# Excitation of Plasma Oscillations*

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The theory of Bohm and Gross and the experiments of Looney and Brown upon the excitation of plasma oscillations by the two-stream mechanism, which appear superficially to be in disagreement, are here shown to be compatible with each other and with related experiments.

## 1. INTRODUCTION

In a paper bearing the above title published in 1954, Looney and Brown[1] reported the results of experiments upon the interaction of an electron beam with an electron plasma. The object and results of these experiments are summarized by the following paragraph taken from that paper:

"In 1949 Bohm and Gross[2] published a theory of plasma electron oscillations based on a one-dimensional analysis of a uniform infinite plasma. The theory postulated that a traveling longitudinal potential field could be excited by a beam of high-velocity electrons in the plasma and that this could be used to explain the energy transfer from a beam of high-energy particles to the oscillation existing in the plasma. A large portion of our experimental effort was devoted to injecting an electron beam into a discharge plasma so as to generate a plasma oscillation by the mechanism proposed by Bohm and Gross. The electron beam injected into a uniform plasma from an electron gun did not excite observable plasma oscillations in any of the experimental tubes constructed to satisfy the conditions of the Bohm and Gross theory. However, as soon as the discharge in the tube...was modified by introducing large sheaths on the beam electrodes, oscillations were immediately found at the same plasma density where they were not observed in the 'infinite' plasma."

The object of the present paper is to resolve the apparent paradox between the theory of Bohm and Gross and the experiments of Looney and Brown.

The aspect of the Bohm and Gross theory which is being questioned is their treatment of perturbations of two infinite uniform interpenetrating electron streams (neutralized by ions), the result of which is summarized by the dispersion relation

$$(\omega_p{}^2/\omega^2)+[\omega_b{}^2/(\omega-vk)^2]=1, \tag{1.1}$$

where $\omega_p$ and $\omega_b$ are the Langmuir frequencies of the plasma and of the beam, respectively, and $v$ is the beam velocity. (The effect of nonzero plasma temperature is not included in this equation, since the temperature term of Bohm and Gross is unsatisfactory. This point will be discussed further in the next section.) $\omega$ and $k$ are the radian frequency and wave number of a Fourier component of the perturbation. It is found that for sufficiently small real values of $k$,

$$k<k_c\equiv v^{-1}(\omega_p{}^{\frac{2}{3}}+\omega_b{}^{\frac{2}{3}})^{\frac{3}{2}}, \tag{1.2}$$

Eq. (1.1) leads to complex values of $\omega$. This implies that the two-stream system is unstable against perturbations of sufficiently large wavelengths. It would therefore appear that an electron beam injected into an electron plasma should be unstable and lead to the generation of oscillations; as Bohm and Gross pointed out, this interpretation of the dispersion relation receives some support from study of energy transfer in the two-stream system. The fact that Looney and Brown failed to observe such oscillations therefore appears to contradict the Bohm-Gross theory.

In analyzing the above discrepancy, our first step will be to examine more closely the inference which one should draw from the dispersion relation (1.1) or from modifications of this dispersion relation which include the effect of temperature. It has been shown in a recent paper[3] that instabilities of propagating media may be divided into two classes which are termed "convective" and "nonconvective." Convective instability provides a mechanism for the spatial amplification of injected disturbances. Nonconvective instability, on the other hand, leads to disturbances which grow in time, ultimately occupying the entire available region of the medium (provided that the dimensions of the system permit the appropriate range of wavelengths to be excited). In Sec. 2, we shall discuss the kinematic properties of the Bohm-Gross dispersion relation as it applies to the Looney-Brown experiment, from which we shall conclude that, in the original experiment in which ion sheaths were absent, the plasma-beam combination should have displayed amplification rather than spontaneous oscillations.

In order to see what the observable effects of such amplification would be, it is necessary to modify the Bohm-Gross theory to apply to a thin beam passing through a plasma. The appropriate dispersion relation is derived in an Appendix and discussed in Sec. 3: we conclude that the spatial amplification was too small to have resulted in amplification of thermal noise to the point at which the beam would display

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[1] D. H. Looney and S. C. Brown, Phys. Rev. **93**, 965 (1954).

[2] D. Bohm and D. Gross, Phys. Rev. **75**, 1851 (1949); **75**, 1864 (1949).

[3] P. A. Sturrock, Phys. Rev. **112**, 1488 (1958).

large-amplitude oscillations, in agreement with observations. We also confirm that the observation of standing-wave oscillations when the plasma is modified by biasing certain electrodes so that the plasma is bounded by reflecting sheaths, and so that there exists a beam of electrons traveling in the direction opposite to that of the principal beam,[4] is in agreement with the predictions of the Bohm-Gross-type dispersion relation.

In Sec. 4, we shall discuss certain related experiments and show them to be in agreement with our interpretation of the Looney-Brown experiment.

## 2. KINEMATIC INTERPRETATION OF THE DISPERSION RELATION

The classification of instabilities of two intersecting streams of charged particles was discussed in reference 3 for the case that the streams are at zero temperature. It was found that the instability is convective if the stream velocities are nonzero and similarly directed, and nonconvective if the stream velocities are nonzero and oppositely directed. The case that one of the stream velocities is zero is singular, and the theory quoted is not applicable without modification. However, one can argue that in this case the instability would be convective. Since the mechanism for instability involves the interaction of the two streams, one would expect that a growing disturbance cannot propagate faster than the faster beam, nor slower than the slower beam, and this may be confirmed by Fourier-transform analysis. If one of the beams is stationary, it follows that any growing disturbance must propagate in the direction of the moving stream.

As has been pointed out by Allis,[5] it is dissatisfying merely to attempt to classify this singular case as convective or nonconvective, since the behavior of the real system will in fact be sensitive to other factors which are ignored in this simple model. The most obvious discrepancy between the Bohm-Gross model and the Looney-Brown experiment is that the former assumes infinite streams while the latter incorporates streams of finite cross section. This point will be discussed further in the next section, where we shall find that its effect is quantitative rather than qualitative; it affects the rate of instability or amplification, but not the kinematic classification of the instability.

The dispersion relation (1.1) ignores the effect of nonzero temperature of the electron plasma. (The temperature of the beam is expected to be small compared with that of the plasma.) The form of the dispersion relation proposed by Bohm and Gross to take account of this effect was

$$\frac{\omega_p^2}{\omega^2}\left(1+\frac{c^2k^2}{\omega^2}\right)+\frac{\omega_b^2}{(\omega-vk)^2}=1, \tag{2.1}$$

[4] E. Gordon, Massachusetts Institute of Technology Research Laboratory of Electronics Quarterly Progress Report B5, (unpublished), pp. 11–13.

[5] W. P. Allis (private communication).

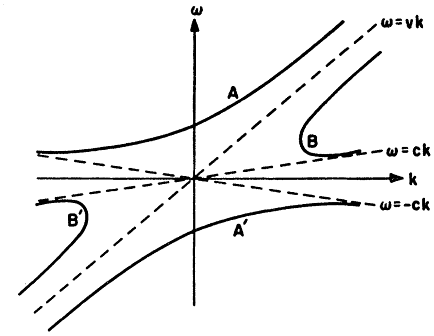


FIG. 1. Dispersion diagram for a thin beam in a thermal plasma.

where we introduce $c$ for the root-mean-square component of electron velocity in the direction considered by our one-dimensional analysis. As those authors noted, this equation is objectionable in that it is of the sixth order in $\omega$, although it is only of the fourth order in $k$. A more serious objection is seen by evaluating the characteristic velocities determined by

$$v_c=\lim_{k\to\infty}(\omega/k), \tag{2.2}$$

which represent the possible velocities of propagation of discontinuities.[6] The wave-front velocities are given by the maximum and minimum values of $v_c$ (evaluated for various modes). These are found from (2.1) to be 0, $v$ from which we infer that it is impossible for any disturbance to propagate other than in the direction of the beam although electrons of the plasma are traveling in both directions. In this respect, the dispersion relation (2.1) does not offer a satisfactory physical description of the plasma-beam system.

A more satisfactory form of the two-stream dispersion relation taking account of the plasma temperature may be obtained by replacing the Maxwell distribution of velocities of the plasma by a "spherical shell" distribution in which it is assumed that electrons of the plasma all have the same speed, $c$, but that the velocities are distributed isotropically.[7] Bohm-Gross-type analysis applied to such a model yields the following dispersion relation:

$$\frac{\omega_p^2}{\omega^2-c^2k^2}+\frac{\omega_b^2}{(\omega-vk)^2}=1. \tag{2.3}$$

This is of the fourth order in both $\omega$ and $k$; moreover, it yields $-c$, $v$ as the minimum and maximum values of $v_c$, satisfying our expectation that disturbances may propagate in both directions.

We now wish to determine whether the dispersion relation (2.3) predicts convective or nonconvective instability. To this end, we construct the dispersion diagram shown in Fig. 1. The diagram is as shown

[6] T. H. Havelock, *The Propagation of Disturbances in Dispersive Media* (Cambridge University Press, New York, 1914).

[7] R. W. Gould, Electron Tube and Microwave Laboratory Technical Report No. 4, California Institute of Technology, 1955 (unpublished).

provided that

$$\omega_b^2 c^2 < \omega_p^2 v^2, \tag{2.4}$$

which will normally be satisfied since $c \ll v$. The limbs $A$, $A'$ of Fig. 1 represent normal propagation since to any real value of $k$ corresponds a real value of $\omega$ and vice versa. The same is not true of the limbs $B$, $B'$: real wave numbers lead to real frequencies only for sufficiently large values of the wave number, and vice versa. Reference to the criteria set out in reference 3 shows that the range of values of $k$ for which $\omega$ is complex represents convective instability; the same physical property of the system may be expressed alternatively by the statement that the range of values of $\omega$ for which $k$ is complex represents amplifying waves.

As a consequence of the above conclusions, we may assert that an electron beam injected into a thermal electron plasma should lead to amplification of noise or other modulation along the length of the beam, but should not result in large-amplitude standing waves along the entire length of the beam.

## 3. MODIFIED DISPERSION RELATION FOR THIN ELECTRON BEAM

We have already drawn attention to one of the most obvious discrepancies between the model of Bohm-Gross theory and the experiment of Looney and Brown: the former deals with infinite streams whereas the latter necessarily employed streams of finite dimensions. The real wave number corresponding to the wave with largest imaginary component of $\omega$ is approximately $k_c$. Hence, if the beam radius $b$ is large compared with $k_c^{-1}$, one may expect that infinite-beam theory provides a useful approximation to the experiment; if, on the other hand, $b$ is appreciably smaller than $k_c^{-1}$, infinite-beam theory is inapplicable. In the Looney-Brown experiment, typical values of the relevant quantities are $b=0.05$ cm, $v=10^9$ cm sec$^{-1}$, $\omega_p=5\times 10^9$ sec$^{-1}$, $\omega_b=10^9$ sec$^{-1}$, so that $b$ is in fact smaller than $k_c^{-1}$ with the consequence that infinite-beam theory is inapplicable.

The dispersion relation which is appropriate to a thin electron beam passing through an infinite plasma may be obtained by slight modification of the calculation presented by Budker in his treatment of electrostatic oscillations in interacting beams of charged particles.[8] An alternative brief derivation of the required relation is given in the Appendix. The result is most conveniently expressed as

$$\frac{\omega_p^2}{\omega^2}+\frac{V^2k^2}{(\omega-vk)^2}=1, \tag{3.1}$$

where

$$V^2=\tfrac{1}{2}\omega_b^2 b^2\{\ln(1/bk)-\gamma\}. \tag{3.2}$$

Since $k$ appears in (3.2) only by way of the logarithmic term, $V$ is insensitive to wave number, so that it is more appropriate to characterize the electron beam by a "plasma velocity" than a "plasma frequency." For present purposes we ignore the nonzero temperature of the plasma.

It is easy to solve (3.1) for $k$ as a function of $\omega$. We find that, for small beam perveance (i.e., $b^2\omega_b^2 \ll v^2$), $k$ has its maximum imaginary part at a frequency just below $\omega_p$ and that the maximum value is given by

$$k_i \simeq \omega_p/2v. \tag{3.3}$$

In the Looney-Brown experiment, we find that $k_i=2.5$ cm$^{-1}$, corresponding to a gain of 20 db/cm. We should therefore expect the beam break-up, if it occurred, to be several centimeters from the point of entry into the plasma, whereas the interaction length was only 1.5 cm.

We now inquire whether the dispersion relation (3.1) is also compatible with the results of the second Looney-Brown experiment, which incorporated a pair of biased, and therefore reflecting, sheaths separated by about 1.5 cm, and which led to oscillations. The distinction between convective and nonconvective instability is now redundant, since even if the principal interaction mechanism between the primary beam and the plasma gives rise to convective instability, the fact that the region of excitation is bounded and the fact that there is a return beam[4] providing a feedback mechanism will ensure that such instability will lead to oscillations throughout this region. Hence we need merely examine the range of wave numbers for which (3.1) leads to complex values of $\omega$. We find the condition for instability to be of the form (1.2), where now

$$k_c \approx \omega_p/v, \tag{3.4}$$

since $V \ll v$. We should expect oscillations to occur if the sheath separation exceeds 0.6 cm, corresponding to half the wavelength of the longest wave allowed by (3.4). The fact that oscillations were observed, therefore, agrees with our theory.

## 4. DISCUSSION

We have seen in the last two sections that Bohm-Gross theory, when applied to the Looney-Brown experiment, leads one to expect that the beam will exhibit amplification in the case that the reflecting sheaths are not present. This is entirely in accord with the analysis of the Merrill and Webb experiment[9] published by Twiss[10] in 1951. Merrill and Webb in fact used a larger beam of lower energy so that $b$ was appreciably larger than $v/\omega_p$. Infinite-beam theory, which was used by Twiss, may be considered appropriate and leads to much higher rates of amplification (about 170 db/cm) than we found for the thin-beam

[8] G. J. Budker, *Proceedings of the CERN Symposium on High-Energy Accelerators and Pion Physics, Geneva,* 1956 (European Organization of Nuclear Research, Geneva, 1956), Vol. I, p. 68.

[9] H. J. Merrill and H. W. Webb, Phys. Rev. **55**, 1191 (1939).

[10] R. Q. Twiss, Proceedings of the Conference on Ionized Media, London, 1951 (unpublished).

model. This leads to amplification of shot-noise to full dc amplitude in less than 1 centimeter, so that Twiss was able to explain why the beam should exhibit large-amplitude oscillations and break up at about this distance from the point of entry of the beam into the plasma.

Two-stream amplification due to the interaction of an electron beam with an electron plasma has more recently been the subject of experimental investigation by Boyd, Field, and Gould.[11] Their experiments fully confirm that, in the case that the plasma is stationary, instability is convective rather than nonconvective.

If the electron plasma is not stationary, we should ascribe velocities $v_p$ and $v_b$ to the plasma and beam, respectively. If thermal velocities also are taken into account by the spherical-shell model, the dispersion relation takes the form

$$\frac{\omega_p^2}{(\omega-v_pk)^2-c^2k^2}+\frac{V^2k^2}{(\omega-v_bk)^2}=1. \tag{4.3}$$

Construction of the appropriate dispersion diagram shows that the instability is convective only if $v_p>-c$, where we assume $v_b>0$. Hence we should expect that if the electron plasma is given a drift velocity exceeding the mean thermal velocity in a direction opposite to that of the electron beam, the instability would become nonconvective so that large-amplitude standing-wave oscillations would be set up.

## 5. ACKNOWLEDGMENT

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## APPENDIX. DISPERSION RELATION FOR A THIN BEAM IN AN INFINITE PLASMA

Consider a thin electron beam, of line-number density $N$ and radius $b$ moving with velocity $v$ through an infinite plasma with Langmuir frequency $\omega_p$. Initially, we ignore the presence of the plasma and the velocity of the beam. We seek an approximate dispersion relation for the case $bk\ll 1$.

Let the beam lie along the $z$ axis of a coordinate set and let us characterize the perturbation of the beam by the displacement $\zeta(z,t)$ of an electron in the perturbed state from its position in the unperturbed state of the beam. The equation of motion is

$$m\frac{\partial^2\zeta}{\partial t^2}=-e\frac{\partial\phi}{\partial z}, \tag{A.1}$$

where the electric potential $\phi$ is given by

$$\phi(z,t)=Ne\int_{-\infty}^{\infty}d\tilde{z}G[z-\tilde{z}-\zeta(\tilde{z},t)], \tag{A.2}$$

if $G(z-\tilde{z})$ is the potential at the point $z$ due to a unit charge at the point $\tilde{z}$. Small-amplitude analysis leads to

$$\frac{\partial^2\zeta(z,t)}{\partial t^2}=\frac{Ne^2}{m}\int_{-\infty}^{\infty}d\tilde{z}\frac{\partial^2\zeta(\tilde{z},t)}{\partial\tilde{z}^2}G(z-\tilde{z}), \tag{A.3}$$

which leads, on Fourier analysis, to the dispersion relation

$$\omega^2=2\pi(Ne^2/m)k^2\tilde{G}(k), \tag{A.4}$$

where

$$\tilde{G}(k)=(1/2\pi)\int_{-\infty}^{\infty}dze^{-ikz}G(z). \tag{A.5}$$

For $|z|>b$, $G(z)$ is approximately $|z|^{-1}$. Hence for $bk\ll 1$, (A.5) may be approximated to

$$\tilde{G}(k)=\frac{1}{\pi}\int_{bk}^{\infty}\frac{dt\cos t}{t}\approx\frac{1}{\pi}\{\ln(1/bk)-\gamma\}, \tag{A.6}$$

where $\gamma$ is Euler's constant, so that the dispersion relation (A.4) becomes

$$\omega^2=(2Ne^2/m)\{\ln(1/bk)-\gamma\}k^2. \tag{A.7}$$

Since

$$N=\pi b^2n, \tag{A.8}$$

(A.7) may be written alternatively as

$$\omega^2=V^2k^2, \tag{A.9}$$

where

$$V^2=\tfrac{1}{2}\omega_b^2b^2\{\ln(1/kb)-\gamma\}. \tag{A.10}$$

The effect of the beam velocity is taken into account by replacing $\omega$ by $\omega-vk$. The effect of the plasma is taken into account by noting that it modifies the dielectric constant to

$$\epsilon=1-\omega_p^2/\omega^2, \tag{A.11}$$

so that the inverse of this quantity should be included on the right-hand side of Eq. (A.2). Hence the dispersion relation for the combined plasma and electron beam becomes

$$(\omega_p^2/\omega^2)+[V^2k^2/(\omega-vk)^2]=1. \tag{A.12}$$

[11] Boyd, Field, and Gould, Phys. Rev. **109**, 1393 (1958).