Euclidean Gauge Transformation*

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The Green's function gauge transformation induced by the elimination of the longitudinal field in Euclidean electrodynamics is discussed.

IN a recent paper[1] the author examined in some detail the transformation from Green's functions of the radiation gauge to Lorentz gauge functions, preliminary to the introduction of a Euclidean metric. It was not sufficiently emphasized, however, that the term "Lorentz gauge," as descriptive of a gauge in which there is no distinguished time-like vector, refers to a class rather than just the special gauge used in the paper. One can also introduce, for example, the transverse[2] Lorentz gauge, characterized by

$$(\partial)_{\xi_\alpha} G_{\pm}{}^{(L)}(x,\xi)=0, \quad \alpha=1\cdots\nu,$$

together with appropriately modified Maxwell differential equations. The radiation gauge functions constructed from Lorentz-gauge Green's functions are clearly independent of the specific Lorentz gauge employed.

Gauge transformations within the class of Lorentz gauges are particularly simple and have received most attention in the published literature.[3] In this note we shall illustrate the Euclidean equivalent of such transformations by decomposing the complementary fields $A_\mu(x)$, $B_\mu(x)$ into their transverse and longitudinal parts:

$$A_\mu={}^TA_\mu+{}^LA_\mu, \quad B_\mu={}^TB_\mu+{}^LB_\mu,$$
$$\partial_\mu\, {}^TA_\mu=\partial_\mu\, {}^TB_\mu=0,$$
$${}^LA_\mu=\partial_\mu D^{\frac{1}{2}}a, \quad {}^LB_\mu=\partial_\mu D^{\frac{1}{2}}b,$$

where

$$D^{\frac{1}{2}}f(x)=(-\partial^2)^{-\frac{1}{2}}f(x)$$
$$=\int\frac{(dk)}{(2\pi)^4}\frac{\exp[ik(x-x')]}{(k^2)^{\frac{1}{2}}}f(x')(dx'),$$

and

$$i[B_\mu(x),A_\nu(x')]={}^T(\delta_{\mu\nu}\delta(x-x'))$$
$$=\int\frac{(dk)}{(2\pi)^4}(\delta_{\mu\nu}-k_\mu k_\nu/k^2)\exp[ik(x-x')]$$
$$i[b(x),a(x')]=\delta(x-x').$$

The Green's functional is given by

$$G[\phi'B']=\langle 0|\exp\{\phi'\psi-iB'A-W[\psi,A]\}|0\rangle$$

in which, apart from the additive constant that assures $G[00]=1$,

$$W[\psi,A]=\tfrac{1}{2}\int(dx)\,[(\partial_\mu A_\nu)^2+\psi(\alpha(\partial-ieqA)+mi\alpha_5)\psi]$$

and the null eigenvalue states refer to the canonical variables $\phi(x)$, $B_\mu(x)$. The dependence of W, the Euclidean action operator, on the longitudinal component of A_μ can be expressed by

$$W[\psi,A]=W[\exp(-ieqD^{\frac{1}{2}}a)\psi,\ {}^TA]+\tfrac{1}{2}aD^{-1}a.$$

After performing the canonical transformation that absorbs $\exp[-ieqD^{\frac{1}{2}}a]$ into ψ, we have

$$\begin{aligned}G[\phi'B']&=\langle 0|\exp\{\phi'\exp(ieqD^{\frac{1}{2}}a)\psi-ib'a\\&\qquad-\tfrac{1}{2}aD^{-1}a-i\,{}^TB'\,{}^TA-W[\psi,{}^TA]\}|0\rangle\\&=\langle 0|\exp\{\phi'\exp[-eqD^{\frac{1}{2}}(\delta/\delta b')]\psi\\&\qquad-i\,{}^TB'\,{}^TA-W[\psi,{}^TA]\}|0\rangle\\&\qquad\times C\langle 0|\exp(-ib'a-\tfrac{1}{2}aD^{-1}a)|0\rangle.\end{aligned}$$

The two factors refer, respectively, to the transverse and longitudinal electromagnetic degrees of freedom, the Dirac field operators appearing entirely in the former. The constant C normalizes the longitudinal factor, which is evaluated as

$$\begin{aligned}C\langle 0|\exp[-\tfrac{1}{2}(a+iDb')D^{-1}(a+iDb')]|0\rangle&\\\times\exp(-\tfrac{1}{2}b'Db')&=\exp(-\tfrac{1}{2}b'Db'),\end{aligned}$$

on performing the canonical transformation[4] $a+iDb'$, $b\rightarrow a, b$. Hence

$$G[\phi'B']=G_T[\exp[eqD^{\frac{1}{2}}(\delta/\delta b')]\phi'\ {}^TB']\exp(-\tfrac{1}{2}b'Db')$$

where $G_T[\phi'\ {}^TB']$ is the Green's functional computed from the transverse field only and

$$b'=-D^{\frac{1}{2}}\partial_\mu B_\mu'.$$

As examples of this connection between the two kinds of Euclidean Green's functions, $G(x_1\cdots x_{2n},\ \xi_1\cdots\xi_\nu)$, we have, for $n=0$, $\nu=2$,

$$G(\xi_1\xi_2)_{\mu_1\mu_2}=G_T(\xi_1\xi_2)_{\mu_1\mu_2}+\partial_{\mu_1}\partial_{\mu_2}'D^2(\xi_1-\xi_2)$$

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[1] J. Schwinger, Phys. Rev. **115**, 721 (1959).
[2] Here "transverse" has a four-dimensional space-time significance.
[3] For example, L. D. Landau and I. M. Khalatnikov, J. Exptl. Theoret. Phys. U.S.S.R. **29**, 89 (1955) [translation: Soviet Phys. JETP **2**, 69 (1956)]; N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), Sec. 40; B. Zumino, Bull. Am. Phys. Soc. **4**, 280 (1959).
[4] Such simple canonical transformations reproduce the entire practical content of the functional integration method.

and for $\nu=0$, arbitrary n,

$$G(x_1\cdots x_{2n})=G_T(x_1\cdots x_{2n})\times\exp\left[-\tfrac{1}{2}e^2\sum_{a,b=1}^{2n}q_aq_bD^2(x_a-x_b)\right],$$

in which

$$D^2(x-x')=\int\frac{(dk)}{(2\pi)^4}\frac{\exp[ik(x-x')]}{(k^2)^2}.$$

On returning to space-time these become examples of the transformation between the original Lorentz gauge and the transverse gauge. The general transformation of this type continues to be expressed compactly by the Green's functionals. By presenting the transformation from Lorentz to radiation gauge in functional form[5]:

$$G^{(R)}[\phi'B']=G^{(L)}[\exp[-eq(\partial^2)^{-1}\partial(\delta/\delta B)]\phi';B]_{B=[1-\partial\partial(\partial^2)^{-1}]B'},$$

one can verify directly that the radiation gauge functions are invariant under a change of Lorentz gauge. In writing this functional relation we have used the equivalence of $i\delta/\delta B(\xi)'$, acting on the Green's functional, with the effect of the symbol $\mathcal{A}(\xi)$ on the Green's functions. The notation ; is a reminder of the location of the functional differential operators on the left in the functional expansion. Now[2]

$$\frac{\delta}{\delta B_\mu}=\frac{\delta}{\delta\,{}^TB_\mu}+\partial^\mu D^{\frac{1}{2}}\frac{\delta}{\delta b},$$

and therefore

$$(\partial^2)^{-1}\partial\frac{\delta}{\delta B}=(\partial^2)^{-1}\partial\frac{\delta}{\delta\,{}^TB}+D^{\frac{1}{2}}\frac{\delta}{\delta b}.$$

[5] This is the form that is produced directly by the external source techniques.

It follows that

$$G^{(L)}[\exp[-eq(\partial^2)^{-1}\partial(\delta/\delta B)]\phi';B]=G_T^{(L)}[\exp[-eq(\partial^2)^{-1}\partial(\delta/\delta\,{}^TB)]\phi';{}^TB]\times\exp(-\tfrac{1}{2}bDb),$$

where b can be directly set equal to zero, since

$$\partial[1-\partial\partial(\partial^2)^{-1}]B'=0,$$

which completes the proof.

It will not have escaped attention that the factor expressing the gauge transformation of the Euclidean Green's functions $G(x_1\cdots x_{2n})$ contains, in the terms with $a=b$, the integral

$$D^2(0)=\int\frac{(dk)}{(2\pi)^4}\frac{1}{(k^2)^2},$$

which is logarithmically divergent at infinity. (It also diverges logarithmically at the origin, but this effect is cancelled in the full summation over a and b.) Thus, against the formal equivalence of all Lorentz gauges must be placed the impossibility of asserting the meaningful existence of the individual Green's functions in an arbitrary gauge. We are committed to the individual existence of the radiation gauge Green's functions[6] and, as a working hypothesis, to the existence of Green's functions in some Lorentz gauge, together with their Euclidean counterparts. From the Euclidean standpoint the transverse gauge is uniquely distinguished, for the longitudinal field is physically superfluous and there is no ambiguity in the identification of a transverse Euclidean field. This is the gauge that will be used in further work on the Euclidean Green's functions of electrodynamics.

Note added in proof.—Related remarks have been published recently by K. Johnson and B. Zumino, Phys. Rev. Letters **3**, 351 (1959).

[6] Subject to the qualifications that are associated with the "infra-red" problem.