $J$ and $\Pi$ is, however, given by the parameters of a single level, and the potential phase shifts are true hard sphere parameters. The scattering matrix element is then given by expression (17) which, for purposes of substitution into (22) we write as

$$S^{J\Pi}(\alpha's'l';\alpha sl)=e^{i\eta(\alpha'l')}e^{i\eta(\alpha l)}\Big(\delta_{\alpha'\alpha}\delta_{s's}\delta_{l'l}$$
$$+\frac{ig^k(\alpha's'l')g^k(\alpha sl)}{E_k-E-\frac{1}{2}i\Gamma_k}\delta_{JJ^k}\delta_{\Pi\Pi^k}\Big). \tag{36}$$

For neutral particles the $\eta(\alpha l)$ are the hard sphere phase shifts themselves; for charged particle reactions Coulomb screening factors appear which, however, drop out of the final formulas.

The differential cross section for a reaction (as distinguished from elastic scattering) $\alpha'\neq\alpha$, or $s'\neq s$, or both, is then given by

$$B_L(\alpha's';\alpha s)$$
$$=(-1)^{s'-s}\sum_{k=1}^{n}\sum_{l^k=|J^k-s|}^{J^k+s}\sum_{l^{k\prime}=|J^{k\prime}-s'|}^{J^k+s'}Z(l^kJ^kl^kJ^k,sL)$$
$$\times Z(l^{k\prime}J^kl^{k\prime}J^k,s'L)[\Gamma^k(\alpha sl)\Gamma^k(\alpha's'l')/\Gamma_k^2]$$
$$\times\cos^2\beta_k+2(-1)^{s'-s}\sum_{k_1=1}^{n}\sum_{l^{k_1}=|J^{k_1}-s|}^{J^{k_1}+s}\sum_{l^{k_1\prime}=|J^{k_1}-s'|}^{J^{k_1}+s'}$$
$$\times\Bigg(\sum_{k_2=k_1+1}^{n}\sum_{l^{k_2}=|J^{k_2}-s|}^{J^{k_2}+s}\sum_{l^{k_2\prime}=|J^{k_2}-s'|}^{J^{k_2}+s'}$$
$$\times Z(l^{k_1}J^{k_1}l^{k_2}J^{k_2},sL)Z(l^{k_1\prime}J^{k_1\prime}l^{k_2\prime}J^{k_2\prime},s'L)$$
$$\times\frac{g^{k_1}(\alpha sl_1)g^{k_1}(\alpha's'l_1')g^{k_2}(\alpha sl_2)g^{k_2}(\alpha's'l_2')}{\Gamma_{k1}\Gamma_{k2}}\cos\beta_{k1}\cos\beta_{k2}$$
$$\times\cos[\eta(\alpha l^{k_1})-\eta(\alpha l^{k_2})+\eta(\alpha' l^{k_1\prime})-\eta(\alpha' l^{k_2\prime})$$
$$+\beta_{k1}-\beta_{k2}]+\sum_{l^{k_2}=l^{k_1}+1}^{J^{k_1}+s}\sum_{l^{k_2\prime}=|J^{k_1}-s'|}^{J^{k_1}+s'}$$
$$\times Z(l^{k_1}J^{k_1}l^{k_2}J^{k_1},sL)Z(l^{k_1\prime}J^{k_1}l^{k_2\prime}J^{k_1},s'L)$$
$$\times\frac{g^{k_1}(\alpha sl_1)g^{k_1}(\alpha's'l_1')g^{k_1}(\alpha sl_2)g^{k_1}(\alpha's'l_2')}{\Gamma_{k1}^2}\cos^2\beta_{k1}$$
$$\times\cos[\eta(\alpha l^{k_1})-\eta(\alpha l^{k_2})+\eta(\alpha l^{k_1\prime})-\eta(\alpha l^{k_2\prime})]$$
$$+\sum_{l^{k_2\prime}=l^{k_1\prime}+1}^{J^{k_1}+s'}Z(l^{k_1}J^{k_1}l^{k_1}J^{k_1},sL)Z(l^{k_1\prime}J^{k_1}l^{k_2\prime}J^{k_1},s'L)$$
$$\times\frac{\Gamma^{k_1}(\alpha sl)g^{k_1}(\alpha's'l_1')g^{k_1}(\alpha's'l_2')}{\Gamma_{k1}^2}$$
$$\times\cos^2\beta_{k1}\cos[\eta(\alpha' l^{k_1\prime})-\eta(\alpha' l^{k_2\prime})]\Bigg). \tag{37}$$

In addition to the restrictions on the ranges of the $l$'s indicated, these orbital angular momentum numbers must satisfy the parity conservation laws. If we denote the channel parity of channel $\alpha$ by $\Pi_\alpha$, that of channel $\alpha'$ by $\Pi_{\alpha'}$, and that of the $k$th level of the compound nucleus by $\Pi^k$, the additional restrictions are

$$\begin{aligned}(-1)^{l^{k_1}}&=\Pi_\alpha\Pi^{k_1}, \quad (-1)^{l^{k_2}}=\Pi_\alpha\Pi^{k_2},\\ (-1)^{l^{k_1\prime}}&=\Pi_{\alpha'}\Pi^{k_1}, \quad (-1)^{l^{k_2\prime}}=\Pi_{\alpha'}\Pi^{k_2}.\end{aligned} \tag{38}$$

The number of terms is again reduced by the requirements (34).

The total cross section for the reaction $\alpha\rightarrow\alpha'$ becomes after introduction of $\Gamma_{\alpha l}=\sum_{s=|I-i|}^{I+i}\Gamma_{\alpha sl}$

$$\sigma_{\alpha';\,\alpha}=[4\pi\lambda\!\!\!^{-}_{\alpha}{}^2/(2I+1)(2i+1)]\times\sum_{k=1}^{n}\sum_{l^k=|J^k-(I+i)|}^{J^k+I+i}\sum_{l^{k\prime}=|J^k-(I'+i')|}^{J^k+I'+i'}(2J^k+1)\times[\Gamma^k(\alpha'l')/\Gamma_k{}^2]\cos^2\beta_k. \quad (39)$$

The expansion coefficient for the differential elastic scattering cross section (without change of channel spin) may again be written as the sum of an $H_L$, an $R_L$, and an $I_L$ term. To obtain the potential scattering term we use the sum rule for $Z$ coefficients[10]

$$\sum_b Z(abcd,ef)Z(abc'd,ef)=\delta_{cc'}(2a+1)(2d+1)\times[ac00\,|\,acf0]^2, \quad (40)$$

where $[ac00\,|\,acf0]$ is a Clebsch-Gordan coefficient, defined as in Condon and Shortley.[16] $H_L$ is then given by

$$H_L(\alpha s;\alpha s)=(2s+1)\sum_{l=0}^{l_{\max}}(2l+1)^2[ll00\,|\,llL0]^2\sin^2\eta(\alpha l)+(2s+1)\sum_{l_1=0}^{l_{\max}}\sum_{l_2=l_1+1}^{l_1+L}(2l_1+1)(2l_2+1)\times[l_1l_200\,|\,l_1l_2L0]^2\sin\eta(\alpha l_1)\sin\eta(\alpha l_2)\times\cos[\eta(\alpha l_1)-\eta(\alpha l_2)]. \quad (41)$$

The resonance term $R_L(\alpha s;\alpha s)$ is given by the entire $B_L(\alpha's';\alpha s)$ for a reaction (expression 37 with $\alpha'$ set equal to $\alpha$ and $s'$ set equal to $s$); the same parity and vector addition restrictions apply. The interference term $I_L$ becomes, by virtue of (40),

$$I_L(\alpha s;\alpha s)=-2\sum_{k=1}^{n}\sum_{l^k=|J^k-s|}^{J^k+s}\sum_{l=|l^k-L|}^{l^k+L}(2J^k+1)(2l+1)\times[ll^k00\,|\,ll^kL0]^2[\Gamma^k(\alpha sl^k)/\Gamma_k]\times\cos\beta_k\sin\eta(\alpha l)\sin[2\eta(\alpha l^k)-\eta(\alpha l)+\beta_k]. \quad (42)$$

The values of $l^k$ in this expression are restricted by the parity rules (38).

Using the fact that $[l_1l_2\,|\,l_1l_200]^2=\delta_{l1l2}(2l_1+1)^{-1}$ and using the notation $\Gamma(\alpha l)=\sum_s\Gamma(\alpha sl)$, the total cross section for pure elastic scattering is given by

$$\sigma_{\alpha;\,\alpha}=4\pi\lambda\!\!\!^{-}_{\alpha}{}^2\sum_{l=0}^{\infty}(2l+1)\sin^2\eta(\alpha l)+[4\pi\lambda\!\!\!^{-}_{\alpha}{}^2/(2I+1)(2i+1)]\Big(\sum_{k=1}^{n}\sum_{l^k=J^k-s}^{J^k+s}(2J^k+1)\times[\Gamma^k(\alpha l^k)/\Gamma^k]\cos\beta_k\{[\Gamma^k(\alpha l^k)/\Gamma^k]\cos\beta_k-2\sin\eta(\alpha l^k)\sin[\eta(\alpha l^k)+\beta_k]\}\Big). \quad (43)$$

[16] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935).

## V. SINGLE-CHANNEL CROSS SECTIONS WITH SQUARE RESOLUTION FUNCTION

We shall now extend our expressions for the cross sections in the presence of overlapping resonances and empirical phase shifts to provide for the energy spread of the incident particles. For the sake of simplicity we choose a square resolution function of width $\Delta$, i.e., we assume that at a nominal channel energy $E_c$, the actual beam has a uniform energy spread from $E_c-\Delta/2$ to $E_c+\Delta/2$. The precise nature of the resolution function is not very important. For example, a simple calculation shows that for an isolated resonance whose width $\Gamma$ equals the energy spread $\Delta$, a square resolution function reduces the maximum value of the total reaction cross section to 0.78 of its uncorrected value, whereas a Gaussian function whose width at half maximum is $\Delta$ yields a value of 0.72 for the same quantity.

We must average every term in expressions (31), (32), (33), and (35) over the interval $E_c-\Delta/2\leq E\leq E_c+\Delta/2$. We shall neglect the energy dependence of $\lambda\!\!\!^{-}$, $\eta$, and $\Gamma_k$ in this integration. The $H_L$ term then remains unchanged. In the first term of $R_L$ we must perform the integral

$$\int_{E_c-\Delta/2}^{E_c+\Delta/2}\cos^2\beta_k dE=\tfrac{1}{2}\Gamma_k\tan^{-1}\left(\frac{\Delta/\tfrac{1}{2}\Gamma_k}{1+(E_c-E_k)^2/(\tfrac{1}{2}\Gamma_k)^2-(\Delta/\Gamma_k)^2}\right)\equiv\tfrac{1}{2}\Gamma_k\Phi_k. \quad (44)$$

The second term of $R_L$ involves the integrals

$$Q_I=\int_{E_c-\Delta/2}^{E_c+\Delta/2}(\cos^2\beta_{k1}\cos^2\beta_{k2}+\cos\beta_{k1}\cos\beta_{k2}\times\sin\beta_{k1}\sin\beta_{k2})dE,$$

and

$$Q_{II}=\int_{E_c-\Delta/2}^{E_c+\Delta/2}(\cos\beta_{k1}\cos^2\beta_{k2}\sin\beta_{k1}-\cos^2\beta_{k1}\cos\beta_{k2}\sin\beta_{k2})dE. \quad (45)$$

The first of these may be transformed into

$$Q_I=\tfrac{1}{2}\Gamma_{k1}\tfrac{1}{2}\Gamma_{k2}\int_{E_c-\Delta/2}^{E_c+\Delta/2}\{\cos\gamma_{k1k2}[(\cos^2\beta_{k1}/\tfrac{1}{2}\Gamma_{k1})+(\cos^2\beta_{k2}/\tfrac{1}{2}\Gamma_{k2})]+\sin\gamma_{k1k2}[(\sin\beta_{k1}\cos\beta_{k1}/\tfrac{1}{2}\Gamma_{k1})-(\sin\beta_{k2}\cos\beta_{k2}/\tfrac{1}{2}\Gamma_{k2})]\}dE, \quad (46)$$

and the second into

$$Q_{II}=\tfrac{1}{2}\Gamma_{k1}\tfrac{1}{2}\Gamma_{k2}\int_{E_c-\Delta/2}^{E_c+\Delta/2}\{\sin\gamma_{k1k2}[(\cos^2\beta_{k1}/\tfrac{1}{2}\Gamma_{k1})+(\cos^2\beta_{k2}/\tfrac{1}{2}\Gamma_{k2})]+\cos\gamma_{k1k2}[(\sin\beta_{k1}\cos\beta_{k1}/\tfrac{1}{2}\Gamma_{k1})-(\sin\beta_{k1}\cos\beta_{k2}/\tfrac{1}{2}\Gamma_{k2})]\}dE, \quad (47)$$

where

$$\tan\gamma_{k1k2}\equiv(E_{k1}-E_{k2})/(\tfrac{1}{2}\Gamma_{k1}+\tfrac{1}{2}\Gamma_{k2}). \quad (48)$$

The integral

$$\frac{2}{\Gamma_k}\int_{E_c-\Delta/2}^{E_c+\Delta/2}\sin\beta_k\cos\beta_k dE=\tfrac{1}{2}\ln\frac{(E_c-E_k+\Delta/2)^2+(\tfrac{1}{2}\Gamma_k)^2}{(E_c-E_k-\Delta/2)^2+(\tfrac{1}{2}\Gamma_k)^2}\equiv\Psi_k. \quad (49)$$

The resonance term therefore becomes

$$R_L(\alpha s;\alpha s)=(1/2\Delta)\sum_{k=1}^{n}Z^2(l^kJ^kl^kJ^k,sL)\Gamma_k\Phi_k+(1/2\Delta)\sum_{k_1=1}^{n}\sum_{k_2=k_1+1}^{n}Z^2(l^{k_1}J^{k_1}l^{k_2}J^{k_2},sL)\Gamma_{k1}\Gamma_{k2}\times\{(\Phi_{k1}+\Phi_{k2})\cos[2\eta^{J^{k_2}}(l^{k_2})-2\eta^{J^{k_1}}(l^{k_1})-\gamma_{k1k2})]+(\Psi_{k1}-\Psi_{k2})\sin[2\eta^{J^{k_2}}(l^{k_2})-2\eta^{J^{k_1}}(l^{k_1})+\gamma_{k1k2}]\}. \quad (50)$$

The integration of the interference term involves only quadratures of the types which have already been done and gives

$$I_L(\alpha s;\alpha s)=(-1/\Delta)\sum_{k=1}^{n}\sum_{l=|l^k-1|}^{(l^k+L)\leq l_{\max}}\sum_{J=|l-s|}^{l+s}Z^2(lJl^kJ^k,sL)\times\Gamma_k\sin\eta^J(l)\{\Phi_k\sin[2\eta^{J^k}(l^k)-\eta^J(l)]+\Psi_k\cos[2\eta^{J^k}(l^k)-\eta^J(l)]\}. \quad (51)$$

The total cross section is

$$\sigma_{\alpha s;\,\alpha s}=[4\pi\lambda_\alpha^2/(2I+1)(2i+1)]\Big(\sum_{l=0}^{l_{\max}}\sum_{J=|l-s|}^{l+s}(2J+1)\times\sin^2\eta^J(l)+(1/2\Delta)\sum_{k=1}^{n}(2J^k+1)\Gamma_k\Phi_k-(1/\Delta)\sum_{k=1}^{n}(2J^k+1)\Gamma_k[\Phi_k\sin^2\eta^{J^k}(l^k)+\Psi_k\sin\eta^{J^k}(l^k)\cos\eta^{J^k}(l^k)]\Big). \quad (52)$$

## VI. OTHER REMARKS

Walton, Clement, and Boreli[17] have worked out a special case of formula (37) and applied it to an analysis of the angular distribution in the reaction $O^{16}(n,\alpha)C^{13}$. The situation is simplified by the uniqueness of the incoming and outgoing orbital angular momentum values for a given $J$ and II. Without making any attempt to say anything about the potential phase shifts, they succeed in establishing the widths, positions, and spins of most of the overlapping levels in oxygen-17, in the region of excitation energy 7 to 9 Mev, as well as the relative parities within groups of levels. The presence or absence of various Legendre polynomial coefficients in the regions of interfering resonances, brought about by the requirements (34), is a powerful tool for the determination of the $J$ values of levels.

Using the formalism developed in this paper, work is currently under way at NDA to determine the parameters of additional levels, to tie down the parities of all the levels, and to find the spin-orbit coupled potential phase shifts for the interaction of neutrons with oxygen-16 for neutrons with energies in excess of 3 Mev.[18] The tools available, in addition to the data of Clement, Walton, Boreli, are a precise measurement of the total cross section and inaccurate measurements of the differential scattering cross section at a few energies.

I should like to thank Dr. M. H. Kalos for helpful discussions.

[17] Walton, Clement, and Boreli, Phys. Rev. **107**, 1065 (1957).
[18] Harry Lustig, Nuclear Development Corporation of America Report NDA-2111-3, 1959 (unpublished), Vol. A.