Analysis of Elastic Cross Sections and Polarization of 10-Mev Protons*

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The differential elastic cross section and polarization of 10-Mev protons scattered by argon and copper have been analyzed using a diffuse surface optical model potential with a spin-orbit term. The model parameters were varied systematically, the best fits with the experimental data being determined by a method of least squares. As in other analyses, it was found that almost equally good fits could be obtained over a range of values of the radius constant, R_0, in this case, for R_0 approximately between 1.20 and 1.30. Only the value of the real part of the central potential was markedly different for the best fits obtained for various R_0 in this range. The experimental polarization data is not precise enough to determine the spin-orbit potentials to within better than 1 or 2 Mev.

I. INTRODUCTION

THE differential elastic cross sections and polarization of 10-Mev protons scattered by argon and copper[1] have been analyzed[2] using a diffuse surface optical model potential with a spin-orbit term of the Thomas type,[3]

$$V(r)=-(V+iW)f(r)+(\hbar/\mu c)^2 \times(V_S+iW_S)(1/r)(df/dr)\boldsymbol{\sigma}\cdot\mathbf{l}+V_C(r). \quad (1)$$

The form factor for both the real and imaginary parts of the central potential is the same, namely

$$f(r)=[1+\exp(r-R)/a]^{-1}, \quad (2)$$

and the derivative of this same form factor appears in the spin-orbit term. The Coulomb potential, V_C, is that corresponding to a uniformly charged sphere of radius R.[4] Altogether there are six parameters, V and W, V_S and W_S, the rounding parameter a, and the radius constant R_0 which appears in the usual expression for the nuclear radius,

$$R=R_0A^{\frac{1}{3}}\times10^{-13}\text{ cm}. \quad (3)$$

II. METHOD FOR DETERMINING THE BEST PARAMETERS

Our method for determining the parameters which give the best agreement with experiment involves using the quantities χ_σ^2 and χ_p^2. The quantity χ_σ^2 is the sum of the squares of the weighted difference between experimental and theoretical cross sections and χ_p^2 is the corresponding quantity for the polarization. The criterion for the best fit is that the quantity $\chi^2=\chi_\sigma^2+\chi_p^2$ be a minimum. It should be pointed out that the quantities χ_σ^2, χ_p^2, and χ^2 as we have defined them are meaningful only when comparing fits corresponding to a given set of data.

It should also be pointed out that there is an element of choice with regard to the weighting factors to be used in determining χ_σ^2 and χ_p^2; the resulting parameters are not entirely insensitive to this choice. In this analysis we have always used the standard deviations which were reported with the experimental data, that is, we have written

$$\chi_\sigma^2=\sum\left[\frac{\sigma_{\text{exp}}(\theta_i)-\sigma_{\text{theo}}(\theta_i)}{\Delta\sigma_{\text{exp}}(\theta_i)}\right]^2,$$

where the sum runs over all experimental points, and similarly for χ_p^2. Other weighting schemes are, of course, possible. For example, an analysis which artifically weights the forward angles more than the backward angles would presumably yield values of the parameters which are somewhat different from those found in the present analysis.

The minimization of χ^2 was carried out in three stages. In the first stage, the parameters V, W, a, and R_0 were kept fixed and only the spin-orbit strengths V_S and W_S were varied until a minimum was obtained for χ^2. The values so obtained appear to be determined primarily by the polarization data. In the second stage, the four parameters V, W, a, and R_0 were varied, keeping V_S and W_S fixed at the values found in the first stage. The third stage consisted in fixing V, W, a, and R_0 at the values found in the second stage and making a final variation on V_S and W_S.

In each case, the minimization of χ^2 was carried out by choosing initial parameter values and setting up an

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[1] The 9.75-Mev experimental cross-section data for argon and copper is that of N. M. Hintz, Phys. Rev. **106**, 1201 (1957); the 10-Mev experimental cross-section and polarization data for argon, as well as the polarization data for copper, is that of L. Rosen, Proceedings of the International Conference on the Nuclear Optical Model, Florida State University, Tallahassee, 1959 (unpublished). We are indebted to Dr. Rosen for furnishing us his results prior to publication.

[2] The calculations were performed on the SWAC, Numerical Analysis Research, Department of Mathematics, and also on the IBM-709, Western Data Processing Center at UCLA.

[3] The Riesenfeld-Watson notation is used here. W. B. Riesenfeld and K. M. Watson, Phys. Rev. **102**, 1157 (1956).

[4] See, for example, Melkanoff, Nodvik, and Saxon, Phys. Rev. **106**, 793 (1957).

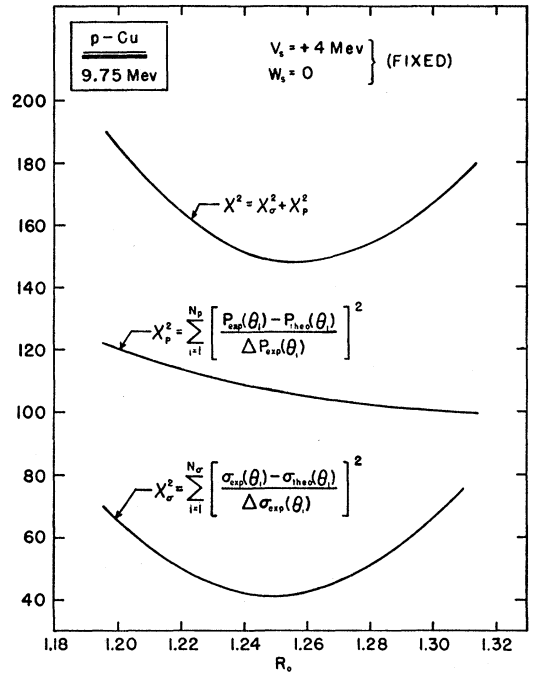

FIG. 1. Minimum values of the sums of the squares of the weighted differences between experimental and theoretical cross sections and polarizations as a function of radius for proton scattering on copper at 9.75 Mev. The spin-orbit parameters were held fixed during variations.

appropriate Taylor series expansion in the parameters involved, keeping terms up to and including the second order. The resulting approximation is of course valid only over a limited region in the parameter space. Once the expansion coefficients have been numerically determined, one can predict the minimum value of χ^2, as well as the parameters which yield this minimum, by differentiating and solving the resulting system of simultaneous linear equations. When it is obvious that the predicted values of the parameters lie outside the region of validity of the Taylor series expansion, one has to choose a new set of initial parameters and repeat the whole process.

When this procedure was carried out for the four parameter variation involved in the second stage, it was found to be quite unsatisfactory as a consequence of the strong compensating effects between the parameters V and R_0. This compensation means that there exists a long valley in the V, R_0 subspace and under this circumstance the minimization procedure becomes very sensitive to the approximations used in determining the expansion coefficients. To avoid these difficulties, it was found expedient to carry out the four parameter variation by fixing R_0 at some value and to vary V, W, and a until a minimum was obtained for χ^2, then to change R_0 to a slightly different value and again vary $V, W,$ and $a,$ and in this manner to explore the physically acceptable range of R_0.

III. COPPER

The first stage analysis for copper indicated the tentative best value of the spin-orbit potential to be $V_S+iW_S=4+0i$. The results of the second stage analysis keeping the spin-orbit potential fixed at this value are shown in Fig. 1, where the minimum values of χ_σ^2, χ_p^2, and χ^2, for fixed R_0 are plotted versus R_0. The absolute minimum for χ^2 occurs at a radius of about $R_0=1.26$; however, this value should perhaps not be taken too seriously inasmuch as the agreement with experiment using the 1.26 parameters is not markedly better than that using the 1.20 or the 1.30 parameters, as can be seen from Fig. 3. The polarization data in this case seem to favor a larger radius and the nominal value of $R_0=1.26$ represents a slight compromise between the polarization data and the cross-section data.

In Fig. 2 are shown the values of V, W, and a which yield the minimum values of χ^2 which are plotted in Fig. 1. The imaginary part of the central potential remains fairly constant at about 8 Mev and the rounding parameter a at about 0.52 fermi. As to be expected, the real part of the central potential varies considerably and in this case seems to follow a VR^n law with $n\cong 2.35$. This point is discussed somewhat further in Sec. V. Also shown in Fig. 2 is the reaction cross section, σ_R, corresponding to the best fits for various R_0. It is clear that in this instance a measurement of the reaction

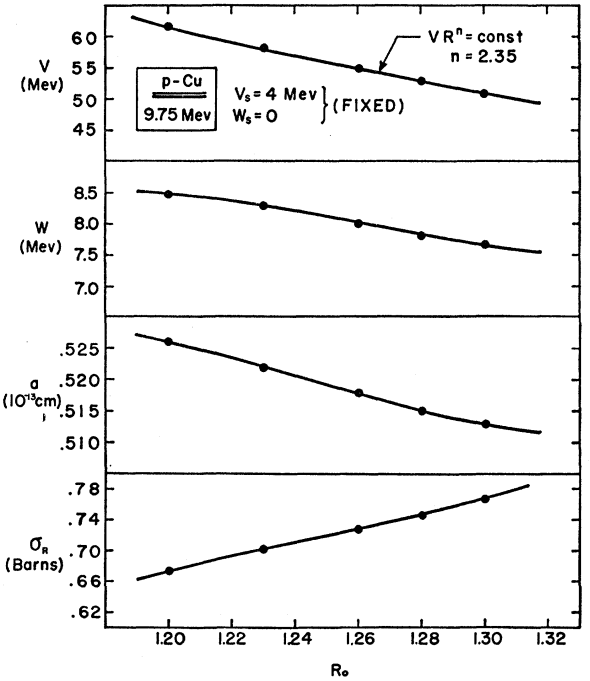

FIG. 2. Values of the real central potential V, the imaginary central potential W, the rounding parameter a, and the reaction cross section σ_R corresponding to the best fits for various radii for proton scattering on copper at 9.75 Mev. The spin-orbit parameters were held fixed during variations.

cross section to within considerably better than 5% is needed to disentangle the $V-R$ ambiguity.

The third stage variation on the spin-orbit potential yielded a value which was only slightly different from that found in the first stage, $V_S+iW_S=3.5\pm1.0i$ instead of $4.0+0i$. The resulting improvement in the agreement with experiment is scarcely noticeable, however. It is our conclusion that the polarization data is not sufficiently precise to specify the spin-orbit potentials to within better than 1 or 2 Mev; in particular, it is not possible to ascribe any significance to the small imaginary part of the spin-orbit potential.

Figure 3 shows the comparisons of the experimental and theoretical cross sections (actually the ratios to the Rutherford cross section) and polarization for copper[5] for the three cases $R_0=1.20$, 1.26, and 1.30 using the corresponding best values of V, W, V_S, W_S, and a. In general, the agreement between the experimental and theoretical cross sections and polarization in the case of copper is quite acceptable except for the forward angles. There seems to be no combination of parameters[6] which will remove the discrepancy which occurs between 20° and 50° and at the same time preserve the fit for angles greater than 50°. The final results for copper are listed in Table I. It might be noted that the values of the parameters V, W, and a, as well as the quality of the fit to the cross section data are not significantly different from that obtained by Glassgold *et al.*[7] using the potential given by (1) but without the spin-orbit term.

IV. ARGON

The first stage analysis for argon yielded a value $V_S+iW_S=8.5+2.0i$ for the spin-orbit potential, approximately twice that found for the case of copper.

Two sets of cross-section data were available, those of Hintz at 9.75 Mev[1] and those of Rosen at 10 Mev.[1]

TABLE I. Result of optical model analysis of Hintz 9.75-Mev cross-section data and Rosen 10-Mev polarization data for proton scattering on copper. Energies are in Mev, lengths in 10^{-13} cm, and cross sections in barns.

R_0	V	W	a	V_S	W_S	χ_σ^2	χ_p^2	χ^2	σ_R
1.20	61.5	8.5	0.526	3.5	1.0	79	111	190	0.671
1.23[a]	58.1	8.3	0.522	4.0	0	45	111	156	0.702
1.26	54.9	8.0	0.518	3.5	1.0	45	98	143	0.725
1.28[a]	52.9	7.8	0.515	4.0	0	51	103	154	0.746
1.30	50.9	7.7	0.513	3.5	1.0	64	90	154	0.762

[a] The third stage variation on the spin-orbit parameters was not carried out.

[5] The theoretical polarizations were computed using an incident energy of 9.75 Mev although the experimental polarizations correspond to an incident energy of 10 Mev. This was done for convenience and it was verified that the resulting error thus introduced is negligible.

[6] The use of a surface absorption instead of a volume absorption may remove some of this discrepancy; F. Bjorkland (private communication).

[7] A. E. Glassgold, W. B. Cheston, M. L. Stein, S. B. Schuldt, and G. W. Erickson, Phys. Rev. **106**, 1207 (1957).

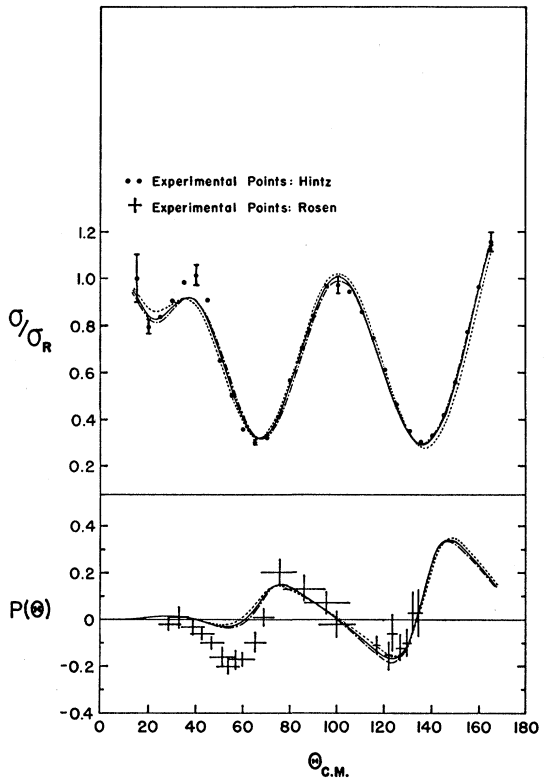

FIG. 3. Best fits obtained in the optical model analysis of the Hintz 9.75-Mev cross-section data and the Rosen 10-Mev polarization data for proton scattering on copper for the three radii $R_0=1.20$, 1.26, and 1.30.

The Rosen data was analyzed for the two radii $R_0=1.20$ and 1.26, the best values of the parameters obtained being listed in Table II. The fits at these two radii are almost equally good, with $R_0=1.20$ being slightly favored. The third stage analysis yielded $V_S+iW_S=8.5+2.0i$ for the 1.26 radius, the same as found in the first stage, and $V_S+iW_S=10.0+2.0i$ for the 1.20 radius. The comparison of experimental and theoretical cross sections and polarizations is shown in Fig. 4. The agreement is remarkably good for both radii.

The Hintz cross-section data (in conjunction with the Rosen polarization data[5]) was analyzed somewhat more completely, five different radii being investigated. The minimum values of χ_σ^2, χ_p^2, and χ^2 are shown in Fig. 5, and the parameters which yield these minimum values are shown in Fig. 6. In contrast to the situation for copper, the polarization data in this case seems to

TABLE II. Final results of optical model analysis of Rosen cross-section and polarization data for 10-Mev proton scattering on argon. Energies are in Mev, lengths in 10^{-13} cm, and cross sections in barns.

R_0	V	W	a	V_S	W_S	χ_σ^2	χ_p^2	χ^2	σ_R
1.20	61.8	8.8	0.415	10.0	2.0	35	28	63	0.595
1.26	54.8	8.0	0.424	8.5	2.0	37	36	73	0.651

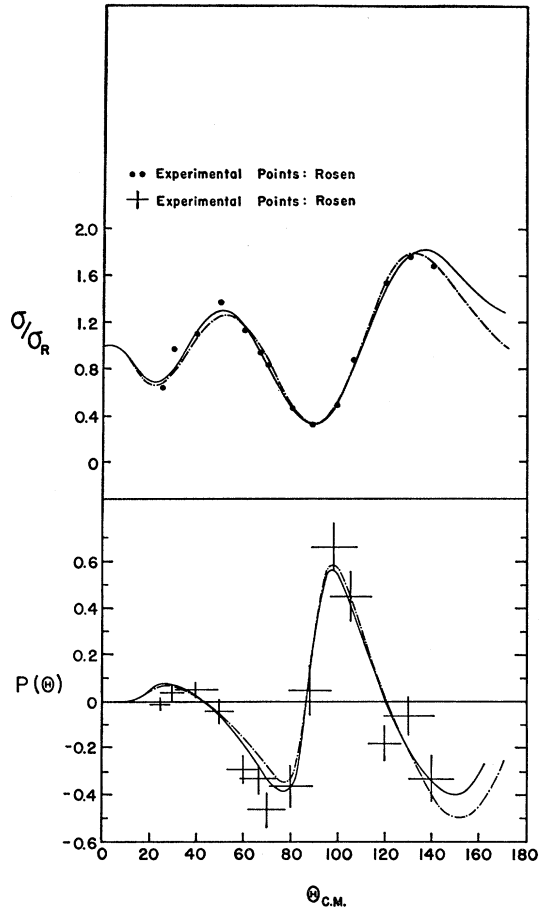

FIG. 4. Best fits obtained in the optical model analysis of the Rosen cross-section and polarization data for 10-Mev proton scattering on argon for the two radii $R_0=1.20$ and 1.26.

favor a smaller radius. The minimum value for χ^2 occurs at approximately $R_0=1.27$, but, as in the case of copper, there is really not much to distinguish one radius from another over the range $R_0=1.20$ to 1.30. The third stage analysis yielded values for the spin-orbit potentials which were insignificantly different from those found in the first stage and the final value $V_S+iW_S=8.5+2.0i$ was adopted for all radii investigated. But again, the fits obtained with potentials which deviate by 1 or 2 Mev from this value are almost equally acceptable. The final values of the parameters are given in Table III and the comparison of experimental and theoretical values for the radii $R_0=1.20$, 1.265, and 1.33 is shown in Fig. 7.

Except for the spin-orbit strengths, the best values of the parameters for argon are quite close to those for copper. Furthermore, the results obtained using the Rosen cross sections are in good agreement with those obtained using the Hintz cross sections except for the rounding parameter, a, and the reaction cross section σ_R, which are somewhat larger for the latter case. These differences appear to be due to the influence of the Hintz cross sections at the backward angles.

TABLE III. Results of optical model analysis of Hintz 9.75-Mev cross section data and Rosen 10-Mev polarization data for proton scattering on argon. Energies are in Mev, lengths in 10^{-13} cm, and cross sections in barns.

R_0	V	W	a	V_S	W_S	χ_σ^2	χ_p^2	χ^2	σ_R
1.20	60.5	8.8	0.469	8.5	2.0	321	33	354	0.656
1.23	57.1	8.2	0.479	8.5	2.0	238	35	273	0.681
1.265	53.6	7.5	0.487	8.5	2.0	174	41	214	0.704
1.30	50.3	6.9	0.496	8.5	2.0	174	49	223	0.729
1.33	47.3	6.5	0.524	8.5	2.0	187	67	254	0.775

V. THE $V-R$ AMBIGUITY

The results of this analysis indicate that the situation with regard to the well-known $V-R$ ambiguity remains practically unchanged when polarization data is included in the optical model analysis. For energies which are not too high the value of V for a given radius R is determined primarily by the positions of the maxima and minima of the cross section and polarization and these positions are affected only slightly by the other parameters.

For the two elements investigated in this analysis, it appears that the data determines only the combination VR^n, with $n\cong 2.35$. It is interesting to note that this is very close to the value of n which is predicted if one assumes that the cross sections and polarization

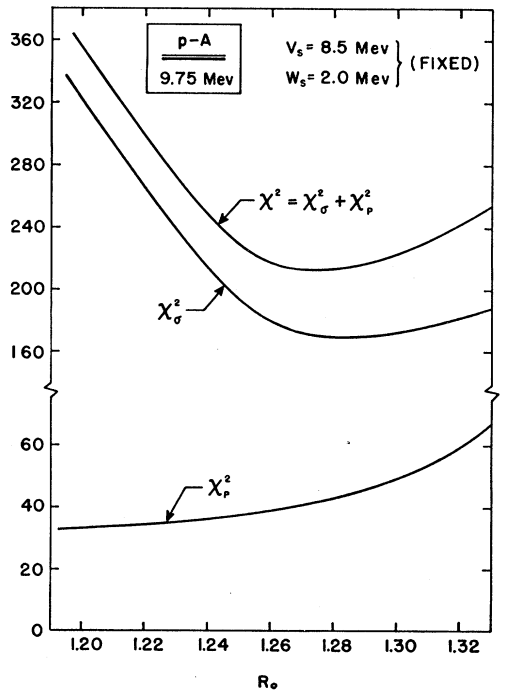

FIG. 5. Minimum values of the sums of the squares of the weighted differences between experimental and theoretical cross sections and polarization as a function of radius for proton scattering on argon at 9.75 Mev. The spin-orbit parameters were held fixed during variations.

are left relatively unchanged when V and R are varied in such a way that the effective number of waves inside the nucleus is a constant,[8] that is, in such a way that κR=constant, where $\kappa=[2M(E+V)/\hbar^2]^{\frac{1}{2}}$ is the effective interior wave number.

In fact, this provides a convenient empirical way of characterizing the information on V and R_0 obtained in this analysis. In Fig. 8 we have plotted the best value of V for a given R_0 versus $1/R_0^2$. These values are taken directly from Tables I and III. As can be seen from Fig. 8, the plotted points exhibit a remarkably linear behavior for both argon and copper, so that the relation between V and R_0 can be taken to be of the form

$$R_0[2M(\epsilon+V)/\hbar^2]^{\frac{1}{2}}=\beta, \quad (4)$$

the constants ϵ and β being determined by fitting a straight line to the points plotted in Fig. 8. Thus we find $\epsilon=9.4$ Mev, $\beta=2.21$ for copper, and $\epsilon=9.2$ Mev, $\beta=2.19$ for argon. Since these values of ϵ are only slightly different from the incident energy, $E=9.75$ Mev, our results for V and R_0 for both copper and argon are fairly precisely summarized by the relation

$$\kappa R=2.20 \text{ (copper and argon, } E=9.75 \text{ Mev)} \quad (5)$$

together with the statement that R_0 probably lies between 1.20 and 1.33.

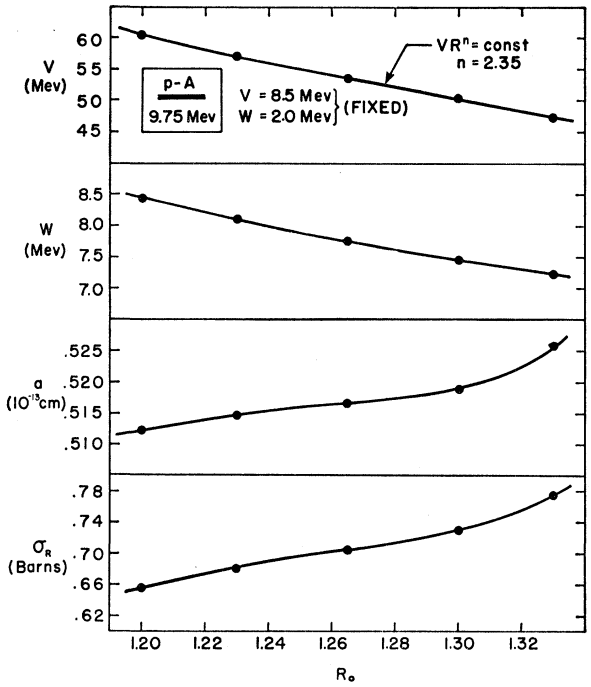

FIG. 6. Values of the real central potential V, the imaginary central potential W, the rounding parameter a, and the reaction cross section α_R corresponding to the best fits for various radii for proton scattering on argon at 9.75 Mev. The spin-orbit parameters were held fixed during variations.

[8] J. S. Nodvik, Proceedings of the International Conference on the Nuclear Optical Model, Florida State University, Tallahassee, 1959 (unpublished).

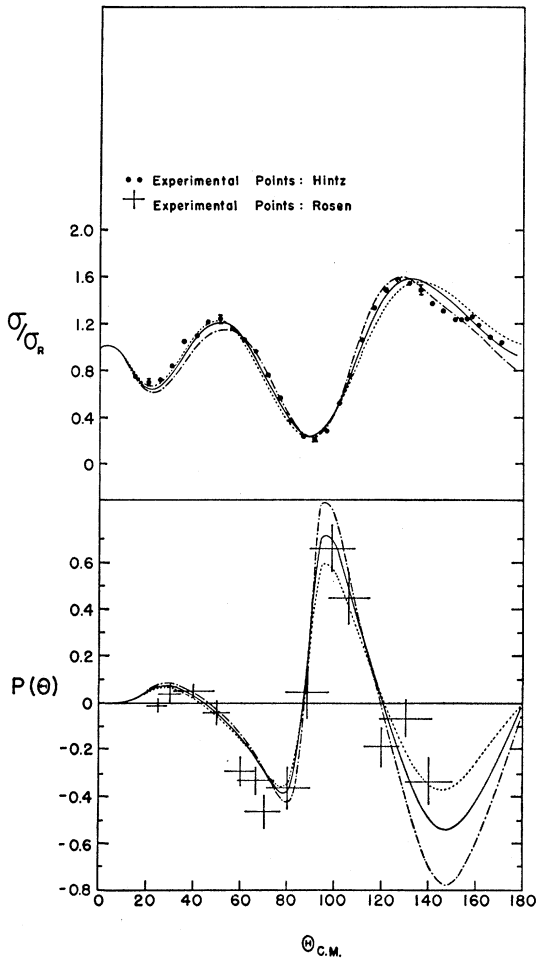

FIG. 7. Best fits obtained in the optical model analysis of the Hintz 9.75-Mev cross section data and the Rosen 10-Mev polarization data for proton scattering on argon for the three radii $R_0=1.20$, 1.265, and 1.33.

It appears that the above method of characterizing the $V-R$ ambiguity also applies quite well at 17 Mev, although for some of the heavier elements the value of the constant ϵ seems to be 2 or 3 Mev lower than the incident energy, in qualitative agreement with what might be expected from Coulomb effects. Figure 8 also shows a plot of the best value of V versus $1/R_0^2$ for the case of proton scattering on copper for an incident energy $E=17.3$ Mev. These values are taken from the analysis of Glassgold and Kellogg.[9] (This analysis does not include spin-orbit effects but as

TABLE IV. Variation of the parameter κR_0 with energy for proton scattering on copper.

E	5.25	9.75	17.3	31.5
κR_0	2.18	2.21	2.27	2.37

[9] A. E. Glassgold and P. J. Kellogg, Phys. Rev. **107**, 1372 (1957).

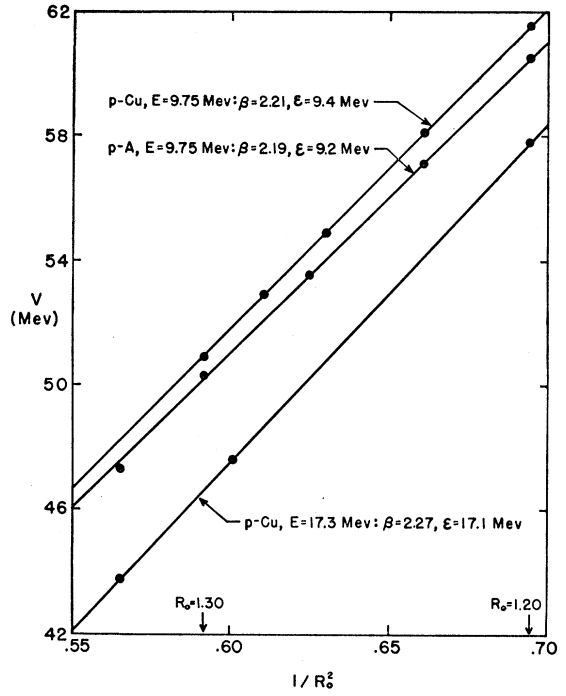

FIG. 8. Best values of real central potential V as a function of $1/R_0^2$ for proton scattering on copper and argon at 9.75 Mev and for proton scattering on copper at 17.3 Mev.

mentioned earlier, these are not expected to cause much change in V). In this case we find $\epsilon=17.1$ Mev, $\beta=2.27$, so that again the constant ϵ is quite close to the incident energy E.

Optical model analyses for proton scattering on copper have also been carried out for incident energies of 5.25 Mev,[10] and 31.5 Mev,[11] however, in each case for only one value of the radius. The results are as follows: $E=5.25$ Mev, $R_0=1.31$, $V=52.5$ Mev; $E=31.5$ Mev, $R_0=1.33$, $V=35$ Mev. Making the assumption that the situation at these energies is not very different from that at 10 and 17 Mev, i.e., that the $V-R$ ambiguity is characterized by Eq. (4) with $\epsilon=E$, enables us to map out the behavior of the parameter κR_0 with energy. This is summarized in Table IV. The dependence of κR_0 on E over the range considered is approximately linear (deviations from

[10] Melkanoff, Nodvik, and Saxon (unpublished).
[11] Melkanoff, Nodvik, Saxon, and Woods, Phys. Rev. **106**, 793 (1957).

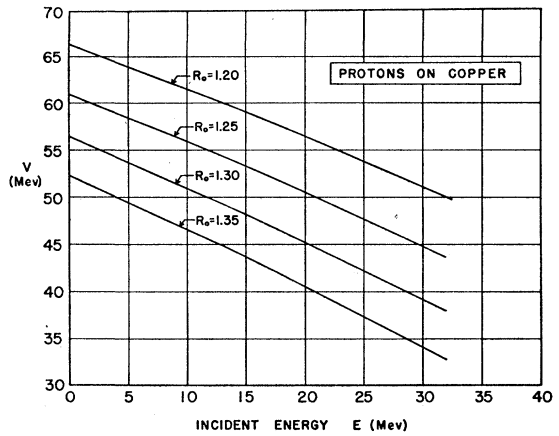

FIG. 9. Behavior of the real central potential V as a function of the incident energy for various radii for proton scattering on copper based on the assumptions made in Sec. V.

linearity seem to begin at roughly $E=30$ Mev) and is represented rather well by the relation

$$\kappa R_0=2.14+(0.0073)E \text{ (copper, } E=5 \text{ to } 30 \text{ Mev}). \quad (6)$$

It would appear that the relation (6) summarizes the only precise quantitative information regarding the values of V and R_0 for copper in this energy range which can be obtained from optical model analyses in the absence of any information on the reaction cross section. Figure 9 shows the behavior of V as a function of the incident energy E for copper for various R_0 assuming that the relation (6) holds over the range $E=0$ to 30 Mev and that R_0 is independent of the energy.

Although the present analysis is perhaps the most comprehensive done thus far, it is limited in that only two elements at a single energy have been investigated. It would be premature at this stage to draw any general conclusions even on the purely empirical level considered here. In addition it may be misleading to regard the value of the potential at the center of the nucleus as having any particular significance in these analyses. It appears that the scattering is actually more sensitive to the surface than the volume features of the potential, at least at higher energies where the mean free path becomes smaller than the nuclear radius.[8] These and other questions related to the shape of the optical model potential are currently under investigation.