methods now available are more sensitive to the form of the interaction.

It is of some interest to compare this work with other calculations of the total capture rate in the closure approximation[17] even though these have a somewhat different purpose, namely to find the gross variation of the rate over the periodic table as well as an estimate of the total effective coupling constant ζ. They ignore the local variations in which we are interested by using an average nuclear model. The rather small variations that we find in a model that, if anything, has too sharp features suggests that for $Z>8$, this is a reasonable approximation. It is still not easy to make a good calculation and in neither case does the predicted Z variation seem soundly based.

Primakoff includes all the angular momentum states in the same inexact fashion used here, that is, neglecting spin-orbit interference effects that arise from the effective pseudoscalar term. This approximation is probably as good as taking one extra term in the multipole expansion. However, he makes a more serious approximation in treating the exclusion principle as a correction term to the capture on Z free protons, just linear in the relative neutron excess. This correction, as one expects, cancels 75–90% of the main term so that a 5% error or variation with Z of the mean nucleon correlation distance d (d^3 is the measure of the exclusion principle that enters) changes the result by 100%. Such an effect can come, for instance, from the change in importance of the nuclear surface as Z increases. Tolhoek has improved on this by assuming that the Pauli cancellation is complete in the zeroth order of the multipole expansion; he then calculates the next order. However, for large Z, electromagnetic effects, which will prevent the cancellation being exact, and the slow convergence of the multipole expansion reduce the reliability of the result. The experimental Z dependence is insensitive and fits both results adequately. This suggests that detailed calculations on a few selected light nuclei may be more reliable for fixing the total effective coupling, the uncertainties due to local fluctuations being smaller than the difficulties inherent in theories which cover the whole range of Z.

[17] H. Primakoff, Revs. Modern Phys. **31**, 802 (1959); and H. A. Tolhoek, Nuclear Phys. **10**, 606 (1959). We are grateful to the authors for private communication of their results.

ACKNOWLEDGMENTS

We should like to acknowledge stimulating conversations with members of the Mathematical Physics Department at the University of Birmingham, and with Dr. R. J. Blin-Stoyle, Professor L. Wolfenstein, and Dr. Hugh Muirhead who first interested us in this problem.

PHYSICAL REVIEW VOLUME 117, NUMBER 5 MARCH 1, 1960

Charge-Dependent Corrections to Pion-Nucleon Scattering*†

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(Received August 7, 1959)

If account is taken of the mass difference between neutral and charged pions and of the possibility that the three coupling constants (π^0-n, π^0-p, $\pi^\pm$ nucleon) may differ, then the pion-nucleon system no longer conserves isotopic spin. This effect has been investigated using Chew-Low theory with a p-state interaction. For each J value there are ten scattering amplitudes, replacing the two of the charge-independent case. Only eight of these amplitudes are independent due to time reversal invariance, and the mass difference effect can be related to a change in the energy scale. The amplitudes are determined as solutions to a set of linear integral equations which may be solved approximately in the one-meson approximation. Corrections to the differential cross sections are then calculated. These corrections go through a maximum at about 125 Mev and can affect the magnitude of the π^- cross sections by as much as 35% in this region, as well as the slope of the π^- cross section in the region 125–175 Mev. The effect on the π^+ cross section is small. Attempts are made to correlate the calculation with available data.

I. INTRODUCTION

CHARGE independence in pion scattering is only an approximation. It is known, for example, that the electromagnetic interaction destroys charge independence, and it is the purpose of this calculation to determine the nature of the contributions to be expected from charge-dependent contributions, without explicitly introducing the e-m field. It is assumed that at low energies these effects will manifest themselves as changes in the pion masses and coupling constants.

The fact that the mass of the neutral pion is about 3% less than that of the charged pions is a clear indication of a breakdown of charge independence. This can

* Work done at the Laboratory for Nuclear Science, Massachusetts Institute of Technology, calculations performed at the MIT computation center and submitted as a doctoral thesis at the University of Illinois.
† Research supported in part by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

affect the charge exchange cross section, which goes as the cube of the outgoing pion momentum (k) at low energy in the laboratory system, by about $9\%/k^2$. This correction comes from phase-space considerations alone and ignores the contribution from the scattering amplitudes themselves. While the pion mass difference causes violations in charge independence, it still preserves charge symmetry, as it affects proton and neutron couplings equally. The existence of a nucleon mass difference proves that even charge symmetry is violated. However, this mass difference is somewhat smaller than the pion mass difference and will be completely ignored in this calculation (though charge asymmetry will be introduced by other means).

The other effect to be included is the possible difference between the coupling constants for the three couplings: charged pion-nucleon, neutral pion-proton, neutral pion-neutron. Time reversal invariance guarantees that $f^{(+)}=f^{(-)}$. (This is shown in the Appendix for the static case.) The two neutral coupling constants are assumed to be almost equal to the charged one, the first-order differences being left as open parameters. After the calculation, the experimental evidence will be examined to see if it can fix them. These pion mass and coupling constant differences are incorporated into a static Hamiltonian, and the calculation performed by the techniques of Chew and Low, in the one-meson approximation.

In the charge-independent case, the scattering amplitude (a) depends on the energy (in a state labelled by T and J) as $a(\omega)\sim e^{i\alpha(\omega)}\sin\alpha(\omega)\sim(e^{2i\alpha(\omega)}-1)$. It is possible to relate the mass difference effect to a change in the energy scale, so that the major corrections to the amplitude have the phase $2\alpha_{33}$. The significance of this lies in the fact that the corrections then go through a maximum at a much lower energy (around 125 Mev) than the cross sections themselves. In fact, if the cross sections are analyzed in the form of Eq. (1.1), then for coupling constant corrections on the order of a few percent, one may expect corrections to the coefficients $B^{(-)}$, $C^{(-)}$ for $\pi^-\rightarrow\pi^-$ scattering to be as large as 35% at energies around 120 Mev, which can seriously affect the values of the s-wave phase shifts as determined by a phase-shift analysis of the scattering data. Such large corrections, from small mass and coupling constant differences, arise from the "resonance" in the corrections at this energy, and are heightened by the circumstances that the cross sections themselves are still quite small at 120 Mev, being well below the resonance, and the π^- cross sections in general are much smaller than the π^+ cross section. It is unfortunate that the data in this region is still too poor to tell whether this possibly large effect actually exists.[1]

A second effect arises from other corrections which go as $e^{i\alpha_{33}}$. These generally are smaller than those going as $e^{2\alpha_{33}}$ and mostly of the opposite sign. When combined, the two effects can produce a significant change in slope for the π^- cross section in the energy range 125–170 Mev (see Fig. 6). This change in slope can have a strong effect on the integrals over cross sections appearing in the dispersion relations, as these are principal value integrals, and might lead to different coupling constants than the uncorrected data do. However, Cini and Agodi[2] have shown that even without charge independence the pion-nucleon system should satisfy approximately the same dispersion relations, so that while charge dependence can affect the value of the coupling constant so obtained, it cannot explain any inability of the dispersion relations to fit the π^- data.[3] However, the greatest changes in $\sigma^{(-)}$ needed to make the dispersion relations work come about in the region where, according to the present calculation, charge independence is least valid.

Actually, even in the charge-dependent case one can isolate the amplitude a_{33} and write it in the form[4] $a_{33}\sim e^{i\alpha_{33}'}\sin\alpha_{33}'$ where α_{33}' will be slightly different from α_{33} of the charge-independent case. However, this α_{33}' is not easily accessible experimentally because the total T matrix is no longer diagonal. One would have to diagonalize it, introducing unfamiliar phase shifts, and then analyze the raw data to find all these phase shifts, one of which would be α_{33}'.

At any rate one certainly cannot force the raw data into a charge-independent analysis and expect the α_{33} which emerges to be simply related to either the α_{33}' above or to a truly charge-independent α_{33} because the functional form imposed is too restrictive and the method of analysis, a least-square fit, say, will then impose a statistical error which is very difficult to interpret theoretically.

In this paper we proceed differently. The object is to isolate the charge-independent part of the scattering cross sections from the measured, charge-dependent data. Then one may analyze just this part of the data, introducing the usual six charge-independent phase shifts. This is done by writing the T matrix essentially as the sum of a charge-independent part plus a part containing first-order corrections. The corrections are then calculated in the static theory and the cross section is determined in the form

$$\begin{aligned} d\sigma/d\Omega &= (A_I-\delta A)+(B_I-\delta B)\cos\theta+(C_I-\delta C)\cos^2\theta \\ &= A_{\text{expt}}+B_{\text{expt}}\cos\theta+C_{\text{expt}}\cos^2\theta. \end{aligned} \tag{1.1}$$

The minus sign is introduced in Eq. (1.1) so that one may merely add the corrections δA, etc., plotted in Figs. 3, 4, and 5, directly to the experimentally measured

[1] Some experiments are in progress to improve the data in this region. See, e.g., York, Kim, Kernan, and Garwin, Bull. Am. Phys. Soc. 4, 274 (1959); U. Kruse and R. Arnold, Phys. Rev. 116, 1008 (1959).

[2] A. Agodi and M. Cini, Nuovo cimento 5, 1256 (1957); Nuovo cimento 6, 686 (1957).

[3] The so-called "Puppi parodox," G. Puppi and A. Stanghellini, Nuovo cimento 5, 1305 (1957). Its existence is still an open question, due to insufficiently accurate data.

[4] R. A. Sorensen, Phys. Rev. 112, 1813 (1958). See this paper for other references on this topic.

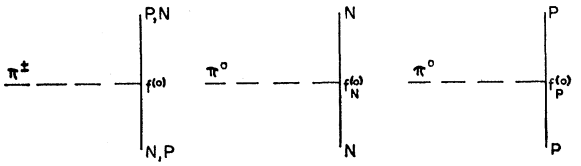

FIG. 1. Basic emission and absorption processes.

quantities $A_{\rm expt}$, etc., to obtain the quantities $A_I=(A_{\rm expt}+\delta A)$, etc., and thus have a charge-independent cross section

$$d\sigma_I/d\Omega=A_I+B_I\cos\theta+C_I\cos^2\theta, \tag{1.2}$$

on which to perform a phase-shift analysis.

The advantage of this method over one which analyzes the raw data directly is that there the charge-dependent phase shift α_{33}' is accessible only if one assumes all significant corrections to charge independence affect the amplitude a_{33} only, which we find to be true only for pure $T=\frac{3}{2}$ scattering. Our results therefore agree with those of Sorensen[4] for the case of π^+ scattering, where our corrections are small. However we find large corrections to scattering involving the $T=\frac{1}{2}$ amplitudes, even though these amplitudes themselves are small, and our corrections to π^- scattering are considerable.

Finally, two attempts to correlate the calculations with available experimental data are made, and a method is given whereby the average neutral coupling constant might be determined from an accurate knowledge of the slope of the π^- cross section from 125–175 Mev.

II. CHARGE DEPENDENT SCATTERING EQUATIONS

A. Coupling Constants

As mentioned, the charge dependence of the scattering in the static limit is assumed to arise from two sources: first, the known mass difference between neutral and charged pions; and second, the inequality of the various coupling constants appearing in the theory. Specifically, three different coupling constants are introduced phenomenologically. They correspond to the three fundamental interactions shown in Fig. 1, where the superscript (0) refers to the fact that these are the unrenormalized coupling constants. Presumably $f_P^{(0)}$ and $f_N^{(0)}$ will differ but little from $f^{(0)}$, the charged constant. The calculation will be carried out to first order in these differences, δ and d, defined by

$$\begin{aligned}\delta&=\{[(f_N^{(0)}+f_P^{(0)})/2]-f^{(0)}\}/f^{(0)},\\ d^{(0)}&=[(f_P^{(0)}-f_N^{(0)})/2]/f^{(0)};\quad \delta,\, d\ll 1.\end{aligned} \tag{2.1}$$

δ represents the difference between the average of the neutral constants and the charged one (we have omitted the superscript as it turns out that $\delta_{\rm renorm.}=\delta^{(0)}$), while $d^{(0)}$ measures the difference between the neutral ones.

While both δ and $d^{(0)}$ destroy charge independence, δ preserves charge symmetry, as does the introduction of the neutral-charged pion mass difference. (These statements will be explained when we have written the Hamiltonian explicitly.) On the other hand, the existence of $d^{(0)}$ violates charge symmetry and in fact introdues a small mass difference between the proton and neutron. We shall ignore this mass difference and assume that the energies of the physical proton and neutron are equal. This is theoretically justifiable because, by introducing a counter term into the Hamiltonian, we could produce whatever mass difference we please. Such a term would not affect the form of the scattering equation other than to change the energy of the physical nucleon states. Since in the present formalism only the renormalized nucleon energies and coupling constants enter into results which are to be checked experimentally,[5] we have the two alternatives of either ignoring the n-p mass difference or introducing the actually measured one. We choose to ignore it both for simplicity and because it is smaller (the order of $\frac{1}{4}$) than the meson mass difference.

The effects of δ and d correspond roughly to electromagnetic corrections to the vertex in the relativistic theory, coming from such diagrams as in Fig. 2(a), and therefore lead to such self-energy diagrams as those of Fig. 2(b), while the counterterm would correspond to such purely electromagnetic corrections as Fig. 2(c). Actually d produces only a very shall n-p mass difference (about 0.2 m_e for $d\sim 1\%$) and if we had chosen

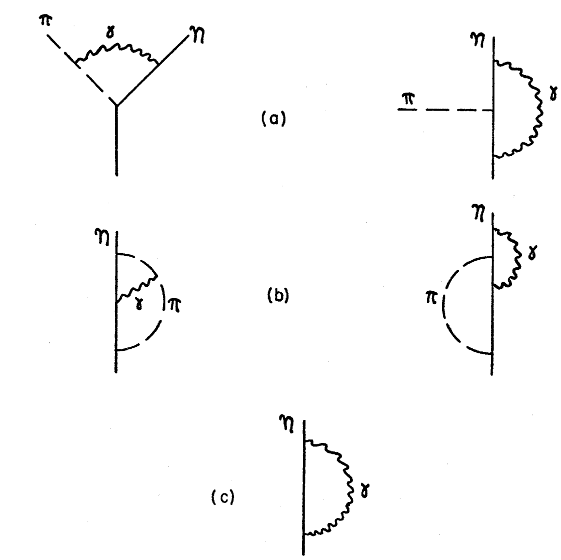

FIG. 2. Electromagnetic contributions to δ, d, and self energy of nucleon. The effects of δ and d are produced by vertex corrections, as in (a), and lead to such self-energy contributions as those in (b). An e-m counter term would produce a purely e-m self energy contribution, as in (c).

[5] It is true that the theory mathematically determines the "unphysical" quantities also, e.g., the unrenormalized coupling constants, but they are not necessary to correlate the experimental scattering data.

to include this effect, the counter term would have to support most of it.

One final caution before we write down the Hamiltonian. The form f/μ, with μ the meson mass, for the static coupling constant, and the dispersion relation form $f=(\mu/2M)g_{\text{rel.}}$ are merely conventions, and cannot be used to predict δ when $\mu_0\neq\mu_\pm$. The coupling constant is determined by the pion-nucleon vertex, and is $g_0\Gamma[(\Delta P)^2]$, where $(\Delta P)^2$ is $-\mu_\pm^2$ for charged meson scattering, and $-\mu_0^2$ for π^0 scattering. Thus the renormalization constant depends in an intrinsic manner on both e-m and mass difference corrections to the vertex itself and is essentially uncalculable by present methods.

B. Form of the Hamiltonian

The static charge-dependent Hamiltonian is

$$H=H_0+H_I;$$

$$H_0=\sum_{k\alpha}\omega_k a_{k\alpha}{}^\dagger a_{k\alpha}, \tag{2.2}$$

$$H_I=\sum_{k\alpha}(a_{k\alpha}{}^\dagger V_{k\alpha}{}^{(0)\dagger}+a_{k\alpha}V_{k\alpha}{}^{(0)}),\quad(\alpha=1,2,3),$$

where

$$V_{k\alpha}{}^{(0)}=if_\alpha{}^{(0)}\tau_\alpha\boldsymbol{\sigma}\cdot\mathbf{k}_\alpha v_k/(2\omega_k)^{\frac{1}{2}}. \tag{2.3}$$

The unrenormalized "coupling constants" $f_\alpha{}^{(0)}$ are defined by

$$\begin{aligned}f_1{}^{(0)}&=f_2{}^{(0)}=f^{(0)}\mathbf{I},\\ f_3{}^{(0)}&=f^{(0)}[(1+\delta)\mathbf{I}+d^{(0)}\tau_3].\end{aligned} \tag{2.4}$$

They are matrices in isotopic spin space. The indices α refer to the type of meson and the sum $\sum_\alpha$ is to be carried out at a particular energy ω_k, $(k\equiv k_1=k_2\neq k_3)$. The momentum involved, k_α, is given by

$$\begin{aligned}k_\alpha(\omega_k)&=(\omega_k{}^2-\mu_\alpha{}^2)^{\frac{1}{2}},\\ \mu_1&=\mu_2=1,\\ \mu_3&=1-\Delta.\end{aligned} \tag{2.5}$$

Here Δ represents the known mass difference (in units of the meson Compton wave number) between the charged and neutral mesons. Throughout this paper the combined symbol $k\alpha$ is to be read as $\mathbf{k}_\alpha,\alpha$—a meson of type α, momentum $\mathbf{k}_\alpha$.

The cutoff function v_k may be considered to be a function of either meson energy or momentum. Since the cutoff will play no role for energies smaller than $\omega\sim6$, any momentum dependence it would introduce will be negligible. This is because the appropriate expansion parameter for k_α is $\Delta/k^2\sim\Delta^2$ for $\omega\sim6$ (and $\Delta\sim3\%$). However we shall consider it a function of ω, for the reasons given in Sec. III-C.

Note that H_0 is different from the H_0 of the charge-independent theory, as $a_{k3}{}^\dagger$ creates particles of different momenta at the same energy. To first order in Δ we have, from Eq. (2.5),

$$k_3(\omega)=\mu_3k(\omega/\mu_3)\approx k(\omega/\mu_3)-\Delta k(\omega), \tag{2.6}$$

so that

$$(1+\delta)\boldsymbol{\sigma}\cdot\mathbf{k}(\omega)\approx\boldsymbol{\sigma}\cdot\mathbf{k}(\omega/\mu_3)+(\delta-\Delta)\boldsymbol{\sigma}\cdot\mathbf{k}(\omega). \tag{2.7}$$

Thus we may anticipate that δ will appear in the calculation only in the form $(\delta-\Delta)$, and that the extra mass corrections will closely resemble the effect of a change in the energy scale.

It is clear from Eq. (2.4) that $f^{(0)}$ and $(1+\delta)f^{(0)}$ renormalize in the same manner, through the nucleon expectation value of $\tau_3\boldsymbol{\sigma}$, while for $f^{(0)}d^{(0)}$ the relevant operator is $\boldsymbol{\sigma}$, which yields a different renormalization constant. From Eqs. (2.4) and (2.7) it follows that corrections due to both Δ and δ have the same effect upon neutrons as protons, while from Eq. (2.4) it is seen that d affects them differently.

A c-number term $(\delta m)\tau_3$ might have been added to the Hamiltonian and adjusted to insure the equality of the masses of the physical nucleons. It has no effect on the scattering equations however, and very small magnitude, and so has been left out though the equality of the physical nucleon masses is assumed.

C. Scattering Equations

The scattering equations can be derived by the methods of Wick[6] and Chew and Low.[7] The lowest eigenstate of H_0 will be denoted by ϕ_{0n}, where the subscript "0" refers to the meson vacuum and "n" to the nucleon charge and spin state. The nucleon states for the free Hamiltonian are Pauli spinors. The latin indices l, m, n will be reserved for nucleon state variables. Then

$$H_0\phi_{0n}=0. \tag{2.8}$$

The free one-meson, one-nucleon states $\phi_{k\alpha n}$ satisfy

$$\begin{aligned}H_0\phi_{k\alpha n}&=\omega_k\phi_{k\alpha n},\\ \mathbf{P}\phi_{k\alpha n}&=\sum\mathbf{k}_\alpha a_{k\alpha}{}^\dagger a_{k\alpha}\phi_{k\alpha n}=\mathbf{k}_\alpha\phi_{k\alpha n}.\end{aligned} \tag{2.9}$$

The greek letters α, β, γ will in general be reserved for meson isotopic spin. A momentum label without a subscript will refer to the momentum of a charged meson at a given energy.

The physical nucleon states ψ_{0n}, solutions of the total Hamiltonian

$$H\psi_{0n}=E_s\psi_{0n}, \tag{2.10}$$

are given by[6]

$$\psi_{0n}=(Z_{2,n})^{\frac{1}{2}}[\phi_{0n}+(E_s-H_0-\Lambda H_I)^{-1}H_I\phi_{0n}], \tag{2.11}$$

where Λ is the projection operator orthogonal to the bare nucleon states. Note that charge and angular momentum conservation require that only the bare nucleon of type n enters Eq. (2.11). The states ψ_{0n} are assumed degenerate.

The ingoing and outgoing scattering states for a meson of type α, momentum $\mathbf{p}_\alpha$, are given by

$$\psi_{p\alpha n}{}^{(\pm)}=a_{p\alpha}{}^\dagger\psi_{0n}+(E_p\pm i\epsilon-H)^{-1}V_{p\alpha}{}^{(0)}\psi_{0n}, \tag{2.12}$$

[6] G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).

[7] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956). G. F. Chew, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958).

where $a_{p\alpha}{}^{\dagger}\psi_{0n}$ is asymptotically a plane wave one-meson state of momentum $\mathbf{p}_\alpha$ and energy $E_p=\omega_p+E_s$. Any admixture of other plane wave states (e.g., involving different type mesons) in the scattering state of Eq. (2.12) would have the wrong asymptotic form.

The important point about Eqs. (2.11) and (2.12) is that, even though charge independence is no longer valid, the states of the exact Hamiltonian are given by the same formal expressions as in the charge-independent theory. In fact they are derived analogously, since the q-number structure of the two theories is identical—the difference lying in the reduced symmetry of the charge-dependent $V_{k\alpha}$ and the more complicated momentum dependence.

The scattering equations now follow as in the charge-independent theory. The equation for the T matrix is

$$(\psi_{p\alpha m}{}^{(-)},V_{q\beta}{}^{(0)}\psi_{0n}) =\sum_r\left[\frac{(\psi_{0m},V_{p\alpha}{}^{(0)\dagger}\psi_r{}^{(-)})(\psi_r{}^{(-)},V_{q\beta}{}^{(0)}\psi_{0n})}{E_p-E_r+i\epsilon} +\frac{(\psi_{0m},V_{q\beta}{}^{(0)\dagger}\psi_r{}^{(-)})(\psi_r{}^{(-)},V_{p\alpha}{}^{(0)}\psi_{0n})}{E_s-\omega_p-E_r}\right], \quad (2.13)$$

where the index r runs over all intermediate incoming states.

In Eq. (2.13) the sum over the four nucleon states can be separated out and written down explicitly. As in the charge-independent case, the matrix element $(\psi_{0l},V_{q\beta}{}^{(0)}\psi_{0n})$ is a multiple of the matrix element of the same operator between bare nucleon states, which defines the renormalized coupling constants. Since the interaction is still invariant with respect to rotations about the T_3 axis, it follows that f_1 will equal f_2 (i.e., the renormalized f_α matrices will have the same structure as the unrenormalized ones, with a charged and two neutrals). This is proved in the Appendix. We may then work with the renormalized constants f, δ, and d. While the unrenormalized constants were rationalized, the renormalized ones will be taken as unrationalized. This means that instead of making the identification $f^{(0)}\rightarrow f$, we will let $f^{(0)}/(4\pi)^{\frac{1}{2}}\rightarrow f$.

Next, we make the one-meson approximation, which involves dropping from the sum in Eq. (2.13) all multimeson states. We then introduce the matrix $T_{q\beta,p\alpha}(z)$ (we are scattering from $q\beta,n\rightarrow p\alpha,m$) as a function of the complex variable z, defined by the following equations:

$$\lim_{z\rightarrow\omega_p{}^+}\langle m|T_{q\beta,p\alpha}(z)|n\rangle=(\psi_{p\alpha m}{}^{(-)},V_{q\beta}{}^{(0)}\psi_{0n})\,; \quad (2.14)$$

$$T_{q\beta,p\alpha}(z)=B(z)-\sum_{k\gamma}\left[\frac{T_{p\alpha,k\gamma}{}^{\dagger}(\omega_k)T_{q\beta,k\gamma}(\omega_k)}{\omega_k-z} +\frac{T_{q\beta,k\gamma}{}^{\dagger}(\omega_k)T_{p\alpha,k\gamma}(\omega_k)}{\omega_k+z}\right], \quad (2.15)$$

$$B(z)=-(1/z)[V_{p\alpha},V_{q\beta}].$$

Equation (2.14) defines $T_{q\beta,p\alpha}(z)$ as a matrix with respect to the nucleon Pauli spinors. In the limit $z\rightarrow\omega_p{}^+$, Eq. (2.15) becomes identical to Eq. (2.13), with r restricted by the one-meson approximation. In the integral over intermediate momenta k_3 in Eq. (2.15), the k_3 spectrum starts at a lower energy than the k spectrum. However, we shall ignore this fact, as the integrals all vanish as k^3 for p waves. So we will feel free in what follows to change with impunity the lower limits of integrals over intermediate momenta, without affecting the results to first order in Δ. (This would not be possible if we had included s waves.[2])

$T_{q\beta,p\alpha}(z)$ is uniquely determined by the five following conditions,[7] which will be explicitly used in solving the scattering equations:

1. $T_{q\beta,p\alpha}(z)$ has branch points at $z=\pm1$, and two cuts running from $z=\pm1$ to $\pm\infty$;
2. There is a pole at $z=0$, given by $B(z)$, and no other singularities;
3. $T_{q\beta,p\alpha}(z)$ behaves like $1/z$ for large z;
4. The S matrix is unitary, where S is given by

$$(\psi_{p\alpha m}{}^{(-)},\psi_{q\beta n}{}^{(+)}) =\lim_{z\rightarrow\omega_p{}^+}[\delta_{p\alpha m,q\beta n}-2\pi i\delta(\omega_p-\omega_q)T_{q\beta,p\alpha}(z)]; \quad (2.16)$$

5. $T_{q\beta,p\alpha}(z)$ possesses, from Eq. (2.15), the two properties

$$\text{a. } T_{q\beta,p\alpha}(z)=T_{p\alpha,q\beta}{}^{\dagger}(z^*), \quad \text{(Hermiticity)}$$
$$\text{b. } T_{q\beta,p\alpha}(z)=T_{p\alpha,q\beta}(-z). \quad \text{(Crossing symmetry)}. \quad (2.17)$$

D. Projection Operators

We want to write $T_{q\beta,p\alpha}$ as a linear combination of matrix operators with respect to the nucleon variables. Since angular momentum is still conserved, it is possible to introduce the angular momentum projection operators

$$\begin{aligned}P_1(\mathbf{p}_\alpha,\mathbf{q}_\beta)&=(\boldsymbol{\sigma}\cdot\mathbf{p}_\alpha)(\boldsymbol{\sigma}\cdot\mathbf{q}_\beta),\\ P_2(\mathbf{p}_\alpha,\mathbf{q}_\beta)&=3(\mathbf{p}_\alpha\cdot\mathbf{q}_\beta)-(\boldsymbol{\sigma}\cdot\mathbf{p}_\alpha)(\boldsymbol{\sigma}\cdot\mathbf{q}_\beta).\end{aligned} \quad (2.18)$$

The subscripts 1 and 2 refer to the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ states, respectively. These operators are projection operators in the sense that

$$\int\frac{d\Omega_k}{4\pi k_\gamma{}^2}P_J(\mathbf{p}_\alpha,\mathbf{k}_\gamma)P_{J'}(\mathbf{k}_\gamma,\mathbf{q}_\beta)=\delta_{JJ'}P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta); \quad (2.19)$$

however their normalization depends on their isotopic subscripts. This will cause no trouble, but will merely introduce extra kinematical corrections into the cross sections.

The isotopic spin dependence is considerably more complicated. Denoting the relevant part of the matrix

dependence by M, one can write

$$\langle\alpha m|M|\beta n\rangle=\sum_{TT'}\langle\alpha m|T\rangle\langle T|M|T'\rangle\langle T'|\beta n\rangle,$$

where T, T' can be any of the six states $|\frac{3}{2},T_3\rangle$, $|\frac{1}{2},T_3\rangle$. However, even in the charge-dependent case T_3 will be conserved, so there will be ten amplitudes of the form $\langle T|M|T'\rangle$. We shall construct matrices $\Lambda_\sigma(\alpha\beta)$ for these ten amplitudes by noting that, besides τ_α and τ_β, the matrix τ_3 was explicitly introduced by the interaction and can enter into the operators Λ_σ in all possible ways. (We shall not use the labelling scheme $|T,T_3\rangle$, which was introduced merely to help count the number of amplitudes. Our operators will have matrix elements between various linear combinations of these states—however, we shall erroneously refer to them as "Projection" operators.)

We classify the ten operators Λ_σ as "even" or "odd," depending upon whether they contain an even or odd number of τ_3's. The five even operators are

$$\begin{aligned}&\Lambda_1=\tfrac{1}{3}\tau_\alpha\tau_\beta,\quad \Lambda_2=\delta_{\alpha\beta}-\tfrac{1}{3}\tau_\alpha\tau_\beta,\quad \Lambda_3=\tau_3\tau_\alpha\tau_\beta\tau_3,\\ &\Lambda_4=\tfrac{1}{2}(\tau_3\tau_\alpha\tau_3\tau_\beta+\tau_\alpha\tau_3\tau_\beta\tau_3),\\ &\Lambda_5=\tfrac{1}{2}(\tau_3\tau_\alpha\tau_3\tau_\beta-\tau_\alpha\tau_3\tau_\beta\tau_3).\end{aligned}\tag{2.20}$$

The first two operators contain no τ_3's and are the projection operators onto the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ states introduced in the charge-independent case. The five odd operators are

$$\begin{aligned}&\Lambda_6=\delta_{\alpha\beta}\tau_3,\quad \Lambda_7=\tau_\alpha\tau_3\tau_\beta,\quad \Lambda_8=\tau_3\tau_\alpha\tau_3\tau_\beta\tau_3,\\ &\Lambda_9=\tfrac{1}{2}(\tau_3\tau_\alpha\tau_\beta+\tau_\alpha\tau_\beta\tau_3),\quad \Lambda_{10}=\tfrac{1}{2}(\tau_3\tau_\alpha\tau_\beta-\tau_\alpha\tau_\beta\tau_3).\end{aligned}\tag{2.21}$$

The ten operators Λ_σ are linearly independent in the sense that the equation $\sum_\sigma C_\sigma\Lambda_\sigma(\alpha\beta)=0$ implies that each $C_\sigma=0$. Actually any independent set of linear combinations of the Λ_σ could be used for our projection operators and later we shall introduce a particular set for computational convenience. Our results will apply to any set of projection operators Γ_σ, where

$$\Gamma_\sigma=\textstyle\sum_\mu a_{\sigma\mu}\Lambda_\mu,\quad \|a_{\sigma\mu}\|\neq 0,\tag{2.22}$$

and where the Γ_σ are chosen subject to three restrictions: they must make the charge-independent limit easily recognizable, they must be even or odd, and they must have specific time reversal properties. Thus we assume $\Gamma_1=\Lambda_1$ and $\Gamma_2=\Lambda_2$; $\Gamma_{1,2,3,4,5}$ are even and $\Gamma_{6,7,8,9,10}$ odd; and $\Gamma_5=\Lambda_5$, $\Gamma_{10}=\Lambda_{10}$ (these operators violate time reversal invariance and do not contribute to the T matrix.)

We can now write

$$\begin{aligned}T_{q\beta,p\alpha}(z)&=-v_pv_q\frac{4\pi}{(4\omega_p\omega_q)^{\frac{1}{2}}}\sum_J[H_{\alpha\beta}{}^J(z)+N_{\alpha\beta}{}^J(z)]\\&\qquad\times P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta),\\ H_{\alpha\beta}{}^J(z)&=\textstyle\sum_\sigma\Gamma_\sigma(\alpha\beta)h_\sigma{}^J(z),\\ h_\sigma{}^J(z)&=0,\quad(\sigma\neq 1,2),\\ N_{\alpha\beta}{}^J(z)&=\textstyle\sum_\sigma\Gamma_\sigma(\alpha\beta)n_\sigma{}^J(z).\end{aligned}\tag{2.23}$$

The functions $h_\sigma{}^J(z)$ are the amplitudes[7] h_{TJ} of the charge-independent theory and the functions $n_\sigma{}^J(z)$ are of first order in the parameters Δ, δ, d and vanish in the limit Δ, δ, $d\rightarrow 0$.

On introducing this expression for $T_{q\beta,p\alpha}$ directly into Eq. (2.15) and keeping only terms of first order in Δ, δ, d we get

$$\begin{aligned}\sum_J N_{\alpha\beta}{}^J(z)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta)=&-\frac{1}{z}[f_\alpha\tau_\alpha(\boldsymbol{\sigma}\cdot\mathbf{p}_\alpha),f_\beta\tau_\beta(\boldsymbol{\sigma}\cdot\mathbf{q}_\beta)]_{\delta,d}\\ &+\frac{1}{\pi}\sum_{\gamma,J}\int_1^{\omega_M}k^3d\omega_k\left(\frac{[H_{\gamma\alpha}{}^{J\dagger}(\omega_k)N_{\gamma\beta}{}^J(\omega_k)+N_{\gamma\alpha}{}^{J\dagger}(\omega_k)H_{\gamma\beta}{}^J(\omega_k)]P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta)}{\omega_k-z}\right.\\ &\left.+\frac{[H_{\gamma\beta}{}^{J\dagger}(\omega_k)N_{\gamma\alpha}{}^J(\omega_k)+N_{\gamma\beta}{}^{J\dagger}(\omega_k)H_{\gamma\alpha}{}^J]P_J(\mathbf{q}_\beta,\mathbf{p}_\alpha)}{\omega_k+z}\right)\\ &+\frac{1}{\pi}\sum_J\int_1^{\omega_M}(k_3{}^3-k^3)d\omega_k\left(\frac{H_{3\alpha}{}^{J\dagger}(\omega_k)H_{3\beta}{}^J(\omega_k)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta)}{\omega_k-z}+\frac{H_{3\beta}{}^{J\dagger}(\omega_k)H_{3\alpha}{}^J(\omega_k)P_J(\mathbf{q}_\beta,\mathbf{p}_\alpha)}{\omega_k+z}\right).\end{aligned}\tag{2.24}$$

In Eq. (2.24) the sum over states has been converted into an integral, and a square cutoff has been assumed.[8] The J values do not mix because of Eq. (2.19), the integral over intermediate meson angles, $d\Omega_{k\gamma}$ having been carried out. In the Born terms the nonrationalized, renormalized coupling constant appears, as explained previously, and the symbol $[\ ,\ ]_{\delta,d}$ refers to the fact that only first-order terms are to be kept in the commutator. The zeroth-order terms in Eq. (2.24) would give the charge-independent Chew-Low equations.

In the first integral of Eq. (2.24) the first-order parts arise from the interference between a first-order matrix (N) and a zero-order matrix (H). We ignore the intermediate π^0-meson phase-space difference in the first integral, letting this effect generate the second integral, which contains only intermediate π^0 contributions. This phase-space difference $(k_3{}^3-k^3)$, is of

[8] The square cutoff should be considered a computational convenience only, as an approximation to a function $v(\omega)$ which contributes negligibly out to $\omega=\infty$. Otherwise it would introduce an extra singularity.

first order in Δ and so it multiplies only zero-order matrices. Thus the second integral has only Δ contributions and may be considered as an additional inhomogeneous term, along with the Born terms, in the integral equations for the $n_\sigma{}^J$.

III. REDUCTION OF THE SCATTERING EQUATIONS

The products of projection operators appearing in Eq. (2.24) can be expanded as linear combinations of single projection operators. This will yield a set of linear integral equations for the functions $n_\sigma{}^J$. First, however, we will investigate some general properties of the theory, as this will lead to enormous simplifications in the form of the equations.

A. Time Reversal Invariance

Under the operation of time reversal, the system represented by $\psi(t)$ transforms into a new system represented by $\psi^T(t)$, also a solution of the Schrödinger equation, and such that the future behavior of this system resembles the past behavior of the original system. Then $\psi^T(t)=R\psi^*(-t)$, where R is unitary, and time reversed observables O^T are given by

$$R^{-1}OR=(O^T)^*. \tag{3.1}$$

The property $(\tau_\pm)^T=\tau_\pm$, with $\tau_\pm=\frac{1}{2}(\tau_1\pm i\tau_2)$, which states that charge is conserved under time reversal, gives the rule

$$\begin{gathered} R^{-1}\tau_\alpha R=\nu_\alpha\tau_\alpha{}^*, \\ \nu_1=\nu_3=1, \quad \nu_2=-1. \end{gathered} \tag{3.2}$$

in the usual representation of the τ_α. Also the properties

$$\boldsymbol{\sigma}^T=-\boldsymbol{\sigma}, \quad (\mathbf{p}_\alpha)^T=-\mathbf{p}_\alpha,$$

which state that spin and momentum change sign under time reversal, give the result

$$R^{-1}P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta)R=P_J{}^*(-\mathbf{p}_\alpha,-\mathbf{q}_\beta). \tag{3.3}$$

The reciprocity relation, in the one-meson approximation,[9] takes the form

$$\begin{gathered} \langle m|T_{q\beta,p\alpha}|n\rangle=\nu_\alpha\nu_\beta\langle n^T|T_{-p\alpha,-q\beta}|m^T\rangle, \\ (\omega_p=\omega_q), \end{gathered} \tag{3.4}$$

where $|m^T\rangle$, $|n^T\rangle$, denote time reversed nucleon states. Writing $T_{q\beta,p\alpha}$ as the sum of projection operators (whose amplitudes we shall denote by $M(\omega)$ for this argument), this gives

$$\begin{aligned} \sum_{\sigma J}\langle m|\Gamma_\sigma(\alpha\beta)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta)|n\rangle M_\sigma{}^J(\omega_q) &= \nu_\alpha\nu_\beta\sum_{\sigma J}\langle n^T|\Gamma_\sigma(\beta\alpha)P_J(-\mathbf{q}_\beta,-\mathbf{p}_\alpha)|m^T\rangle \\ &\quad\times M_\sigma{}^J(\omega_q). \end{aligned} \tag{3.5}$$

Then, using the relations

$$\begin{gathered} R^{-1}\Gamma_\sigma(\beta\alpha)R=\nu_\alpha\nu_\beta\Gamma_\sigma{}^*(\beta\alpha), \\ P_J{}^\dagger(\mathbf{q}_\beta,\mathbf{p}_\alpha)=P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta), \end{gathered}$$

the right-hand side of Eq. (3.5) becomes

$$\sum_{\sigma J}\langle m|\Gamma_\sigma{}^\dagger(\beta\alpha)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta)|n\rangle M_\sigma{}^J(\omega_q).$$

Inspection of Eqs. (2.20) and (2.21) shows that

$$\begin{aligned} \Gamma_\sigma{}^\dagger(\beta\alpha)&=\epsilon_\sigma\Gamma_\sigma(\alpha\beta), \\ \epsilon_\sigma&=+1, \quad (\sigma\neq 5, 10) \\ &=-1, \quad (\sigma=5, 10), \end{aligned}$$

from which it follows that $M_5=M_{10}=0$, because of the independence of the Γ_σ. The requirement for $\epsilon_\sigma=+1$ is that the τ_3's be symmetrically placed with respect to τ_α and τ_β, as is the case for all the Λ_σ (and therefore Γ_σ) except for $\sigma=5$, 10.

B. Properties of the Γ_σ Operators

The operators Λ_σ, and therefore Γ_σ, form a complete set in the sense that every operator composed of τ_α, τ_β, and τ_3 can be expanded in terms of them. Their completeness and independence guarantee the existence and uniqueness of the following expansions.

The crossing matrix $T_{\mu\sigma}$ is defined by

$$\Gamma_\mu(\beta\alpha)=\textstyle\sum_\sigma T_{\mu\sigma}\Gamma_\sigma(\alpha\beta). \tag{3.6}$$

Applying crossing twice, we obtain

$$\textstyle\sum_\sigma T_{\mu\sigma}T_{\sigma\nu}=\delta_{\mu\nu}. \tag{3.7}$$

$T_{\mu\sigma}$ possesses the following two properties, which we state without proof:

(a) $T_{\mu\sigma}$ does not mix states with different time reversal properties.

(b) $T_{\mu\sigma}$ connects only even states to even, odd to odd.

The definition of $T_{\mu\sigma}$ can be extended to include sums over the angular momentum projection operators:

$$\begin{gathered} \Gamma_\mu(\alpha\beta)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta)=\sum_{\nu,J} T_{\mu\nu}T^{JJ'}\Gamma_\nu(\beta\alpha)P_{J'}(\mathbf{q}_\beta,\mathbf{p}_\alpha), \\ T_{\mu\nu}T^{JJ'}\equiv T_{\mu\nu}{}^{JJ'}. \end{gathered} \tag{3.8}$$

Products of Γ_μ's can be expanded as follows:

$$\textstyle\sum_\gamma \Gamma_\mu(\alpha\gamma)\Gamma_\nu(\gamma\beta)=\sum_\sigma C_\sigma{}^{\mu\nu}\Gamma_\sigma(\alpha\beta). \tag{3.9}$$

Taking the adjoint of both sides of Eq. (3.9) yields the following symmetry property for the $C_\sigma{}^{\mu\nu}$:

$$\begin{aligned} C_\sigma{}^{\mu\nu}&=\epsilon_\mu\epsilon_\nu\epsilon_\sigma C_\sigma{}^{\nu\mu}, \\ C_\sigma{}^{\mu\nu}&=\epsilon_\sigma C_\sigma{}^{\nu\mu}, \quad (\mu,\nu\neq 5, 10). \end{aligned} \tag{3.10}$$

This symmetry property simplifies the first integral of the scattering Eq. (2.24). Keeping only the time reversible parts, we have

$$\begin{aligned} \sum_{J,\gamma}(H_{\gamma\alpha}{}^{J\dagger}N_{\gamma\beta}{}^J+N_{\gamma\alpha}{}^{J\dagger}H_{\gamma\beta}{}^J)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta) \\ =\sum_{J,\lambda\mu\sigma}2C_\sigma{}^{\lambda\mu}\operatorname{Re}\{h_\lambda{}^{J*}n_\mu{}^J\}\Gamma_\sigma(\alpha\beta)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta). \end{aligned} \tag{3.11}$$

[9] The quantities ν_α, ν_β appear because we are using the symmetric theory, and are absent in the charged theory.

Similarly

$$\sum_{J,\gamma} (H_{\gamma\beta}{}^{J\dagger}N_{\gamma\alpha}{}^{J}+N_{\gamma\beta}{}^{J\dagger}H_{\gamma\alpha}{}^{J})P_J(\mathbf{q}_\beta,\mathbf{p}_\alpha) = \sum_{JJ',\lambda\mu\rho\sigma} 2C_\rho{}^{\lambda\mu}T_{\rho\sigma}{}^{J'J}\operatorname{Re}\{h_\lambda{}^{J'*}n_\mu{}^{J'}\} \times\Gamma_\sigma(\alpha\beta)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta). \quad (3.12)$$

In the last integral in Eq. (2.24) there is no sum over intermediate mesons, so we must introduce products of the form

$$\Gamma_\lambda(\alpha 3)\Gamma_\mu(3\beta)=\textstyle\sum_\sigma D_\sigma{}^{\lambda\mu}\Gamma_\sigma(\alpha\beta). \quad (3.13)$$

The $D_\sigma{}^{\lambda\mu}$ have the same symmetry property as the $C_\sigma{}^{\lambda\mu}$, namely,

$$\begin{aligned} &D_\sigma{}^{\lambda\mu}=\epsilon_\lambda\epsilon_\mu\epsilon_\sigma D_\sigma{}^{\mu\lambda},\\ &D_\sigma{}^{\lambda\mu}=\epsilon_\sigma D_\sigma{}^{\mu\lambda}, \quad (\mu,\lambda\neq 5,10). \end{aligned} \quad (3.14)$$

In the last integral of Eq. (2.24) we have

$$\sum_J H_{3\alpha}{}^{J\dagger}H_{3\beta}{}^{J}P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta) = \sum_{J,\lambda\mu\sigma} D_\sigma{}^{\lambda\mu}h_\lambda{}^{J*}h_\mu{}^{J}\Gamma_\sigma(\alpha\beta)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta), \quad (3.15)$$

and

$$\sum_J H_{3\beta}{}^{J\dagger}H_{3\alpha}{}^{J}P_J(\mathbf{q}_\beta,\mathbf{p}_\alpha) = \sum_{JJ',\lambda\mu\rho\sigma} D_\rho{}^{\lambda\mu}T_{\rho\sigma}{}^{J'J}h_\lambda{}^{J'*}h_\mu{}^{J'}\Gamma_\sigma(\alpha\beta)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta). \quad (3.16)$$

The matrices $C_\sigma{}^{\lambda\mu}$, $D_\sigma{}^{\lambda\mu}$, $T_{\mu\nu}$ satisfy many interesting identities[10] which are, however, unnecessary for our purposes. We will need one identity later though and will derive it now. From the defining equations, Eqs. (2.20), it is seen that $\sum_{\mu=1,2}\Gamma_\mu(\alpha\beta)=\delta_{\alpha\beta}$ so that this sum commutes through all Γ_μ and is symmetrical in α and β. Thus

$$\sum_{\mu=1}^{2}\Gamma_\lambda(\alpha 3)\Gamma_\mu(3\beta)=\sum_{\mu=1}^{2}\sum_\rho D_\rho{}^{\lambda\mu}\Gamma_\rho(\alpha\beta) =\sum_{\mu=1}^{2}\sum_{\rho\sigma} D_\rho{}^{\lambda\mu}T_{\rho\sigma}\Gamma_\sigma(\beta\alpha). \quad (3.17)$$

But

$$\Gamma_\lambda(\alpha 3)\sum_{\mu=1}^{2}\Gamma_\mu(3\beta)=\sum_{\mu=1}^{2}\Gamma_\mu(\beta 3)\Gamma_\lambda(\alpha 3) =\sum_{\mu=1}^{2}\sum_\rho T_{\lambda\rho}\Gamma_\mu(\beta 3)\Gamma_\rho(3\alpha) =\sum_{\mu=1}^{2}\sum_{\rho\sigma} T_{\lambda\rho}D_\sigma{}^{\mu\rho}\Gamma_\sigma(\beta\alpha). \quad (3.18)$$

Defining

$$D_\sigma{}^{\rho}\equiv\sum_{\mu=1}^{2}D_\sigma{}^{\rho\mu}=\sum_{\mu=1}^{2}D_\sigma{}^{\mu\rho}, \quad (\rho,\sigma\neq 5,10), \quad (3.19)$$

we find, by equating coefficients in Eqs. (3.17) and (3.18), that

$$\textstyle\sum_\rho D_\rho{}^{\lambda}T_{\rho\sigma}=\sum_\rho T_{\lambda\rho}D_\sigma{}^{\rho}. \quad (3.20)$$

We shall need the commutivity of T and D later.

Before solving Eq. (2.24) for the $n_\sigma{}^J$, we must first choose a convenient set of Γ_σ. Then unitarity will determine the phase of the $n_\sigma{}^J$, and the energy-scaling properties of the theory will simplify the last integral in Eq. (2.24).

Since the first integral of Eq. (2.24) connects only zero-order terms (h_λ) to first-order terms (n_μ) it is clear that $C_\sigma{}^{1\mu}$ and $C_\sigma{}^{2\mu}$ are the only elements of $C_\sigma{}^{\lambda\mu}$ that will be needed. Thus we will choose our Γ_σ in such a way as to simplify these two matrices as much as possible. In fact, we can make them both diagonal in their time reversible parts. The second integral connects only zero-order terms to zero-order terms, so we need worry about $D_\sigma{}^{\lambda\mu}$ for $\lambda,\mu=1,2$ only. We define:

$$\begin{array}{lll} \Gamma_1=\Lambda_1, & & \Gamma_6=\Lambda_6+\Lambda_8,\\ \Gamma_2=\Lambda_2, & \Gamma_3=\Lambda_1+3\Lambda_3+2\Lambda_4, & \Gamma_7=3\Lambda_6+\tfrac{1}{3}\Lambda_7-2\Lambda_9,\\ \Gamma_5=\Lambda_5, & \Gamma_4=\Lambda_1+\Lambda_4, & \Gamma_8=\tfrac{1}{3}\Lambda_7,\\ \Gamma_{10}=\Lambda_{10}, & & \Gamma_9=\Lambda_9-\tfrac{1}{3}\Lambda_7. \end{array} \quad (3.21)$$

This defines the matrix $a_{\mu\nu}$ of Eq. (2.22), and it may be checked that $\|a_{\mu\nu}\|\neq 0$. The matrices $C_\sigma{}^{1\mu}$, $C_\sigma{}^{2\mu}$ are given in Tables I and II. Only components where $\mu\neq 5$, 10 have been considered, since $n_5{}^J=n_{10}{}^J=0$. Notice that except for $\sigma=4$, 9 the states appear in one but not both matrices as unity along the diagonal; the states $\sigma=4$, 9 connect to the nonreversible states, and appear in both matrices as $\frac{1}{2}$ along the diagonal.

The matrix $T_{\mu\nu}$ is given in Table III. It possesses the expected properties of separating even, odd, and irreversible states, and $T^2=1$. Finally, $D_\sigma{}^{\lambda\mu}$ (for $\lambda,\mu=1,2$) is determined by the equations

$$\begin{aligned} &\Gamma_1(\alpha 3)\Gamma_1(3\beta)=\tfrac{1}{3}\Gamma_1(\alpha\beta),\\ &\Gamma_2(\alpha 3)\Gamma_2(3\beta)=\tfrac{1}{12}\Gamma_3(\alpha\beta),\\ &\Gamma_1(\alpha 3)\Gamma_2(3\beta)=\tfrac{1}{6}[\Gamma_4(\alpha\beta)-\Gamma_5(\alpha\beta)],\\ &\Gamma_2(\alpha 3)\Gamma_1(3\beta)=\tfrac{1}{6}[\Gamma_4(\alpha\beta)+\Gamma_5(\alpha\beta)]. \end{aligned} \quad (3.22)$$

Even if the $C_\sigma{}^{1\mu}$ and $C_\sigma{}^{2\mu}$ matrices were not diagonal, the following general statements could be made: since Γ_1 and Γ_2 are even, μ and σ in $C_\sigma{}^{1\mu}$, $C_\sigma{}^{2\mu}$ must be both even or both odd; only even Γ_σ can be reached by $D_\sigma{}^{\lambda\mu}$ for $\lambda,\mu=1$, 2, and therefore the inhomogeneous mass terms [the second integral in Eq. (2.24)] are even;

TABLE I. The matrix $C_\sigma{}^{1\mu}$.

$\mu \backslash \sigma$	1	4	8	9	5	10
1	1	0	0	0	0	0
4	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0
8	0	0	1	0	0	0
9	0	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$

[10] For example, any "crossing symmetrized" matrix $\lambda_\sigma{}^J$ of the form $\lambda_\sigma{}^J=\kappa_\sigma{}^J+\sum_{J',\mu}T_{\mu\sigma}{}^{J'J}\kappa_\mu{}^{J'}$, will obey the identity $\sum_{J\sigma}\lambda_\sigma{}^J T_{\sigma\mu}{}^{JJ'}=\lambda_\mu{}^{J'}$. (Any Born term is of this form.)

TABLE II. The matrix $C_\sigma{}^{2\mu}$.

$\mu \backslash \sigma$	2	3	4	6	7	9	5	10
2	1	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0
4	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0
6	0	0	0	1	0	0	0	0
7	0	0	0	0	1	0	0	0
9	0	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$

since in the δ and d Born terms, the contributions from δ are even (they come from the even part of f_3) while those from d are odd (they come from the odd τ_3 part of f_3), it follows that Eqs. (2.24) break into two sets, one involving the even $n_\sigma{}^J$ and containing the Δ and δ contributions and the other involving the odd $n_\sigma{}^J$ and containing the d contributions.

C. Energy-Scaling Properties

The Chew-Low equations are the zero-order contributions to the T matrix. In our notation[11] they take the form

$$h_\sigma{}^J(\omega)=\frac{f^2\lambda_\sigma{}^J}{\omega}+\frac{1}{\pi}\sum_{J\lambda\mu}\int d\omega_k\, k^3 \times\left(\frac{C_\sigma{}^{\lambda\mu}h_\lambda{}^{J*}(\omega_k)h_\mu{}^J(\omega_k)}{\omega_k-\omega-i\epsilon}+\sum_{J',\rho}\frac{C_\rho{}^{\lambda\mu}T_{\rho\sigma}{}^{J'J}h_\lambda{}^{J'*}(\omega_k)h_\mu{}^{J'}(\omega_k)}{\omega_k+\omega}\right),$$
$$(\lambda,\mu,\sigma,\rho=1,2) \quad (3.23)$$

where $\lambda_\sigma{}^J$ is defined by

$$\sum_{\sigma J} f^2\lambda_\sigma{}^J\Gamma_\sigma(\alpha\beta)P_J(\mathbf{p},\mathbf{q})=f^2[\tau_\alpha\boldsymbol{\sigma}\cdot\mathbf{p},\tau_\beta\boldsymbol{\sigma}\cdot\mathbf{q}]. \quad (3.24)$$

Since $C_1{}^{11}=C_2{}^{22}=1$ and no other terms contribute, this becomes

TABLE III. The matrix $T_{\mu\nu}$.

$\mu \backslash \nu$	1	2	3	4	6	7	8	9	5	10
1	$-\frac{1}{3}$	$\frac{2}{3}$	0	0						
2	$\frac{4}{3}$	$\frac{1}{3}$	0	0			0		0	
3	16/3	8/3	$-\frac{1}{3}$	8/3						
4	0	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$						
6					1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{2}{3}$		
7		0			0	$-5/9$	28/9	14/9	0	
8					0	$-2/9$	13/9	2/9		
9					0	$-8/9$	$-16/9$	$\frac{1}{9}$		
5		0					0		1	0
10									0	1

[11] Our $h_2{}^2$ is the h_3 of reference 7. Also $h_1{}^2=h_2{}^1\rightarrow h_2$, $h_1{}^1\rightarrow h_1$, $T_{\rho\sigma}{}^{J'J}\rightarrow A_{\rho\sigma}$ of that reference.

$$h_\sigma{}^J(\omega)=f^2\frac{\lambda_\sigma{}^J}{\omega}+\frac{1}{\pi}\int_1^\infty d\omega'\, k^3(\omega')v^2(\omega')\frac{|h_\sigma{}^J(\omega')|^2}{\omega'-\omega-i\epsilon}+\text{C.T.} \quad (3.25)$$

In Eqs. (3.25) we have included the cutoff $v(\omega)$ explicitly in order to show that the scaling relations hold rigorously in its presence, though in all calculations a square cutoff has been assumed.[8] The crossing terms (C.T.) have been left out in these equations because for our purposes they have the same form as the direct terms and do not affect the argument.

If all the mesons had mass $\mu\neq 1$, Eqs. (3.25) would become

$$h_\sigma{}^J(\omega,\mu)=\left(\frac{f}{\mu}\right)^2\frac{\lambda_\sigma{}^J}{\omega}+\frac{1}{\pi}\int_\mu^\infty d\omega'\, k_\mu{}^3(\omega')v_\mu{}^2(\omega')\frac{|h_\sigma{}^J(\omega',\mu)|^2}{\omega'-\omega-i\epsilon}+\text{C.T.}$$

Here we assume the convention that mesons of mass μ possess a coupling constant (f/μ), and also that the cutoff function becomes effective at an energy equal to so many rest masses; i.e., $v_\mu(\omega)=v(\omega/\mu)$ for $\omega\geqslant\mu$. Then since $k_\mu{}^3(\omega)=\mu^3k^3(\omega/\mu)$, from Eq. (2.6), we find

$$\mu^3h_\sigma{}^J(\mu\omega,\mu)=f^2\frac{\lambda_\sigma{}^J}{\omega}+\frac{1}{\pi}\int_1^\infty d\nu\, k^3(\nu)v^2(\nu)\frac{|\mu^3h_\sigma{}^J(\mu\nu,\mu)|^2}{\nu-\omega-i\epsilon}+\text{C.T.}$$

Comparison with Eq. (3.25) yields the result

$$\begin{aligned}\mu^3h_\sigma{}^J(\mu\omega,\mu)&=h_\sigma{}^J(\omega,1)\equiv h_\sigma{}^J(\omega),\\ h_\sigma{}^J(\omega,\mu)&=\frac{1}{\mu^3}h_\sigma{}^J(\omega/\mu).\end{aligned} \quad (3.26)$$

This "scaling property," relating the scattering amplitude for mesons of mass μ to those of mass 1, is merely a consequence of the dimensionality of the p-wave amplitudes. However, it is a nontrivial result that we can relate these amplitudes (to first order) when only the π^0 meson has mass $\mu\neq 1$. This is done using the unitarity requirement of the next section; however, we will sketch the general idea here.

We define the differential scaling functions $K_\sigma{}^J$ by the equations

$$\begin{aligned}\Delta K_\sigma{}^J(\omega)&=h_\sigma{}^J(\omega,\mu_3)-h_\sigma{}^J(\omega,1)\\ &=\frac{1}{\mu_3{}^3}h_\sigma{}^J(\omega/\mu_3)-h_\sigma{}^J(\omega),\quad(\sigma=1,2)\end{aligned} \quad (3.27)$$

$$K_\sigma{}^J(\omega)=0,\quad(\sigma\neq 1,2).$$

$\Delta K_\sigma{}^J(\omega)$ is of first order in Δ, which has been explicitly factored out. Unitarity will give the following equation for those $n_\sigma{}^J$ which are first order in Δ:

$$\mathrm{Im}n_{\sigma,\Delta}{}^J=\sum_{\lambda\mu}\left[2k^3C_\sigma{}^{\lambda\mu}\,\mathrm{Re}(h_\lambda{}^{J*}n_{\mu,\Delta})+(k_3{}^3-k^3)D_\sigma{}^{\lambda\mu}h_\lambda{}^{J*}h_\mu{}^J\right],\quad(3.28)$$

at a given energy ω_k. However, it will be possible to satisfy this equation identically by writing

$$n_{\sigma,\Delta}{}^J=u_{\sigma,\Delta}{}^J+\textstyle\sum_\lambda D_\sigma{}^\lambda K_\lambda{}^J-\eta^J\delta_{\sigma,4}.\quad(3.29)$$

The $\delta_{\sigma,4}$ is a Kronecker δ symbol, the matrix $D_\sigma{}^\lambda$ is that introduced in Eq. (3.19) and the function η^J is a known real function. The functions $u_\sigma{}^J$ will have a definite known phase, and their amplitudes will be the only unknowns in Eq. (3.29). The point in introducing the functions $u_\sigma{}^J$ is that they satisfy simpler equations than the $n_\sigma{}^J$, as the second integral in Eq. (2.24) will not occur.

The function η^J appears because of our choice of operators Γ_σ. They have been chosen to simplify both time reversal and the matrix $C_\sigma{}^{\lambda\mu}$. However the matrix $D_\sigma{}^{\lambda\mu}$ connects both $\Gamma_1\Gamma_2$ and $\Gamma_2\Gamma_1$ to Γ_4, and η^J arises as an interference term. This could have been avoided only by complicating time reversal and introducing an extra nonzero amplitude.

The functions $u_\sigma{}^J$ will satisfy the equations

$$u_\sigma{}^J(\omega)+\sum_\lambda D_\sigma{}^\lambda K_\lambda{}^J(\omega)-\eta^J(\omega)\delta_{\sigma,4}=B_\sigma{}^J/\omega+\frac{1}{\pi}\int d\omega'\left[\frac{\mathrm{Im}(u_\sigma{}^J+\sum_\lambda D_\sigma{}^\lambda K_\lambda{}^J)}{\omega'-\omega-i\epsilon}+\frac{T_{\rho\sigma}{}^{J'J}\,\mathrm{Im}(u_\rho{}^{J'}+\sum_\lambda D_\rho{}^\lambda K_\lambda{}^{J'})}{\omega'+\omega}\right],\quad(3.30)$$

where $B_\sigma{}^J$ is the Born term of Eq. (2.24), defined by

$$-[f_\alpha\tau_\alpha(\boldsymbol{\sigma}\cdot\mathbf{p}_\alpha),f_\beta\tau_\beta(\boldsymbol{\sigma}\cdot\mathbf{q}_\beta)]_{\delta,d}=\sum_{\sigma,J}B_\sigma{}^J\Gamma_\sigma(\alpha\beta)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta).\quad(3.31)$$

Now because T and D commute, by Eq. (3.20), we can collect all the terms in $K_\lambda{}^J$ on the right-hand side:

$$\sum_\lambda D_\sigma{}^\lambda\left[-K_\lambda{}^J+\frac{1}{\pi}\int d\omega'\left(\frac{\mathrm{Im}K_\lambda{}^J}{\omega'-\omega-i\epsilon}+\frac{T_{\rho\lambda}{}^{J'J}\,\mathrm{Im}K_\rho{}^{J'}}{\omega'+\omega}\right)\right].\quad(3.32)$$

But $K_\lambda{}^J$ is just the difference $[h(\omega,\mu_3)-h(\omega)]$ so that the unitarity condition for the h's, $\mathrm{Im}h=k^3|h|^2$, says that expression (3.32) is just the difference between the Chew-Low equations for mesons of mass μ_3, and those of mass 1—but with the Born term missing. So expression (3.32) reduces to

$$-\sum_\rho D_\sigma{}^\rho\frac{\lambda_\rho{}^J}{\omega}\left[\left(\frac{f}{\mu_3}\right)^2-f^2\right]=-\sum_\rho D_\sigma{}^\rho\lambda_\rho{}^Jf^2\frac{2\Delta}{\omega}\equiv\Delta\frac{\xi_\sigma{}^J}{\omega}.\quad(3.33)$$

Thus the second integral in Eq. (2.24) for the functions $n_\sigma{}^J$ becomes, with the introduction of the functions $u_\sigma{}^J$, merely a Born term of the form $1/\omega$, due to the properties of the scaling functions $K_\sigma{}^J$.

D. Unitarity

The unitarity condition on the S matrix, Eq. (2.16), together with Hermiticity, Eq. (2.17), gives

$$T_{p\alpha,q\beta}{}^\dagger(\omega)-T_{q\beta,p\alpha}(\omega)=2\pi i\sum_\gamma\int\frac{d^3\mathbf{k}_\gamma}{(2\pi)^3}\delta(\omega_k-\omega)T_{p\alpha,k\gamma}(\omega_k)T_{q\beta,k\gamma}(\omega_k),\quad(\omega_p=\omega_q\equiv\omega).\quad(3.34)$$

Writing $T_{q\beta,p\alpha}$ as in Eq. (2.23) and making use of Eqs. (3.11), and (3.15) this becomes

$$\mathrm{Im}n_\sigma{}^J=2k^3\sum_{\lambda\mu}C_\sigma{}^{\lambda\mu}\,\mathrm{Re}(h_\lambda{}^{J*}n_\mu{}^J)+(k_3{}^3-k^3)\sum_{\lambda\mu}D_\sigma{}^{\lambda\mu}h_\lambda{}^{J*}h_\mu{}^J.\quad(3.35)$$

Recalling the discussion at the end of Sec. II-B, we write, explicitly factoring out the dependence on the parameters Δ, δ, d,

$$\begin{aligned}n_\sigma{}^J&=\Delta n_{\sigma,\Delta}{}^J+\delta u_\sigma{}^J,\quad(\sigma=1\cdots4),\\ n_\sigma{}^J&=du_\sigma{}^J,\quad(\sigma=6\cdots9).\end{aligned}\quad(3.36)$$

Equation (3.28) is just Eq. (3.35) for the Δ parts. For the δ and d parts the last term is absent, and Eq. (3.35) takes a very simple form. Using the fact that $C_\sigma{}^{\lambda\mu}$ is of the form $C_\sigma{}^{\lambda\sigma}$, for $\lambda=1$, 2 and $\sigma\neq5$, 10, and making explicit use of the following form for the Chew-Low amplitudes[7]

$$h_\sigma{}^J=(1/k^3)\exp(i\alpha_\sigma{}^J)\sin\alpha_\sigma{}^J,\quad(\sigma=1,2),\quad(3.37)$$

Eq. (3.35) is identically satisfied by

$$u_\sigma{}^J=A_\sigma{}^J(\omega)\sum_{\lambda=1}^2C_\sigma{}^{\lambda\sigma}\exp[2i\alpha_\lambda{}^J(\omega)].\quad(3.38)$$

For convenience, we write these phases down explicitly [using the fact that $(e^{2i\alpha_1}+e^{2i\alpha_2})$ has phase $(\alpha_1+\alpha_2)$]:

$$u_\sigma{}^J=A_\sigma{}^J\exp(iB_\sigma{}^J);$$
$$(B_\sigma{}^J,\sigma=1\cdots10)=(2\alpha_1{}^J,2\alpha_2{}^J,2\alpha_2{}^J,\alpha_1{}^J+\alpha_2{}^J,-,2\alpha_2{}^J,2\alpha_2{}^J,2\alpha_1{}^J,\alpha_1{}^J+\alpha_2{}^J,-),\quad(3.39)$$

so that unitarity completely determines the phase of the coupling constant contributions.

For the terms in Δ, for which the unitarity condition is more complicated, we get the result (3.29). To see how this comes about, consider Eq. (3.28) for $\sigma=1$:

$$\cos 2\alpha_1 \, \mathrm{Im} n_1 = \sin 2\alpha_1 \, \mathrm{Re} n_1 + \tfrac{1}{3}[(1/k^3)-(1/k_3{}^3)]\sin^2\alpha_1, \quad (3.40)$$

where we have used the fact that

$$(k_3{}^3-k^3)/k^6 = [(1/k^3)-(1/k_3{}^3)] + O(\Delta^2).$$

If we let ξ_1 be any first-order change in $\alpha_1(\omega)$, then to first order

$$\sin^2\alpha_1(\omega) = \sin(\alpha_1-\xi_1)\sin(\alpha_1+\xi_1) = \mathrm{Im}[e^{-i(\alpha_1+\xi_1)}\sin(\alpha_1+\xi_1)e^{2i\alpha_1}].$$

Now choose ξ_1 such that

$$\xi_1 = \Delta\omega(\partial\alpha_1/\partial\omega),$$
$$\alpha_1(\omega)+\xi_1 = \alpha_1(\omega/\mu_3)+O(\Delta^2).$$

Then, from Eqs. (2.6) and (3.37),

$$(1/k_3{}^3)\sin^2\alpha_1(\omega) = \mathrm{Im}[(1/\mu_3{}^3)h_1{}^*(\omega/\mu_3)e^{2i\alpha_1(\omega)}],$$
$$[(1/k_3)-(1/k_3{}^3)]\sin^2\alpha_1(\omega) = -\mathrm{Im}[K_1{}^*(\omega)e^{2i\alpha_1(\omega)}].$$

Thus Eq. (3.40) yields

$$\mathrm{Im}[e^{2i\alpha_1}(n_1-\tfrac{1}{3}K_1)^*]=0,$$
$$n_1-\tfrac{1}{3}K_1 = A_{1\Delta}e^{2i\alpha_1} \equiv u_{1\Delta}. \quad (3.41)$$

The other equations, for $\sigma=2, 3, 4$, give similar results, except that for $\sigma=4$ both α_1 and α_2 appear and the first-order corrections ξ_1 and ξ_2 give rise to an interference term, which leads to the introduction of the term η^J in Eq. (3.29), given by

$$\eta^J = -\frac{\Delta\omega}{6k^3}\left[\frac{\partial}{\partial\omega}(\alpha_2{}^J-\alpha_1{}^J)\right]\frac{\sin(\alpha_2{}^J-\alpha_1{}^J)}{\sin(\alpha_2{}^J+\alpha_1{}^J)}. \quad (3.42)$$

In Eq. (3.29), the functions $u_{\sigma,\Delta}{}^J$ have the same phase as the functions $u_\sigma{}^J$ of Eq. (3.39) for the δ corrections.

To recapitulate, the $n_\sigma{}^J$ written in the form (3.29) satisfy the unitarity condition identically, and the $u_\sigma{}^J$ satisfy the Eq. (3.30), which has been reduced to the form

$$u_\sigma{}^J = \Delta\eta^J\delta_{\sigma,4} + B_\sigma{}^J/\omega + \Delta\xi_\sigma{}^J/\omega + \frac{1}{\pi}\int d\omega'\left(\frac{\mathrm{Im} u_\sigma{}^J(\omega')}{\omega'-\omega-i\epsilon} + \sum_{\rho,J'}\frac{T_{\rho\sigma}{}^{J'J}\,\mathrm{Im} u_\rho{}^{J'}(\omega')}{\omega'+\omega}\right). \quad (3.43)$$

We will now explicitly determine the Born terms $B_\sigma{}^J$, defined by Eq. (3.31). The zeroth-order terms of the commutator in Eq. (3.31) are the Born term for the Chew-Low Eqs. (3.23) and (3.24):

$$f^2\sum_{\sigma J}\lambda_\sigma{}^J\Gamma_\sigma P_J = -3f^2[P_1(\mathbf{p}_\alpha,\mathbf{q}_\beta)\Gamma_1(\alpha\beta) - P_1(\mathbf{q}_\beta,\mathbf{p}_\alpha)\Gamma_1(\beta\alpha)], \quad (3.44)$$
$$\lambda_\sigma{}^J = -3(\delta_{\sigma 1}\delta_{J1}-T_{1\sigma}{}^{1J}).$$

The first-order charge-symmetric part of Eq. (3.31) is [with f_α defined by (2.4)]

$$-f^2[\tau_\alpha\tau_3\delta_{\beta 3}+\delta_{\alpha 3}\tau_3\tau_\beta]P_1 + \mathrm{C.T.}$$

Then, using $\delta_{\alpha\beta}=\sum_{\mu=1,2}\Gamma_\mu(\alpha\beta)$ and the identity (3.20),

$$B_{\sigma,\delta}{}^J = -6f^2\delta(\delta_{J1}D_\sigma{}^1-\sum_\rho T_{1\rho}{}^{1J}D_\sigma{}^\rho) = 2f^2\delta\sum_\rho D_\sigma{}^\rho\lambda_\rho{}^J = -\delta\xi_\sigma{}^J, \quad (3.45)$$

where $\xi_\sigma{}^J$ is defined by Eq. (3.33). The first-order nonsymmetric part of (3.31) is

$$f^2 d(\delta_{\alpha 3}\tau_\beta+\tau_\alpha\delta_{\beta 3})P_1+\mathrm{C.T.} = f^2 d(\Gamma_{10}+4\Gamma_8)P_1+\mathrm{C.T.} \quad (3.46)$$
$$B_{\sigma,d}{}^J \equiv d\zeta_\sigma{}^J.$$

Thus the complete Born term of Eq. (3.43) reads:

$$\Delta\eta^J(\omega)\delta_{\sigma,4}+(\Delta-\delta)\xi_\sigma{}^J/\omega+d\zeta_\sigma{}^J/\omega.$$

The dependence on $(\Delta-\delta)$ was predicted in the discussion following Eq. (2.7). The matrices $\xi_\sigma{}^J$ and $\zeta_\sigma{}^J$ are given in Table IV. The equations for the $u_\sigma{}^J$ have now been completely determined.

Now (and not until now) we shall make the assumption that the resonance (i.e., the amplitude $h_2{}^2$) dominates the scattering. This means we shall ignore in the scattering equations all phases but $\alpha_2{}^2$ ($\equiv\alpha_{33}$). Then, using Eq. (3.39), we may say that: (a) all the $u_\sigma{}^J$ for $J=1$ are real; (b) $u_1{}^2$ and $u_8{}^2$ are real; (c) $u_2{}^2$, $u_3{}^2$, $u_7{}^2$ have phase $2\alpha_2{}^2$; (d) $u_4{}^2$, $u_9{}^2$ have phase $\alpha_2{}^2$. There are left only five amplitudes that must be solved for; the rest are real[12] and can be determined in terms of these five. From Eq. (3.42),

$$\eta^1=0, \quad \Delta\eta^2 = -\frac{\Delta\omega}{6k^3}\frac{\partial\alpha_2{}^2}{\partial\omega} \equiv f^2\Delta\eta. \quad (3.47)$$

Introducing the two functions

$$b_1 = \frac{1}{k^3}\exp(i\alpha_2{}^2)\sin\alpha_2{}^2, \quad (\equiv h_2{}^2(\omega)),$$
$$b_2 = \frac{1}{k^3}\exp(2i\alpha_2{}^2)\sin 2\alpha_2{}^2, \quad (3.48)$$

TABLE IV. The matrices $\xi_\sigma{}^J$ and $\zeta_\sigma{}^J$.

$J \backslash \sigma$	1	2	3	4	$J \backslash \sigma$	6	7	8	9
1	16/9	0	$\frac{1}{9}$	10/9	1	0	0	−16/3	$-\frac{4}{3}$
2	4/9	0	−2/9	−2/9	2	0	0	9/3	$\frac{2}{3}$
		$\xi_\sigma{}^J$						$\zeta_\sigma{}^J$	

[12] The equations for the $u_6{}^J$ connect only to each other and have no Born terms, so we may set $u_6{}^J=0$, leaving fourteen amplitudes.

and noting that $\mathrm{Im}u_2^2 = A_2^2(\omega)\sin 2\alpha_2^2 = k^3 b_2^* u_2^2$, etc., we can write explicitly the five equations to be solved:

$$\begin{aligned}
u_2^2 &= \frac{1}{\pi}\int \frac{d\omega' k'^3 b_2^* u_2^2}{\omega'-\omega-i\epsilon} + \frac{1}{9\pi}\int d\omega' \frac{k'^3(b_2^*u_2^2+8b_2^*u_3^2-4b_1^*u_4^2)}{\omega'+\omega},\\
u_3^2 &= -\frac{2}{9}\frac{f^2}{\omega}(\Delta-\delta)+\frac{1}{\pi}\int \frac{d\omega' k'^3 b_2^* u_3^2}{\omega'-\omega-i\epsilon} + \frac{1}{9\pi}\int d\omega' \frac{k'^3(-b_2^*u_3^2+b_1^*u_4^2)}{\omega'+\omega},\\
u_4^2 &= -\frac{2}{9}\frac{f^2}{\omega}(\Delta-\delta)+f^2\Delta\eta+\frac{1}{\pi}\int \frac{d\omega' k'^3 b_1^* u_4^2}{\omega'-\omega-i\epsilon} + \frac{1}{9\pi}\int d\omega' \frac{k'^3(8b_2^*u_3^2+b_1^*u_4^2)}{\omega'+\omega},\\
u_7^2 &= \frac{1}{\pi}\int \frac{d\omega' k'^3 b_2^* u_7^2}{\omega'-\omega-i\epsilon} + \frac{1}{27\pi}\int d\omega' \frac{k'^3(-5b_2^*u_7^2+8b_1^*u_9^2)}{\omega'+\omega},\\
u_9^2 &= \frac{2}{3}\frac{f^2}{\omega}d+\frac{1}{\pi}\int \frac{d\omega' k'^3 b_1^* u_9^2}{\omega'-\omega-i\epsilon} + \frac{1}{27\pi}\int d\omega' \frac{k'^3(14b_2^*u_7^2+b_1^*u_9^2)}{\omega'+\omega}.
\end{aligned} \tag{3.49}$$

There are two sets of coupled equations in Eqs. (3.49). In each set, a function of phase $2\alpha_2^2$ is coupled to one of phase α_2^2. The method of solution will be sketched in the next section.

E. Solutions of the Scattering Equations

Equations (3.49) are linear in the functions u_σ^J, and we shall use a simple extension of a method developed by Chew-Low for photoproduction[13] to solve them. First we solve the equations with all crossing terms neglected and then determine the crossing terms by a "self-consistent" procedure. Then Eqs. (3.49) are of either of the following forms[8]:

$$u(z)=B(z)+\frac{1}{\pi}\int_1^{\omega_M} d\omega' \frac{k'^3 b_1^*(\omega')u(\omega')}{\omega'-z}, \tag{3.50}$$

$$v(z)=C(z)+\frac{1}{\pi}\int_1^{\omega_M} d\omega' \frac{k'^3 b_2^*(\omega')v(\omega')}{\omega'-z}, \tag{3.51}$$

[13] G. Chew and F. Low, Phys. Rev. **101**, 1579 (1956).

where the solution is wanted in the limit $z\to\omega+$. In Eq. (3.50) the function $B(z)$ is arbitrary, with the restriction that $zB(z)$ is bounded as $z\to\infty$. But in Eq. (3.51) the function $C(z)$ can have at worst a first-order pole for our solution to work.

Since we have eliminated all amplitudes but h_2^2, we will henceforth use the standard notation $h_2^2\equiv h_3$ and the phase $\alpha_2^2\equiv\alpha_{33}$. Equation (3.50) is solved by noting that $u(z)$ has phase α_{33}, the singularities of $B(z)$ plus a branch point at $\omega=+1$ with a cut running[14] from $+1$ to $+\infty$ and is otherwise analytic, and behaves as $1/z$ at ∞. All we need do then is construct a function possessing these properties, which we do with the aid of the function g_3 of Chew-Low, defined by

$$g_3(z)=\lambda_3/zh_3(z),\quad \lambda_3=\tfrac{4}{3}f^2. \tag{3.52}$$

We assume the effective range approximation to be valid even for negative energies.[15] This is equivalent to neglecting the singularity of the crossing contribution to it, so that g_3 has the following properties: it possesses a branch point at $z=+1$ with a cut running to $+\infty$, it is $=+1$ at the origin, goes to a constant as $z\to\infty$, and $g_3(\omega)$, $\omega>1$ has phase $-\alpha_{33}$. Thus we can write

$$\begin{aligned}
u(\omega) &= \lim_{z\to\omega+}\left[B(z)+\frac{\lambda_3}{g_3(z)}\int_1^{\omega_M} d\omega' \frac{k'^3 B(\omega')}{\omega'(\omega'-z)}\right]\\
&= B(\omega)e^{i\alpha_{33}}\cos\alpha_{33}+\frac{\lambda_3}{\pi g_3}P\int_1^{\omega_M} d\omega' \frac{k'^3 B(\omega')}{\omega'(\omega'-\omega)}.
\end{aligned} \tag{3.53}$$

In Eq. (3.53) both terms have the proper phase and the function possesses the appropriate singularities and asymptotic behavior.[16]

Similarly, for the solution of Eq. (3.51), where $v(\omega)$ has phase $2\alpha_{33}$, we may write

$$v(\omega)=\lim_{z\to\omega+}\left(\frac{g_3(\beta)}{g_3(z)}\right)\times\left[C(z)+\frac{\lambda_3}{\pi g_3(z)}\int_1^{\omega_M} d\omega' \frac{k'^3 C(\omega')}{\omega'(\omega'-z)}\right], \tag{3.54}$$

where $C(z)$ has a pole at $z=\beta$. The extra $1/g_3(z)$ provides the proper phase, but must be normalized to give the correct residue at $z=\beta$. Equation (3.54) can be easily generalized if $C(z)$ contains more than one pole or higher-order poles.

[14] Neglecting the crossing terms is equivalent to neglecting the branch line singularity for $\omega\leqslant -1$.

[15] It has been shown by G. Saltzman and F. Saltzman, Phys. Rev. **108**, 1619 (1957) that it is not safe to extend the effective range approximation to negative energies for $g_3(\omega)$; however, this is because of the crossing contributions we have ignored. For the approximate $g_3(\omega)$ on which solutions (3.53) and (3.54) are based, the effective range approximation can be extended. The only difference between the approximate and exact $g_3(\omega)$ in the physical region is that the approximate one needs a higher cutoff to produce the effect of the crossing terms.

[16] Our numerical solutions use the parameters $\omega_M=6$, $f^2=0.08$, $1/r_3=\omega_0=2.1$, $g(-1.7)=1.81$, in the notation of reference 7.

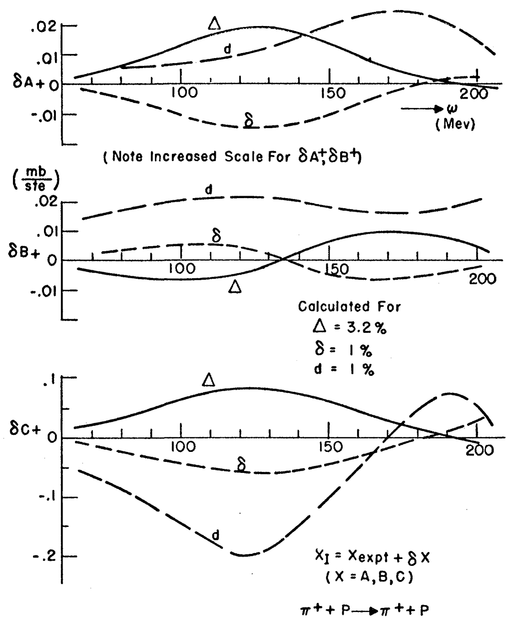

FIG. 3. Corrections to $d\sigma^{(+)}/d\Omega$. These corrections are to be added directly to the experimental quantities to yield the charge-independent ones. Δ, δ, d are defined by Eqs. (2.1) and (2.5).

If we use the solutions (3.53) and (3.54) as first approximations to the solutions of Eqs. (3.49), and place them into the crossing integrals, we find that the crossing integrals containing b_1 and b_2 can both be well approximated in the region of interest $(1\leqslant\omega<4)$ by a function of the form $A/(\omega+1.7)$. Furthermore the crossing integral of $1/(\omega+1.7)$ can be approximated by $A/(\omega+1.7)$. This suggests the following procedure for solving the equations. First, replace all crossing terms by functions of the form $A_i/(\omega+1.7)$ so that they become extra Born terms. Then the equations have exactly the form of Eqs. (3.50) and (3.51), and can be solved. The crossing integrals are then performed and, being proportional to $1/(\omega+1.7)$, they give a set of linear equations in the A_i.

Once these five $u_\sigma{}^J$ have been determined, the other nine real $u_\sigma{}^J$ are given by crossing integrals over these five and have the form $(A/\omega)+B/(\omega+1.7)$. Then the functions $n_\sigma{}^J$ are calculated by Eq. (3.29), with

$$\Delta K_2{}^2(\omega)=\Delta[-(3h_3/k^2)-6\eta e^{2i\alpha_{33}}],\qquad K_1{}^1=K_1{}^2=K_2{}^1=0. \tag{3.55}$$

The T matrix for a particular scattering process, say $\pi^++p\rightarrow\pi^++p$ is then given by Eq. (2.23) as

$$\langle\pi^+p|T_{q\beta,p\alpha}(\omega)|\pi^+p\rangle=-\frac{2\pi}{\omega}\sum_{J,\sigma}\langle\pi^+p|\Gamma_\sigma(\alpha\beta)|\pi^+p\rangle \times(h_\sigma{}^J+n_\sigma{}^J)P_J(\mathbf{p}_\alpha\mathbf{q}_\beta),$$

where the matrix element has been evaluated with respect to the isotopic variables only. If now the s-wave contributions are phenomenologically placed into the T matrix above, it assumes the form

$$T=-\frac{2\pi}{p^3\omega}\sum_{J,T}[(a_{TJ}+\epsilon_T{}^J)P_J(\mathbf{p}_\alpha,\mathbf{q}_\beta)+a_T\delta_{J,\frac{1}{2}}pq] \equiv\frac{2\pi}{p^3\omega}X, \tag{3.56}$$

where the $\epsilon_T{}^J$ are our calculated corrections and the amplitudes, a, have the form $e^{i\alpha}\sin\alpha$. Then the cross section is given by

$$\sigma=\frac{2\pi\omega}{q_\beta}\tfrac{1}{2}\operatorname{Tr}[\sum_p\delta(\omega_p-\omega_q)T_{q\beta,p\alpha}{}^\dagger T_{q\beta,p\alpha}],\qquad d\sigma/d\Omega=(p_\alpha/q_\beta)\tfrac{1}{2}\operatorname{Tr}[(1/p^6)X^\dagger X]. \tag{3.57}$$

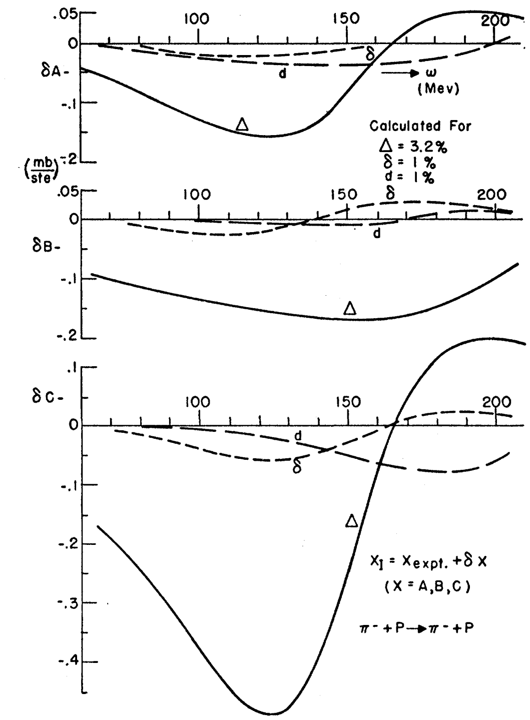

FIG. 4. Corrections to $d\sigma^{(-)}/d\Omega$.

Thus, besides the corrections we have calculated, there is a kinematical correction due to the factor p_α/q_β and also to the p-wave projection operators. So for p-wave charge exchange scattering there is, at low energy, a correction of $9\%/p^2$ due to kinematical corrections alone.

When a laboratory experiment is analyzed in the form $d\sigma/d\Omega=A+B\cos\theta+C\cos^2\theta$, there are no theoretical prejudices built into it other than the limitation to s and p waves. If now the charge independence hypothesis is made, the coefficients A, B, and C are forced into a prescribed functional dependence on the six phase shifts which is too restrictive, because charge

independence is only approximate. It has been the purpose of this calculation to write the quantities A_{expt}, etc., as first order corrections to quantities $A_I = A_{\text{expt}} + \delta A$ on which it would be valid to perform a charge-independent analysis. That is, we have isolated the charge-independent part of the cross section. The quantities δA, δB, δC are plotted in Figs. 3, 4, and 5 for the three experimental scattering processes, including the effects of the charge-exchange kinematical corrections. In each of the figures the corrections due to Δ, δ, d are plotted separately. The numerical values are determined on the basis of $\Delta = +3.2\%$ (the known value) and δ, $d = +1\%$. The correction to the total cross section $\sigma_T^{(-)}$ is plotted in

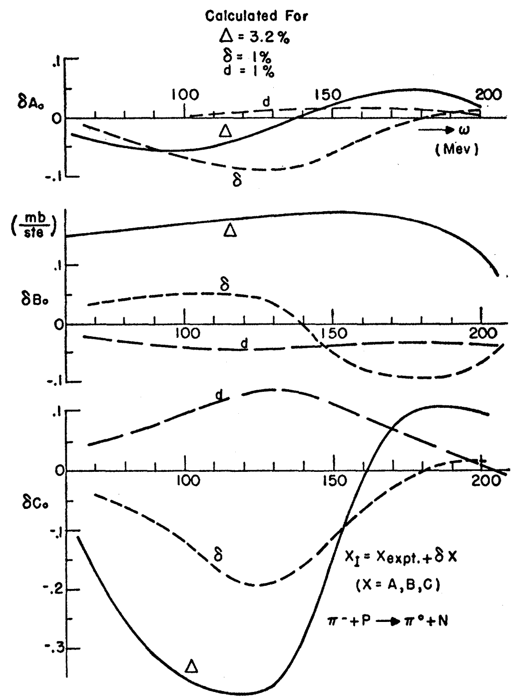

FIG. 5. Corrections to $d\sigma^{(0)}/d\Omega$.

Fig. 6, calculated from the formula

$$\sigma_T^{(-)} = 4\pi\{[A^{(-)} + (C^{(-)}/3)] + [A^{(0)} + (C^{(0)}/3)]\}. \quad (3.58)$$

The contribution from d is very small, due to cancellations, and has not been included in Fig. 6. The corrections to $\sigma^{(+)}$ are all very small and have not been plotted there either.

IV. COMPARISON WITH EXPERIMENT

The cross-section corrections calculated above were incorporated into a phase-shift analysis of some of the more accurate scattering data. In this analysis, the parameters δ and d had to be determined along with the phase shifts. The problem was programmed on the IBM 704 computer at MIT. The machine was first

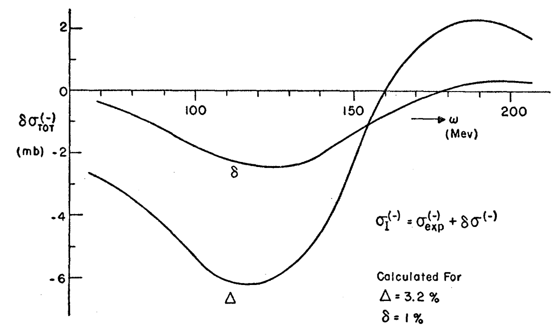

FIG. 6. Corrections to $\sigma_{\text{total}}^{(-)}$. This includes both π^- elastic and charge exchange scattering.

told to find a set of Fermi type phase shifts[17] by minimizing[18] the quantity $M = \sum_i [N_i(\alpha) - N_i]^2/\epsilon_i$, where the N_i are the nine experimental quantities (A^x, B^x, C^x; $x = +, -, 0$), the $N_i(\alpha)$ are calculated from the phase shifts, and the ϵ_i are the mean square experimental errors. The phase shifts so obtained are "uncorrected," i.e., the raw data is forced into a charge-independent fit. The results are listed as "Run 1" in Table V.

Next, the N_i were changed to include the known mass corrections. Then the phase shifts were again varied to give a minimum in M, listed as "Run 2." Finally, δ and d were allowed to vary, thus changing the N_i—while the α's varied, changing the $N_i(\alpha)$. At this stage the minimum was found as a function of eight variables, giving a best overall minimum for M, and a set of values for δ, d and the six α, listed as "Run 3." These α's should be charge independent, and the δ, d should come out the same at any energy. Of course experimental errors will lead to a spread of values for δ and d, when analysis of many experiments is carried out. However, analysis of a sufficient number of experiments should tend to produce consistent results. No statistical analysis was carried out, as only the qualitative nature of the results was wanted.

To perform this analysis a complete set of experiments (all nine N_i) must be available at the energy involved. Only the data[19] at 150 Mev and 170 Mev were accurate enough to give any meaningful correlation. The coupling constant corrections were too small to use the data[20] at 220 Mev. The data at 150 and 170 Mev gave quite consistent values for δ at about (-4%) and for $d \sim 0$. An analysis was also attempted using the early poor data[18] of Fermi at 135 Mev, and while it gave a consistent value for δ, it gave about 5% for d.

Examination of Table V shows that the only significant change takes place in α_1, which becomes much

[17] These are described in H. Bethe and F. de Hoffman, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), Vol. II.

[18] The analysis is somewhat similar to that described by Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).

[19] Ashkin, Blaser, Feiner, and Stern, Phys. Rev. **101**, 1149 (1956).

[20] Ashkin, Blaser, Feiner, and Stern, Phys. Rev. **105**, 724 (1957).

TABLE V. Corrections to phase-shift analysis. Run 1—Raw data. Run 2—Mass corrections made. Run 3—Mass plus δ, d corrections made.

Energy (Mev)	Run	α_3	α_1	α_{33}	α_{31}	α_{13}	α_{11}	Δ	δ	d
135	1	−14.8°	9.5°	39.6°	−1.5°	1.6°	−2.0°	0%	0%	0%
	2	−13.2	7.5	39.8	−2.0	−1.0	−2.3	+3.2	0	0
	3	−13.2	8.3	41.5	−1.5	−2.0	−1.5	+3.2	−5.3	−5.1
150	1	−13.2	9.0	51.8	−3.9	2.1	−2.0	0	0	0
	2	−12.5	6.8	51.2	−4.5	0.7	−2.0	+3.2	0	0
	3	−12.4	5.9	53.3	−5.4	0.3	−1.8	+3.2	−3.1	−0.3
170	1	−13.3	9.0	65.4	−3.4	3.4	−2.4	0	0	0
	2	−14.7	7.2	65.6	−3.2	3.9	−2.2	+3.2	0	0
	3	−15.2	4.2	67.3	−5.9	3.6	−1.2	+3.2	−5.4	−0.1

less positive at higher energy. For the p waves: α_{13} goes negative at low energy, the others are quite stable, and all the small phase shifts remain small. The s-wave phase shifts are strongly affected because the s-p interference term, δB, is strongly changed for π^- scattering.

We have used the results of Table V together with low-energy data from Bethe and de Hoffman,[17] corrected for charge dependence, to draw an effective range plot in Fig. 7. This is a graph of $\frac{4}{3}(\eta^3/\omega^*)\cot\alpha_{33}$ *vs* ω^*, where η and ω^* are the center-of-mass momentum and energy (in units of the pion mass). The effective range approximation[7] assumes that this will be a linear function and extrapolate to $1/f^2$ at $\omega^*=0$. The charge-dependent effects definitely tend to decrease f^2, as determined by this method. To see why, note from Table V that for $135\rightarrow 170$ Mev these corrections tend to increase α_{33} by a few degrees, thus decreasing $\cot\alpha_{33}$ by a few percent. At low energies however, the corrections tend to decrease the charge-independent α_{33}. This is because the extra phase space for the lighter π^0 makes the charge-exchange cross section appear larger than it should. This large cross section needs an artificially large α_{33} to explain it (since $\sigma\sim\sin^2\alpha_{33}$). The corrected α_{33} will be smaller and so have a larger cotangent (by $\sim$10%). Both these effects increase the slope of the line in Fig. 7 and thus decrease f^2. For comparison, the line drawn by Puppi[3] is included in the figure.

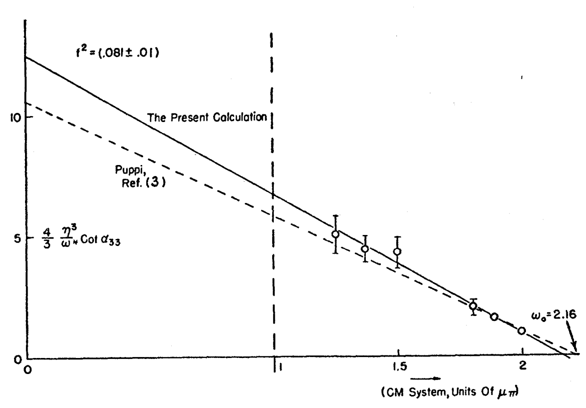

FIG. 7. Effective range plot for the pion coupling constant.

Finally, we present a method for determining δ from the total cross sections in the energy range 130–170 Mev. The method will not determine d, as d has little effect on the total cross sections.

Experimentally the cross sections are quite linear in this region, as are our corrections $\delta\sigma^{(-)}$ (we ignore $\delta\sigma^{(+)}$). We write

$$[\sigma^{(-)}-(\sigma^{(+)}/3)]=[\sigma_I^{(-)}-(\sigma_I^{(+)}/3)]-\delta\sigma^{(-)}. \quad (4.1)$$

But[17]

$$[\sigma_I^{(-)}-(\sigma_I^{(+)}/3)]=(8\pi/3)\lambda^2(\sin^2\alpha_1+2\sin^2\alpha_{13}+\sin^2\alpha_{11})$$
$$\approx(8\pi/3)\lambda^2\sin^2\alpha_1\sim(8\pi/3)\lambda^2(\alpha_1)^2,$$

where we ignore the small p-wave terms as they are of second order (assuming the small p waves are of the same order as our corrections). In this equation, λ is the wavelength divided by 2π. Now at low energies s-wave phase shifts go linearly with the momentum, and in pion scattering this linearity holds approximately[21] in the region we are considering, so that $\lambda^2(\alpha_1)^2$ should be constant and

$$(d/d\omega)[\sigma_I^{(-)}-(\sigma_I^{(+)}/3)]\sim 0.$$

What is meant by zero in this equation is the assumption that $(d/d\omega)[\lambda^2(\alpha_1)^2]\ll(d/d\omega)[\delta\sigma^{(-)}]$, which will have to be checked experimentally when the data permits; however, $\delta\sigma^{(-)}$ is strongly energy dependent and the assumption probably holds. Then Eq. (4.1) becomes

$$(d/d\omega)[\sigma^{(-)}-(\sigma^{(+)}/3)]\approx-(d/d\omega)\delta\sigma^{(-)}$$
$$\approx-0.18\Delta+0.053\delta, \quad \text{(in mb/Mev)}. \quad (4.2)$$

The values in the right-hand side are taken from the slope of the curves in Fig. 4, which are approximately linear in this region. Since the cross sections themselves are approximately linear, Eq. (4.2) becomes an equation

[21] J. Orear, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, New York, 1957).

for δ. At the present time the slope of these cross sections in this region are not well known at all.

APPENDIX. FORM OF THE RENORMALIZED COUPLING CONSTANTS

In the charge-independent theory

$$(\psi_{0n},\tau_\alpha\boldsymbol{\sigma}\psi_{0m})=\rho\langle n|\tau_\alpha\boldsymbol{\sigma}|m\rangle,\quad \rho=f/f^{(0)}. \tag{A.1}$$

The constant ρ defines the renormalized coupling constant. In the charge-dependent theory, the most general possible form for this matrix element is

$$(\psi_{0n},\tau_\alpha\boldsymbol{\sigma}\psi_{0m})=\rho_\alpha\langle n|(\tau_\alpha+A\tau_3\tau_\alpha+B\tau_\alpha\tau_3+C\tau_3\delta_{\alpha 3}+D\mathbf{I}+E\tau_3)\boldsymbol{\sigma}|m\rangle \tag{A.2}$$

where ρ_α, A, B, C, D, and E are constants. The term $\delta_{\alpha 3}\tau_3$ in Eq. (A.2) is equivalent to $\tau_3\tau_\alpha\tau_3$.

The generator of finite rotations in isotopic spin space is

$$U=\exp(-i\mathbf{T}\cdot\boldsymbol{\theta}),\quad \mathbf{T}=\mathbf{t}+\boldsymbol{\tau}/2, \tag{A.3}$$

where an explicit form for $\mathbf{t}$ is given by

$$\mathbf{t}=\textstyle\sum_k i\mathbf{a}_k\times\mathbf{a}_k^\dagger. \tag{A.4}$$

One can show that $\mathbf{T}$ commutes with H_0 and H_i of the charge-independent theory. Our charge-dependent Hamiltonian, Eqs. (2.2) and (2.3), is still invariant to rotations about T_3. For the physical nucleon, $\psi_{0n}=X\phi_{0n}$, where X is given by Eq. (2.11),

$$\begin{aligned} UXU^{-1}&=X,\\ e^{-i\theta T_3}\psi_{0n}&=Xe^{-i\theta T_3}\phi_{0n}=e^{-i(\theta/2)\lambda_n}\psi_{0n},\\ \lambda_n&=+1,\quad n=P\\ &=-1,\quad n=N.\end{aligned} \tag{A.5}$$

Letting $\theta=\pi/2$, we can show that

$$(\psi_{0n},\tau_1\boldsymbol{\sigma}\psi_{0m})=e^{i(\pi/4)(\lambda_n-\lambda_m)}(\psi_{0n},\tau_2\boldsymbol{\sigma}\psi_{0m}). \tag{A.6}$$

Thus this matrix element rotates like a vector, for $n\neq m$. The equation says nothing for $n=m$. Using Eq. (A.2) we have

$$\begin{aligned}(\psi_{0P},\tau_1\boldsymbol{\sigma}\psi_{0N})&=i(\psi_{0P},\tau_2\boldsymbol{\sigma}\psi_{0N}),\\ \rho_1(1+A-B)&=\rho_2(1+A-B),\\ \rho_1&=\rho_2\equiv\rho.\end{aligned} \tag{A.7}$$

Time reversal gives the result

$$(\psi_{0n},\tau_\alpha\boldsymbol{\sigma}\psi_{0m})=-\nu_\alpha(\psi_{0m}{}^T,\tau_\alpha\boldsymbol{\sigma}\psi_{0n}{}^T), \tag{A.8}$$

where $\psi_{0n}{}^T$ is a time-reversed nucleon and ν_α is defined in Eq. (3.2).

First, we eliminate the extaneous terms D and E of Eq. (A.2). Specializing Eq. (A.8) to $\alpha=2$ between two protons, and remembering that ψ_{0P} and $\psi_{0P}{}^T$ have opposite spin, yields $\rho_3(D+E)=-\rho_3(D+E)$. Had we used two neutrons, this would read $\rho_3(D-E)=-\rho_3(D-E)$. Thus $D=E=0$.

Writing Eq. (A.8) in the form (A.2) gives, for $\alpha=1$, 2,

$$\begin{aligned}\rho\langle n|(\tau_\alpha+A\tau_3\tau_\alpha+B\tau_\alpha\tau_3)\boldsymbol{\sigma}|m\rangle &=-\nu_\alpha\rho\langle m^T|(\tau_\alpha+A\tau_3\tau_\alpha+B\tau_\alpha\tau_3)\boldsymbol{\sigma}|n^T\rangle\\ &=\rho\langle n|(\tau_\alpha+A\tau_\alpha\tau_3+B\tau_3\tau_\alpha)\boldsymbol{\sigma}|m\rangle.\end{aligned} \tag{A.9}$$

Thus $A=B$, and Eq. (A.2) now reads

$$(\psi_{0n},\tau_\alpha\boldsymbol{\sigma}\psi_{0m})=\rho\langle n|(\tau_\alpha+A\delta_{\alpha 3}+C\delta_{\alpha 3}\tau_3)\boldsymbol{\sigma}|m\rangle. \tag{A.10}$$

Only the first term contributes to charged scattering and therefore $f^{(+)}=f^{(-)}$. Comparison with Eq. (2.4) shows that the renormalized coupling constants have the same structure as the unrenormalized ones.

ACKNOWLEDGMENTS

The author is indebted to Professor Francis Low well beyond his guidance in this work. He is also very grateful to Professor G. M. Almy of the University of Illinois for making possible his stay at the Laboratory for Nuclear Science at MIT, to Professor Weisskopf and Professor Feshbach for the hospitality extended during his stay, and to the MIT Computation Center for making their facilities available. He also wishes to thank Professor G. F. Chew for giving a brief report of the work at the 1958 CERN conference.