the optical model fitting, and considerably smaller than the value of $r_0 \sim 1.5$ fermis which is obtained from reaction cross sections.

ACKNOWLEDGMENTS

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Single Particle Motion in a Deformed, Nonlocal Potential Well*

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The effects of the nonlocal character of the average nucleon-nucleus interaction on single particle motion in a strongly deformed field are examined by using a simple phenomenological description of the nonlocal interaction and introducing an effective mass approximation for a finite nuclear system.

The eigenvalues and eigenfunctions of the anisotropic harmonic oscillator potential are used as a starting point for a perturbative treatment of the nonlocal interaction, and the resulting energy level scheme is given as a function of the nuclear deformation. A conventional spin-orbit interaction has also been included in these calculations.

The main effects of the nonlocal interaction appear as an increase in the level spacing of the unperturbed oscillator states combined with an additional interaction energy which can be interpreted as an effective angular momentum dependence of the average potential field.

Calculations of nuclear equilibrium deformations based on the computed level schemes are presented. It is found that the preponderance of prolate nuclear shapes found empirically can be accounted for quite well by the nonlocal model and results essentially from the favoring of high angular momentum substates that are introduced by the nonlocal interaction.

1. INTRODUCTION

IT is well known that the collective rotational and vibrational excitations observed in nuclei throughout most of the periodic table find a ready interpretation in terms of the Bohr-Mottelson collective model which in general embodies a complicated interweaving of single particle motion and collective motion of the nucleus as a whole.[1,2] However, this unified description of nuclear motion simplifies considerably in the two limiting cases when the nucleus is almost spherical or has a strongly deformed equilibrium shape. Because of the close relation between the nuclear equilibrium shape and nucleon configuration, nuclei assume a near spherical equilibrium shape in the vicinity of closed shells and the collective excitations are predominantly of a vibrational character about the spherical shape. On the other hand, nuclei lying in regions between closed shells[3] have a strongly deformed equilibrium shape as evidenced by the large quadrupole moments and pure rotational spectra observed for such nuclei.[4] For large equilibrium deformations of this type one finds that the nuclear motion separates into collective and intrinsic single particle modes which are approximately independent of each other.[5] The wave function of the nucleus can then be expressed as the product

$$\Psi = \Phi_{\text{coll}} \chi, \tag{1}$$

apart from some symmetrization terms.[2] Here Φ_{coll} describes the collective rotational motion of the nucleus and zero-point vibrations about the nonspherical equilibrium shape; χ denotes the intrinsic wave function for single particle motion in the average deformed field of the nucleus, which we will denote by $V(\mathbf{r})$ and which can be regarded as static if the period of rotation of the nucleus is much longer than a single particle period in the average field.[5]

The present discussion is confined to this latter region where the nuclear motion is adequately described by Eq. (1). In particular, we examine how the velocity dependence of the average nuclear field affects the intrinsic motion of the individual nucleons.

According to Eq. (1) the main features of nuclear rotational states are independent of the details of the intrinsic motion of the nucleons themselves. However,

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[1] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 14 (1953).
[2] S. A. Moskowski, *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. XXXIX (a comprehensive recent review).
[3] Large deformations are known to occur in nuclei with $155 < A < 185$ and $A > 225$. Nuclei around $A \sim 25$ also appear to exhibit otational spectra, [G. Rakavy, Nuclear Phys. **4**, 375 (1957)].
[4] The experimental data is summarized by K. Alder, A. Bohr, T. Huus, B. Mottelson, and A. Winther, Rev. Modern Phys. **28**, 432 (1956).
[5] This is the strong-coupling approximation of A. Bohr.

a more complete application of the collective model to other nuclear properties requires explicit knowledge of the intrinsic wave functions for single particle motion in the nonspherical field of the nucleus. Calculations of this type have been carried out by several authors,[6] perhaps most extensively by Nilsson,[7] using simple spheroidal wells to approximate the deformed nuclear field. The predictions of ground-state spins and other nuclear properties based on simple potentials of this type are in good agreement with observation.[8] However, the recent self-consistent field methods of Brueckner and collaborators[9] which take into account the correlations in the nuclear wave functions, show that the average nuclear potential is of a nonlocal nature. Thus according to the Brueckner theory, the average nucleon-nucleus interaction is represented by a nonlocal potential $(\mathbf{r}|V|\mathbf{r}')$ in coordinate space, and the wave equation describing nucleon motion in the nucleus is modified to

$$(\hbar^2/2m_0)\nabla^2\psi+E\psi=\int(\mathbf{r}|V|\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}', \qquad (2)$$

for a particle of mass m_0 and energy E.

Accordingly we assume (2) to be the basic wave equation describing the intrinsic single particle motion in the deformed nucleus. In principle $(\mathbf{r}|V|\mathbf{r}')$ should now be the self-consistently determined nonlocal potential for the nucleus under consideration. However, as we will only be interested in the qualitative effects of including nonlocality in the nuclear potential, it is reasonable at this point to adopt a simple phenomenological approach used elsewhere[10] and assume a physically reasonable form

$$(\mathbf{r}|V|\mathbf{r}')=V[(\mathbf{r}+\mathbf{r}')/2]\delta_a(\mathbf{r}-\mathbf{r}'), \qquad (3)$$

for the nonlocal potential appearing in Eq. (2). As before, $\delta_a(\mathbf{r}-\mathbf{r}')$ represents some normalized approximation to the δ-function and $V(\mathbf{r})$ is an ordinary potential. The specific form (3) is primarily based on the observations (i) that the nonlocal interaction must be Hermitian, and (ii) that the approximate validity of calculations employing local potentials indicates that nonlocal effects should not be too large. The parameter a is thus a measure of the nonlocality in the potential, $a=0$ corresponding to the local case. By expanding the interaction term appearing in the wave equation (2) about $\mathbf{r}$ when the nonlocal potential is given by (3), one finds that the single particle wave function χ satisfies the modified wave equation[10]

$$\frac{1}{4}\left[\frac{1}{2m}\mathbf{p}^2+2\mathbf{p}\frac{1}{2m}\mathbf{p}+\mathbf{p}^2\frac{1}{2m}\right]\chi+V(\mathbf{r})\chi=E\chi, \qquad (4)$$

where

$$m=m_0[1-(a^2m_0/2\hbar^2)V(\mathbf{r})]^{-1}$$

is the effective nucleon mass appearing in a fully symmetrized kinetic energy operator. It is important to note that the form of Eq. (4) is altogether independent of the particular function used to represent δ_a, and depends only on the assumed separability of the nonlocal potential into a "center of mass" and relative coordinate dependence as used in (3).

The potential $V(\mathbf{r})$ appearing in (4) is now a local potential, which for a real nucleus should be of a Woods-Saxon shape,[11] having a diffuse edge. In our case $V(\mathbf{r})$ must also be nonspherical. Thus to determine a realistic set of χ's one would have to find the self-consistent solutions of Eq. (2) for a nonspherical nucleus, a procedure which would represent an extremely involved problem in general. However, if the nonlocality is described in a phenomenological way by assuming that Eq. (4) is valid, the nonlocal problem is only slightly more difficult than the corresponding local case, provided $V(\mathbf{r})$ has a simple form. Obviously the self-consistency supposed in (2) is entirely lost this way.

The introduction of a diffuse edge for the deformed field is, however, equally difficult to treat, whether we include nonlocal effects or not. We will neglect details of this type here arguing that edge effects might only be important for loosely bound particle states, leaving the deeper states relatively unchanged.

Hence we consider single particle motion described by the wave equation (4) where $V(\mathbf{r})$ is a spheroidal oscillator potential to which we add a spin-orbit term of the usual type to produce the correct single particle level scheme in the spherical limit. A simple perturbation treatment of the nonlocal problem defined by (4) is then possible if one expands the χ's in terms of anisotropic oscillator wave functions. A brief outline of the perturbation procedure is presented in the next section, together with the resulting level scheme. We discuss in Sec. 3 the main effects of the nonlocal interaction in the deformed field as embodied in the modified wave equation (4), and the close connection this type of nonlocal interaction has with the Nilsson potential.[7] Calculations of nuclear equilibrium deformations based on the energy level scheme presented here will be found in Sec. 4.

2. NONLOCAL OSCILLATOR POTENTIAL

In this section we consider the eigenvalues and eigenfunctions of the modified wave equation (4) when $V(\mathbf{r})$ is given by the anisotropic oscillator potential of

[6] A list of authors who have studied nucleon motion in a nonspherical potential is given by K. Gottfried, [Phys. Rev. **103**, 1017 (1956)].

[7] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 16 (1955).

[8] B. R. Mottelson and S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Skrifter **1**, No. 8 (1959).

[9] H. A. Bethe, Phys. Rev. **103**, 1353 (1956) (this paper contains references to earlier work).

[10] W. E. Frahn and R. H. Lemmer, Nuovo cimento **5**, 1564 (1957).

[11] R. D. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954).

depth V_0. Considering only deformations with axial symmetry, we have

$$V(\mathbf{r})=-V_0+W(\mathbf{r});\quad W(\mathbf{r})=\tfrac{1}{2}m_0\omega_0^2 \times[(x^2+y^2)\Lambda+z^2\Lambda^{-2}]. \quad (5)$$

As usual, ω_0 is the classical frequency and $\mathbf{r}=(x,y,z)$ denotes the coordinates of the particle relative to a set of axes fixed in the nucleus. According to (5) the nuclear volume is independent of the deformation and the equipotentials are spheroids of revolution about the nuclear symmetry axis which is taken as the z axis. Λ is the deformation parameter, $\Lambda>1$ corresponding to a prolate and $\Lambda<1$ to an oblate deformation.

Substituting for $V(\mathbf{r})$ from (5) in the wave equation (4) one can write

$$(H_0+H')\chi=E\chi, \quad (6a)$$

where

$$H_0=-(\hbar^2/2m^*)\nabla^2+V(\mathbf{r}), \quad (6b)$$

and

$$H'=(a^2/16)[W\nabla^2+2\nabla W\nabla+\nabla^2 W], \quad (6c)$$

m^* is the reduced nucleon mass at the center of the nucleus and is given by

$$m^*=m_0[1+(a^2m_0/2\hbar^2)V_0]^{-1}. \quad (6d)$$

The Hamiltonian H_0 is identical with the corresponding local Hamiltonian except that the free nucleon mass m_0 is replaced by the effective mass m^*. The operator H' represents the nonlocal effects coming from the spatial dependence of the nuclear potential and will be treated as a perturbation on the operator H_0. This is consistent with the assumptions made in the derivation of wave equation (4). In addition we also include a spin-orbit interaction $C(\mathbf{l}\cdot\mathbf{s})$ of the type used by Nilsson, so that the total perturbation on H_0 becomes

$$U=H'+C(\mathbf{l}\cdot\mathbf{s}). \quad (7)$$

It might be pointed out that since $(\mathbf{l}\cdot\mathbf{s})$ has simple commutation relations with the momentum operators appearing in the symmetrized kinetic energy operator in (4), inclusion of this spin-orbit interaction as part of the local potential in (5) instead of adding it on directly as we have done in (7), only changes the coefficient C, which is fixed empirically in any event. The final result would thus be completely equivalent to Eq. (7).

As pointed out by Nilsson,[7] the basic set of wave functions belonging to H_0 which are most convenient to use in a perturbation calculation depends upon the magnitude of the deformation of the potential field. For small deformations the particles' orbital angular momentum $\mathbf{l}$ is still an approximate constant of motion so that the isotropic harmonic oscillator wave functions form a useful basic set in which to expand the single particle wave functions χ. On the other hand for a strongly deformed field, there is an approximate separation of the nucleon motion into oscillation modes along the symmetry axis and in a plane perpendicular to it, and the wave functions of the anisotropic oscillator are more convenient to use as a basic set. Recently, the Nilsson potential has been treated in the latter representation by Rassey.[12] It also provides the most appropriate representation to use for the present problem.

We consider then single particle motion in the strongly deformed spheroidal force field given by (5). The total orbital angular momentum $\mathbf{l}^2$ now no longer commutes with H_0 and is not conserved. The only constants of motion are the components $\mathbf{l}_z$ and $\mathbf{s}_z$ of the orbital and spin angular momentum along the nuclear symmetry axis with quantum numbers m and $m_s=\pm\frac{1}{2}$, respectively. These constants of motion are not altered when the nonlocality is included since the operator H' defined in (6c) always has the same symmetry as the unperturbed Hamiltonian H_0. When the spin-orbit interaction is included, however, the only strict constants of motion are the parity w and the total projection $\mathbf{j}_z=\mathbf{l}_z+\mathbf{s}_z$ of the particles' angular momentum along the nuclear axis with quantum number $\Omega=m+m_s$. The remaining quantum numbers which characterize the single particle motion are n and n_3, denoting the number of oscillation quanta along and in a plane perpendicular to the nuclear symmetry axis.[7]

To put our results in a dimensionless form, we introduce the variables $\rho=(m^*\omega^*/\hbar)^{\frac{1}{2}}\Lambda^{\frac{1}{2}}r$, $\eta=(m^*\omega^*/\hbar)^{\frac{1}{2}}\Lambda^{-\frac{1}{2}}z$ and ϕ, where (r,z,ϕ) are the cylindrical coordinates of the particle, ϕ being the angle of rotation about the nuclear symmetry axis. The anisotropic oscillator wave equation with the modified Hamiltonian H_0 is separable in these variables and the eigenvalues are easily shown to be[13]

$$E_0=[(n+1)\Lambda^{\frac{1}{2}}+(n_3+\tfrac{1}{2})\Lambda^{-1}]\hbar\omega^*, \quad (8)$$

measured from the bottom of the well. Here $\hbar\omega^*=(m/m^*)^{\frac{1}{2}}\hbar\omega_0$ gives the increased level spacing due to the nonlocal interaction.[2,14]

The corresponding single particle wave functions, including spin, are

$$\psi(nn_3mm_s)=\rho^{|m|}R_{nm}(\rho)H_{n_3}(\eta)\exp[-\tfrac{1}{2}(\rho^2+\eta^2)] \times\exp(im\phi)\times(\alpha \text{ or } \beta), \quad (9)$$

where R_{nm} and H_{n_3} are Laguerre and Hermite polynomials.[15] For a given n the quantum number m can take on the values $-n$, $-(n-2)$, $\cdots$, $(n-2)$, n. As usual, α and β denote the spin functions belonging to $m_s=\frac{1}{2}$ and $-\frac{1}{2}$, respectively.

In order to investigate the effects of the nonlocality and spin-orbit perturbation terms appearing in U, we must diagonalize the total Hamiltonian $H=H_0+U$

[12] A. J. Rassey, Phys. Rev. **109**, 949 (1958).

[13] For details see L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1935).

[14] W. E. Frahn and R. H. Lemmer, Nuovo cimento **6**, 1221 (1957).

[15] See P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I.

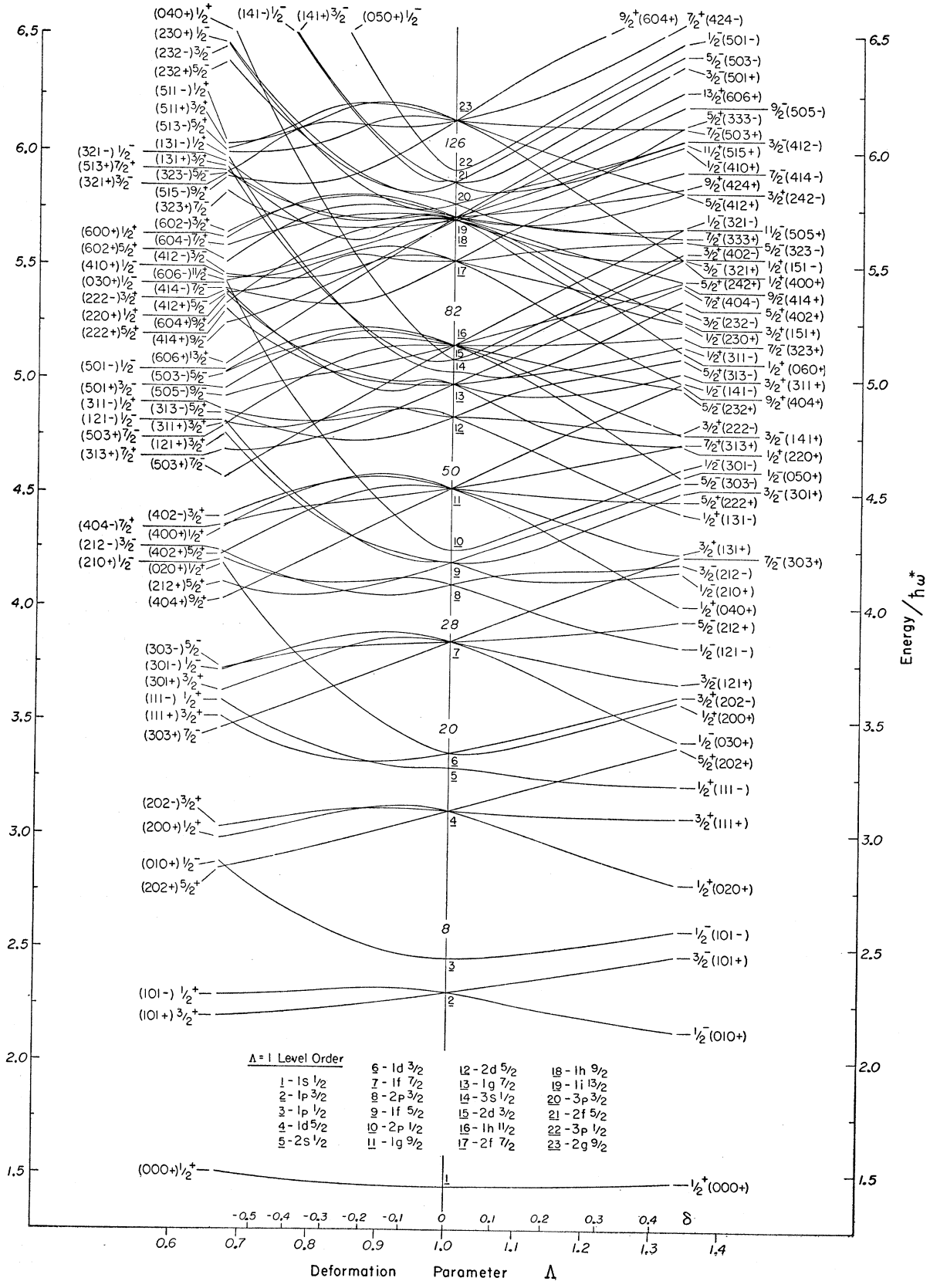
Energy / ħω*
Deformation Parameter Λ
δ
Λ = l Level Order
1 - 1s 1/2
2 - 1p 3/2
3 - 1p 1/2
4 - 1d 5/2
5 - 2s 1/2
6 - 1d 3/2
7 - 1f 7/2
8 - 2p 3/2
9 - 1f 5/2
10 - 2p 1/2
11 - 1g 9/2
12 - 2d 5/2
13 - 1g 7/2
14 - 3s 1/2
15 - 2d 3/2
16 - 1h 11/2
17 - 2f 7/2
18 - 1h 9/2
19 - 1i 13/2
20 - 3p 3/2
21 - 2f 5/2
22 - 3p 1/2
23 - 2g 9/2

FIG. 1.

in the (nn_3mm_s) representation formed by the wave functions (9). Since the spin and parity are still strict constants of motion when all perturbations considered here are included, we need only consider matrix elements of H connecting states of the same spin and parity. In addition we confine ourselves to a given major oscillator shell defined by the quantum number $N=n+n_3$, neglecting thereby the small intershell couplings.

The matrix elements of H_0+U are simple to obtain in the representation (9). H_0 is diagonal and is equal to E_0 given in Eq. (8). The nonlocal operator H' is diagonal in m and m_s but not in n or n_3. The nonvanishing elements turn out to be (see Appendix)

$$H'=\kappa^2 K,$$

$$\langle nn_3|K|nn_3\rangle = -\tfrac{1}{2}\{[n(n+2)+2]\Lambda+4(n+1)(n_3+\tfrac{1}{2})\Lambda^{-\frac{1}{2}} +[n_3(n_3+1)+\tfrac{1}{2}]\Lambda^{-2}\}-\tfrac{1}{2}\Lambda m^2, \quad (10)$$

$$\langle n-2,n_3+2|K|nn_3\rangle = -\tfrac{1}{2}\Lambda^{-\frac{1}{2}}(n^2-m^2)^{\frac{1}{2}}(n_3+1)^{\frac{1}{2}}(n_3+2)^{\frac{1}{2}},$$

$$\kappa^2=\tfrac{1}{8}m_0\omega_0^2a^2.$$

The matrix elements of $(\mathbf{l}\cdot\mathbf{s})$ in this representation are given by Rassey.[12]

The parameter κ^2 in (10) is a measure of the strength of the nonlocal interaction and can be expressed[14] in terms of the mass reduction at the center of the nucleus through Eq. (6d). Introducing the dimensionless parameters μ and γ through $\kappa^2=\mu\hbar\omega^*$ and $C=\gamma\hbar\omega^*$, we finally get

$$H=[(n+1)\Lambda^{\frac{1}{3}}+(n_3+\tfrac{1}{2})\Lambda^{-1}+\mu K+\gamma(\mathbf{l}\cdot\mathbf{s})]\hbar\omega^*, \quad (11)$$

for the total Hamiltonian. The choice of the parameters μ and γ is discussed below.

As pointed out previously, we need only consider matrix elements of (11) that connect states lying in the same oscillator shell and which have the same spin and parity. The energy eigenvalues of the perturbed system are then obtained by an exact diagonalization of the dimensionless matrices $(H/\hbar\omega^*)$ for each value of the deformation. We call these eigenvalues $E(k\Omega w)$ and denote the associated eigenvectors by $C_{k,\Omega}$. The extra quantum number k distinguishes between the different roots of the matrix, i.e., between different levels having the same $|\Omega|$ and w. Due to axial symmetry each level is still degenerate in $\pm\Omega$. The eigenvectors $C_{k,\Omega}$ also provide the expansion coefficients for the intrinsic wave functions χ in terms of the basic set given in Eq. (9). One can write

$$\chi_{k,\Omega}=\sum C_{k,\Omega}(nn_3mm_s)\psi(nn_3mm_s), \quad (12)$$

where the summation goes over all unperturbed states belonging to the same N, Ω, and w.

We have calculated the dimensionless eigenvalues and eigenvectors of (11) by machine for all oscillator shells up to $N=6$ for 13 values of Λ ranging from 0.6 to 1.4. The resulting level scheme is plotted in Fig. 1 as a function of the deformation; an additional scale shows the more usual deformation parameter δ defined[12] by $\Lambda=\exp(\frac{2}{3}\delta)$. The quantum number k referred to above labels the spherical level sequence as indicated. The set (k,Ω,w) then identifies a deformed level completely. In Fig. 1 we have labelled each level by its spin and parity only, followed by the appropriate asymptotic quantum numbers in brackets (see Sec. 3).

In these computations we have taken $\mu=0.03$, and $\gamma=-0.10$ for the nonlocality and spin-orbit parameters. This value of μ corresponds to a mass reduction $m^*=0.5m_0$ in a medium weight nucleus,[14] and the spin-orbit strength γ is identical with that used previously.[7] A complete table of the $C_{k,\Omega}$ is not given here.[16] However, their general behavior as a function of the deformation is indicated in the following section.

3. CONNECTION WITH THE NILSSON POTENTIAL

The main effects of the nonlocal interaction considered in the previous section divide up into modifications introduced by the reduced mass appearing in the unperturbed Hamiltonian H_0, and the perturbative effects of the H' interaction given in (6c) which depends on the spatial variation of the nuclear potential. As seen from Eq. (8) the reduced mass in H_0 leads to the familiar result[2,14] that the level spacing of a nonlocal oscillator must increase due to the increased effective kinetic energy. By comparison, the H' perturbation has the smaller but nevertheless more important effect of splitting up the unperturbed levels in each shell in a characteristic way, and hence determines the main features of the nonlocal level scheme.

Apart from the change in energy scale due to the increased oscillator spacing, the nonlocal level scheme shown in Fig. 1 with spin-orbit coupling included turns out to be very similar to that given by Nilsson.[7] In particular, the spherical level order in our calculation is identical with the ordering arrived at in Nilsson's calculations with the same spin-orbit strength, except that our upper levels are more strongly bound. The reason for this equivalence can be seen directly by examining the structure of the H' matrix given in (10) for the limiting cases of large and small deformations.

[16] A table of coefficients is available on request from the Physics Department, Florida State University, Tallahassee, Florida.

FIG. 1. Energy levels in a nonlocal anisotropic oscillator potential with spin-orbit coupling for nonlocality and spin-orbit constants $\kappa^2=0.03\hbar\omega^*$ and $C=-0.1\hbar\omega^*$. The level spacing parameter can be estimated as $\hbar\omega^*=82A^{-\frac{1}{3}}$ Mev. [V. F. Weisskopf, Nuclear Phys. 3, 423 (1957).] The horizontal scale shows the deformation parameter Λ with an additional scale for the more usual parameter δ. Each level is labelled by its spin and parity quantum numbers $|\Omega|$ and w, followed in brackets by the asymptotic quantum numbers nn_3mm_s which are appropriate for large deformations. The two possible values $\pm\frac{1}{2}$ for m_s are indicated by sign only. In addition, the quantum number k referred to in the text is given along the vertical axis at $\Lambda=1$ for each state and indicates the spherical level from which a deformed level originates, as shown in the inset. Neutron and proton closed shells are also indicated.

When Λ differs considerably from unity only the diagonal elements of H' are important, and as written consist of a sum of two terms which have rather different effects on the unperturbed energy levels. The first term has exactly the same degeneracy as the unperturbed energy E_0 and thus only depresses each oscillator shell as a whole, becoming more important in the higher oscillator shells. On the other hand the splitting of each unperturbed state is caused by the term $-\frac{1}{2}\kappa^2\Lambda m^2$ which separates substates of different angular momentum m. This term is formally equivalent to the interaction $-\frac{1}{2}\kappa^2\Lambda \mathbf{l}_z^2$ in the total Hamiltonian and strongly favors orbits of high angular momentum, especially for prolate deformations. The resulting stronger binding of high angular momentum states has an important effect on total energies and calculated equilibrium deformations (see Sec. 4).

For small deformations the nondiagonal elements of H' also become important and must be taken into account by diagonalizing the entire interaction (10). In this case it is easier to work directly in spherical coordinates with isotropic oscillator wave functions as a basic set. Then H' is diagonal and again separates into terms only affecting the energy of a shell as a whole, and a term which is equivalent to $-\frac{1}{2}\kappa^2\mathbf{l}^2$ in the interaction.[17] Thus we see that the interaction $\Lambda \mathbf{l}_z^2$ appearing for the strongly deformed field goes over into $\mathbf{l}^2$ as $\Lambda \rightarrow 1$, and has the same effect as Nilsson's $D\mathbf{l}^2$ interaction. If we further identify D with our value for $-\frac{1}{2}\kappa^2$, the two interactions have the same order of magnitude.[17]

We wish to point out that the appearance of the angular momentum operators $\mathbf{l}^2$ or $\mathbf{l}_z^2$ in the nonlocal interaction H' only depends on the rotation symmetry that $V(\mathbf{r})$ has and not on the spatial dependence of this potential. Hence we conclude that these operators are characteristic of the type of velocity dependent interaction described by the wave equation (4).

Inspection of the table of C's shows that one coefficient in the expansion (12) rapidly becomes much larger than the rest as the deformation increases. This result has also been obtained for the Nilsson potential[12] and indicates that any wave function $\chi_{k,\Omega}$ can be well approximated by the most dominant state occurring in its expansion in terms of the wave functions (9) when the deformation is large.

The rapid dominance of a particular unperturbed state in the expansion of χ is readily understood in terms of the degeneracies occurring in the unperturbed energy E_0 and the selection rules for the matrix elements of H' and $C(\mathbf{l}\cdot\mathbf{s})$ in the representation (9). For from Eq. (8) it follows that in the small deformation limit E_0 is degenerate in n and n_3 and only depends on the total quantum number N; the zero order wave function belonging to this state must necessarily be a linear combination of the type (12) when all perturbations are included. However, as the deformation increases this degeneracy is lifted. For large deformations such originally degenerate states become well separated and a perturbative treatment of the nonlocal and spin-orbit interactions will involve matrix elements which belong to the same n and n_3 and which can at the most be nondiagonal in m or m_s. The selection rules for H' and $C(\mathbf{l}\cdot\mathbf{s})$ show that all nondiagonal elements of this type vanish, and accordingly the interaction U is already diagonal if we neglect all matrix elements between nondegenerate states. No diagonalization is necessary and the eigenvalues of H are just the diagonal values given in (11) which can now be classified by the asymptotic quantum numbers[18] (nn_3mm_s) as indicated in Fig. 1. The wave functions (12) obviously reduce to a single term $\chi=\psi(nn_3mm_s)$. These are usually called the asymptotic solutions[7] and are expected to hold as long as the spacing of the unperturbed levels is much larger than the off-diagonal elements of the interaction U which connect them.[19]

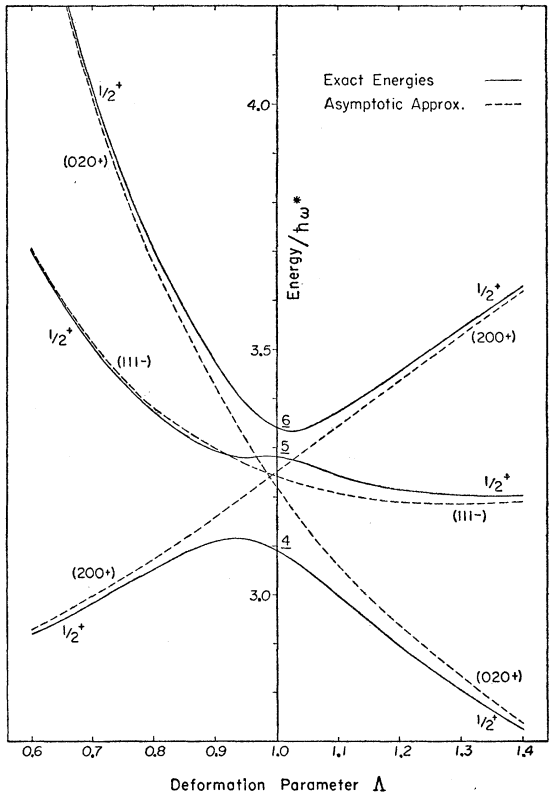

FIG. 2. Comparison of the exact energy values for the $|\Omega|=\frac{1}{2}$ states in the $N=2$ shell of the nonlocal oscillator potential with the approximate solutions given by the diagonal elements of Eq. (11) alone. The asymptotic quantum numbers for each state are given in brackets, and the nonlocality and spin-orbit constants are the same as for Fig. 1.

[17] W. E. Frahn and R. H. Lemmer, Nuovo cimento **6**, 664 (1957).

[18] G. Alaga, Phys. Rev. **100**, 432 (1955).

[19] See for instance E. M. Corson, *Perturbation Methods in the Quantum Mechanics of n-Electron Systems* (Blackie and Son, Limited, London, 1953).

The exact and asymptotic solutions of the eigenvalue problem (11) are compared in Figs. 2 and 3 for the $(4,\frac{1}{2},+)$, $(5,\frac{1}{2},+)$ and $(6,\frac{1}{2},+)$ deformed states belonging to the $N=2$ shell. The asymptotic quantum numbers for each state are indicated in brackets.

The broken curves in Fig. 2 show the diagonal values of H, and the solid curves the exact eigenvalues calculated by diagonalization of the complete energy matrix for $N=2$ and $|\Omega|=\frac{1}{2}$. One observes that the asymptotic approximation is rather reliable from $|\Lambda-1|\gtrsim 0.2$, but can be very misleading at smaller deformations when the nondiagonal elements become increasingly important. We also point out that the $(4,\frac{1}{2},+)$ and $(5,\frac{1}{2},+)$ levels must exchange their approximating asymptotic states as shown in Fig. 2 since the unperturbed levels given by E_0 invert as Λ passes through unity, and the asymptotic quantum numbers of a deformed state must change accordingly (see Fig. 1). The expansion coefficients shown in Fig. 3 likewise reflect the asymptotic properties of the wave functions χ belonging to the deformed levels in Fig. 2, and provide a direct indication of the relative importance of the nondiagonal elements of H which the asymptotic approximation neglects. One sees that a particular asymptotic wave function always predominates at deformations where its eigenvalue is a good approximation to the exact one. We illustrate with the $(5,\frac{1}{2},+)$ level. For this state one finds

$$\chi_{5,\frac{1}{2}}=0.58\psi(020+)+0.82\psi(200+), \quad (13)$$

at $\Lambda=1$, changing to

$$\chi_{5,\frac{1}{2}}=0.35\psi(020+)+0.91\psi(111-)+0.23\psi(200+), \quad (14)$$

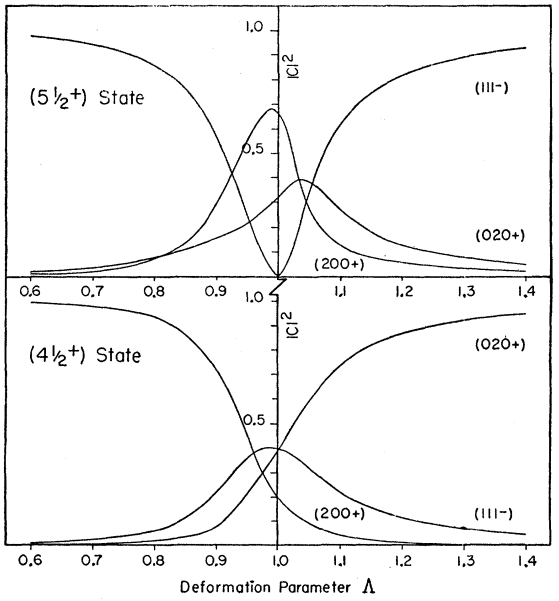

FIG. 3. Squares of the normalized expansion coefficients for the wave functions of the $(4,\frac{1}{2},+)$ and $(5,\frac{1}{2},+)$ deformed states in the $N=2$ oscillator shell. The bracketed numbers indicate the quantum numbers of the wave function in the basic set (9) to which each coefficient belongs.

when $\Lambda=1.2$. The predominance of $\psi(111-)$ at this deformation is obvious. We note that the ϕ-dependence of $\psi(111-)$ does not allow this wave function to appear in (13) which is just an expression for the $2s$-isotropic oscillator wave function in cylindrical coordinates.

The connections between the exact and asymptotic solutions illustrated for the $N=2$ shell are generally true for all the deformed levels given in Fig. 1. In particular, all states turn out to be predominantly asymptotic in character for $|\Lambda-1|\gtrsim 0.2$, or $|\delta|\gtrsim 0.27$, i.e., for deformations which are of interest in strongly deformed nuclei. Thus one expects the simple asymptotic approximation given by the diagonal elements in (11) to be a useful representation for the energy levels of such nuclei.

4. NONLOCAL INDEPENDENT PARTICLE MODEL

We next consider nonlocal effects on the predicted equilibrium shapes of nuclei within the framework of the independent particle model. The nuclear equilibrium shape is dependent on the nucleon configuration and can be calculated if we know the variation with deformation of the total energy as the deformed levels are filled in order of increasing energy. Different orbits become available at different deformations and the total energy varies accordingly. The minimum value of the total energy for a given number of particles defines the equilibrium deformation.

Calculations of this type[20] based on a simple oscillator model of the nucleus do lead to equilibrium deformations of the correct order of magnitude, but predict a wrong behavior as a function of mass number and do not produce the experimentally observed preponderance of prolate shapes. We proceed with similar calculations based on the nonlocal model.

First we consider a system consisting of an equal number of neutrons and protons moving independently in the nonlocal field described by the Hamiltonian in Eq. (11) but without spin-orbit interaction, and we also ignore any difference between neutron and proton states due to the Coulomb interaction. While a simple model of this type will not reproduce closed shells at the correct magic numbers, it does allow a simple and unambiguous identification of the changes introduced by the nonlocality in the assumed average potential felt by each nucleon. One notes, however, that the inclusion of the spin-orbit force is essential before any comparison with empirical data can be made.

Assuming that the average field originates from two-body forces only, the total energy of the nucleons in the nonlocal potential is

$$E_T(\Lambda)=\sum_i\langle H_i\rangle - \tfrac{1}{2}\sum_i\{\langle W_i\rangle+\langle H_i'\rangle+[(m_0/m^*)-1]\langle T_i\rangle\}, \quad (15)$$

where $\langle T_i\rangle$ is the average kinetic energy of the ith

[20] K. Gottfried, Phys. Rev. **103**, 1017 (1956).

TABLE I. Values of the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ that minimize the total energy when the $N=4$ shell of the nonlocal oscillator (without spin) is consecutively filled up with two particles per available state, and the calculated equilibrium deformations Λ_∞ from Eq. (18).

No. of particles	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	Λ_∞
40	100	100	832	832	208	1.00
42	110	102	1000	872	210	0.94
44	120	104	1168	912	212	0.88
46	110	146	888	1128	390	1.21
48	116	156	944	1248	416	1.22
50	122	166	1000	1368	442	1.23
52	128	176	1040	1488	468	1.26
54	136	182	1144	1584	478	1.22
56	144	188	1248	1680	488	1.20
58	182	134	1864	1416	258	0.82
60	188	144	1920	1536	284	0.84
62	194	154	1976	1656	310	0.85
64	200	164	2016	1776	336	0.88
66	204	178	2040	1888	386	0.91
68	200	208	1968	2032	516	1.03
70	210	210	2072	2072	518	1.00

nucleon, and H is the Hamiltonian in (11) with $\gamma=0$. We have neglected any contributions from Coulomb or surface energies in the expression (15). No serious error is committed by this neglect as long as one is only interested in the variation of the total energy with deformation.[20] Using the asymptotic wave functions to evaluate the expectation values in (15) (this amounts to making the asymptotic approximation for the single particle energies) one finds

$$E_T(\Lambda)=[\tfrac{3}{4}(\mathcal{A}\Lambda^{\frac{1}{3}}+\tfrac{1}{2}\mathcal{B}\Lambda^{-1}) -\tfrac{1}{8}\mu(\mathcal{C}_1\Lambda+\mathcal{C}_2\Lambda^{-\frac{1}{3}}+\mathcal{C}_3\Lambda^{-2})]\hbar\omega^*, \quad (16)$$

where

$$\mathcal{A}=\sum(n+1), \quad \mathcal{B}=\sum(2n_3+1),$$
$$\mathcal{C}_1=2\sum[2+n(n+1)+m^2], \quad \mathcal{C}_2=4\sum(n+1)(2n_3+1),$$
$$\mathcal{C}_3=\sum[2n_3(n_3+1)+1]. \quad (17)$$

The deformation for which (16) has a minimum value defines an equilibrium deformation Λ_∞. Differentiating (16) with respect to Λ and setting the result equal to zero, one has to the first order in μ

$$\Lambda_\infty=\Lambda_0+\tfrac{1}{9}\mu\mathcal{B}^{-1}[2\mathcal{C}_1\Lambda_0^3-\mathcal{C}_2\Lambda_0^{\frac{2}{3}}-4\mathcal{C}_3], \quad (18)$$

where $\Lambda_0=(\mathcal{B}/\mathcal{A})^{\frac{3}{2}}$ is the equilibrium deformation given by the local oscillator model.[20] Substituting back into (16) one obtains the minimum total energy $E_T(\Lambda_\infty)$. It is clear that the sums (17) which determine the value of $E_T(\Lambda_\infty)$ depend on the order in which the single particle orbits are occupied. As the deformation changes different levels become available, so that the total energy has several minima. The smallest value of $E_T(\Lambda_\infty)$ determines the equilibrium deformation. Table I gives the values of the sums (17) which lead to the lowest total energy as the $N=4$ shell of the nonlocal oscillator model fills up progressively with two particles per state. The corresponding equilibrium deformations calculated from (18) are given in the last column.

The behavior of the equilibrium deformation as a function of particle number shown in Table I is rather different from that predicted by a local oscillator model.[20] The most interesting difference is the definite preponderance of prolate shapes to well past the middle of the shell. This behavior comes from the $-\Lambda m^2$ interaction energy in the single particle spectrum and can be qualitatively understood by observing (i) that this interaction is stronger when the deformation is prolate and (ii) that more states with high angular momentum components occur in the upper half of a shell when the shape is prolate rather than oblate. These factors cooperate to lower the total energy considerably in the case of prolate deformations and lead to a favoring of prolate equilibrium shapes. However, the nonlocal corrections to the magnitude of the equilibrium deformations given by the second term in the expression (18) are always small, amounting to a few percent of the local value Λ_0. The equilibrium deformations given in Table I already include this correction.

Another point of difference is the behavior of the predicted equilibrium deformations in the vicinity of a closed shell which is opposite to that found previously.[20] Although the change of preferred shape from prolate to oblate as the particle number passes through a closed shell is now in accordance with shell model predictions, the strong coupling scheme is not valid in this vicinity and the agreement may be quite fortuitous. Moreover, residual two-body interactions which we have ignored entirely can have important effects when the equilibrium shape is nearly spherical.[2] Nevertheless, it is interesting to observe that Eq. (18) does reduce correctly to $\Lambda_\infty=1$ at completely filled oscillator shells, since one then always finds that $\mathcal{A}=\mathcal{B}$ and $\mathcal{C}_1=\mathcal{C}_2=4\mathcal{C}_3$.

When the spin-orbit force is included in the single particle Hamiltonian it is better to use the exact wave functions (12) to compute the expectation values of the operators in the expression (15) for the total energy. In this case it is convenient to rewrite (15) in the equivalent form

$$2E_T(\Lambda)=\sum_i E_i(k\Omega w)+(m^*/m_0)\sum_i\langle W_i\rangle \quad (19)$$

by using the virial theorem associated with the symmetrized Hamiltonian in the wave equation (4) (see Appendix), after the spin-orbit interaction has been added to H in (15). Then the first term in (19) is just a sum over the single particle energies shown in Fig. 1, and the expectation value of W is straight forward to obtain. The minimum value of (19) now defines another deformation parameter which we call Λ_e, and which can only be found by evaluating the sums shown in Eq. (19) for different Λ and determining the minimum graphically.

The predicted deformations found in this manner are indicated in Fig. 4 by open circles through which a smooth curve has been drawn to show the general trend. The deformations shown are for consecutive filling of the single particle levels of Fig. 1 with two particles

per state as a function of the number of neutrons. For comparison the broken curves give the values of Λ_∞ plotted on a different particle number scale so that the $N=82$ shell coincides with the $N=70$ oscillator shell.

The behavior of Λ_∞ and Λ_e is seen to be essentially the same, except that the closed shells appear at the correct magic numbers when the spin-orbit force is taken into account also. The predominance of prolate deformations in regions between the magic numbers still persists and their magnitudes are not significantly altered. It is interesting to observe that in spite of the rather extreme simplifying assumptions on which the calculations of the equilibrium deformations Λ_e are based, the behavior of the predicted deformations given in Fig. 4 as a function of the number of particles is almost quantitatively correct, especially with respect to the sudden increase of deformation just after a shell closure followed by a gradual decrease in deformation as the next shell is filled. This behavior of the nuclear deformations seems to be well established experimentally.[21]

Finally we consider different numbers of neutrons and protons moving in the average field. Then strong deformations can only develop when both the neutron and proton numbers lie at about the middle of a shell filling and can hence cooperate to produce a stable deformed shape. One such situation occurs in the rare earth region when the 126-neutron shell and 82-proton shell are being filled simultaneously. Our calculations for Λ_e shown in Fig. 4 were in just this mass number region, but for equal neutron and proton numbers.

If we neglect the Coulomb interaction as before and assume that the neutron and proton level schemes are

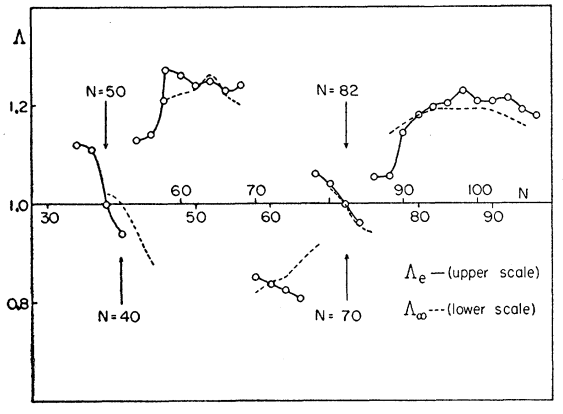

FIG. 4. Calculated equilibrium deformations as a function of the number of particles for the consecutive filling up of the nonlocal oscillator potential with two particles per available state. The broken curve gives the equilibrium deformations for the nonlocal oscillator without spin-orbit coupling and the open circles denote the corresponding values when the spin-orbit term is included. The calculated values of Λ_∞ and Λ_e have been plotted on different horizontal scales so that the closing of the $N=82$ shell corresponds to the $N=70$ oscillator shell. The vertical arrows indicate the empirical and oscillator closed shells.

[21] N. P. Heydenburg and G. M. Temmer, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, 1956), Vol. 6, p. 77.

TABLE II. Calculated equilibrium deformations from total energies based on the nonlocal level scheme presented in Fig. 1 for some odd neutron nuclei in the region $155<A<185$. The asymptotic estimates, δ_∞, were evaluated from Eq. (18). The last column lists the asymptotic quantum numbers of the odd neutron orbit that is available at the measured deformation according to the level scheme in Fig. 1, and which is in accord with the experimental ground-state spin. The empirical data is from references 4 and 22.

Nucleus	Spin	δ_{exp}	δ_e	δ_∞	Odd neutron state
Gd^{155}	3/2	0.31	0.28	0.37	321+
Gd^{157}	3/2	0.31	0.29	0.35	321+
Dy^{161}	5/2	0.31	0.30	0.35	242+
Er^{167}	7/2	0.29	0.29	0.34	333+
Yb^{171}	1/2	0.29	0.28	0.32	321−
Yb^{173}	5/2	0.29	0.28	0.31	412+
Hf^{177}	7/2	0.26	0.27	0.31	414−
Hf^{179}	9/2	0.27	0.24	0.30	424+

identical, the total energy is still given by the expression (19) except that summations break up into partial sums for the neutrons and protons separately. Table II presents some calculated equilibrium deformations for odd neutron nuclei in the rare earth region. We have given the values of the more usual deformation parameter δ_e, defined earlier, rather than Λ_e to allow a direct comparison with experiment and with the results of Nilsson and Mottelson[8] who have made extensive calculations of nuclear equilibrium shapes based on the Nilsson potential.

Thus δ_e is the equilibrium deformation found graphically by evaluating the total energies (19) for the consecutive occupation of the nonlocal energy levels with the appropriate numbers of neutrons and protons for each nucleus considered. For comparison we have also obtained the asymptotic estimate δ_∞ of the equilibrium deformation from the expression (18) for Λ_∞ by evaluating the sums (17) over all the occupied states in question. It is clear that the sequence in which the deformed levels are occupied is irrelevant as far as filled shells are concerned, and only matters for the partially filled upper shell. Hence we have calculated the sums (17) by taking the level sequence occurring in the vicinity of the predicted deformation δ_e for each nucleus. The last column in Table II gives the asymptotic quantum numbers of the orbit with the correct spin available to the odd neutron of each nucleus according to the level sequence in Fig. 1 taken at the observed value of δ.

The close agreement of both estimates of the equilibrium deformation with the empirical values is rather interesting, especially since all the δ_e turn out positive. The reason for this is the same as before, since the proton and neutron numbers of the nuclei given in Table II separately fall into regions of prolate deformations shown in Fig. 4. Furthermore, the assignment of the odd neutron orbitals based on the nonlocal level scheme which also give the correct nuclear spin is quite effortless and merely reproduces the assignments of Mottelson and Nilsson. In fact, the results presented in Table II

are essentially those given previously by these authors using the Nilsson potential.[8,22]

5. CONCLUSION

Our principal aim has been to pursue the possible effects of allowing the average nucleon-nucleus field to be nonlocal in the sense required by the Brueckner theory, but by replacing the full self-consistency problem by a much more limited phenomenological approach (which has of course no claim to self-consistency whatsoever) that can be discussed in analytic terms.

The basic assumption in our treatment of the nonlocality in the nuclear field lies in the particular spatial dependence used in the expression (3) for the nonlocal potential which was based on physical plausibility arguments alone. However, the further approximations involved in deriving the effective mass wave equation (4) are not very serious. As was proved elsewhere,[17] the form of this equation is entirely independent of the specific function used to represent δ_a, since in the expansion of the nonlocal interaction term in Eq. (3) only the even moments of δ_a appear (odd moments vanish identically) and may be regarded as parameters which must be empirically determined. Dropping all moments higher than the second, we obtain the effective mass equation given in the Introduction. The second moment is then proportional to the parameter a^2 given there, which in turn is fixed empirically by the mass reduction. The small value of a^2 (or rather μ which is proportional to it) indicates that this approximation is probably sufficient for a qualitative study of nonlocal effects within the limitations of the present approach. Actually, if the next higher order moment of δ_a is retained, the nonlocal equation can still be reduced to a symmetrized wave equation like (4), but the expression for the effective mass becomes considerably more complicated.[23] Thus one might conjecture that the effective mass equation is a quite general form, the only difference arising in the definition of the effective mass. However, the magnitudes of such additional terms lie entirely within the uncertainties associated with our specific choice of nonlocal potential and the shape of the potential $V(\mathbf{r})$ appearing in the effective mass equation (4) and can be safely neglected for the present.

We also pointed out that the interaction described by the effective mass equation is approximately equivalent to letting the average nuclear potential depend on the particles' orbital angular momentum in a simple way and is a direct manifestation of what is usually referred to as the velocity dependence of the nuclear potential for infinite nuclear matter. This result was shown to depend only on the rotation symmetry assumed for the average field in the present approximation and hence is probably representative of the type of effects embodied in the full nonlocal interaction.[24]

Further detailed analysis of nuclear ground-state properties in terms of the nonlocal level scheme is not particularly interesting, since we have shown that the nonlocal potential used in these calculations closely approximates the Nilsson potential in many respects. However, the favoring of prolate nuclear shapes found here in equilibrium deformation calculations was much easier to interpret in terms of the characteristic angular momentum dependence of the effective potential in the present formulation. The angular momentum dependence appeared mainly as a result of the particular rotational invariance of the assumed potential forms so that these predictions are probably not too dependent on the particular choice of an oscillator well shape to generate the basic level scheme and wave functions. It is important to note, however, that any distortion of the pure oscillator well shape (such as that effectively produced by the nonlocal interaction), or the use of a more realistic average potential shape embodying a diffuse edge will lead to a single particle spectrum similar to that generated in the present calculation. Hence any results dependent on the actual level ordering (like the equilibrium deformations) should not be regarded as being characteristic of a nonlocal interaction alone. However, recent calculations[25] of deformed single particle states in a diffuse nonlocal potential show that the characteristic effects of nonlocality pointed out in this paper are retained in the more realistic case also, and that the present formulation does appear to be capable of simulating the effects of a full nonlocal interaction in an approximate, but nevertheless reasonable manner.

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APPENDIX

(a) Matrix Elements

The matrix elements of the operator H' defined in Eq. (6c) can be computed by operating directly on the anisotropic oscillator wave functions, but this method

[22] B. R. Mottelson and S. G. Nilsson, Phys. Rev. **99**, 1615 (1955).

[23] R. H. Lemmer (unpublished).

[24] K. A. Brueckner, *Proceedings of the International Conference on the Nuclear Optical Model* (Florida State University, May, 1959), p. 145.

[25] R. H. Lemmer and A. E. S. Green (to be published).

is very tedious and a considerable simplification results by employing matrix methods first. Expressing H' in the dimensionless variables given in the text, one has $H'=\frac{1}{8}m_0\omega_0^2a^2K$ in terms of the dimensionless operator

$$K=(\rho^2\nabla_\rho^2-1)\Lambda+\left(\rho^2\frac{d^2}{d\eta^2}+\eta^2\nabla_\rho^2\right)\Lambda^{-\frac{1}{2}}+\left(\eta^2\frac{d^2}{d\eta^2}-\tfrac{1}{2}\right)\Lambda^{-2}, \quad (20)$$

where ∇_ρ is the two dimensional Laplace operator in cylindrical coordinates. Consider the diagonal elements of K first in the asymptotic representation. Recalling that the separated differential equations for the anisotropic oscillator may be written in the operator form

$$-\nabla_\rho^2+\rho^2=2(n+1); \quad -(d^2/d\eta^2)+\eta^2=2(n_3+\tfrac{1}{2}), \quad (21)$$

the matrix elements of the differential operator in (20) can be found in terms of the well-known matrices $\langle\rho^2\rangle$ and $\langle\eta^2\rangle$ by appropriate premultiplication of the relations (21). In this manner the required diagonal elements are found to be

$$\begin{aligned}\langle\rho^2\nabla_\rho^2\rangle&=-\tfrac{1}{2}(n+1)^2-\tfrac{1}{2}(m^2-1),\\ \langle\rho^2d^2/d\eta^2\rangle&=\langle\eta^2\nabla_\rho^2\rangle=-(n+1)(n_3+\tfrac{1}{2}),\\ \langle\eta^2d^2/d\eta^2\rangle&=-\tfrac{1}{2}(n_3+\tfrac{1}{2})^2+\tfrac{3}{8},\end{aligned} \quad (22)$$

and on making the relevant substitutions in (20) the diagonal elements of K given in Eq. (10) in the text are obtained.

The off-diagonal elements of K are easy to obtain by the above method since only the middle term in (20) involving both ρ and η contributes. One finds with the help of (21) that

$$\langle n-2, n_3+2, m|K|nn_3m\rangle=2\Lambda^{-\frac{1}{2}}\langle n-2|\rho^2|n\rangle\langle n_3+2|\eta^2|n_3\rangle, \quad (23)$$

which immediately reduces to the result given in the text.

(b) Virial Theorem

A virial theorem appropriate to the velocity dependent Hamiltonian appearing in Eq. (4) can be established in the usual manner by calculating the time rate of change of $\langle\mathbf{p}\cdot\mathbf{r}\rangle$ and setting the result equal to zero.

The Hamiltonian of Eq. (4) may be written as

$$H=T_{\text{sym}}+V(\mathbf{r}), \quad (24)$$

where T_{sym} denotes the symmetrized kinetic energy operator appearing in (4). The virial theorem then follows by requiring that the average value of the commutator bracket $[H,(\mathbf{p}\cdot\mathbf{r})]$, where H is given by (24), must vanish. Evaluating the commutator in the usual way one finds

$$\langle T_{\text{sym}}\rangle=\tfrac{1}{2}\langle\mathbf{r}\cdot\nabla V\rangle, \quad (25)$$

which is identical with the virial theorem for static potentials except that the kinetic energy operator must be symmetrized as in Eq. (4). However, the effective nucleon mass appearing in (25) is not the same as in (4) but is given by the interesting expression

$$m=m_0[1-(a^2m_0/2\hbar^2)(V(\mathbf{r})-\tfrac{1}{2}\mathbf{r}\cdot\nabla V)]^{-1}. \quad (26)$$

In particular, for an oscillator type potential like that given in (5), one finds that (26) reduces to

$$m=m_0[1+(a^2m_0/2\hbar^2)V_0]^{-1}=m^*, \quad (27)$$

and the virial theorem (25) assumes the simple form

$$m_0/m^*\langle T_0\rangle=\langle V\rangle, \quad (28)$$

where $T_0=p^2/2m_0$ is the ordinary kinetic energy. Eq. (28) can also be verified by calculating average kinetic and potential energies directly, and is responsible for the last term in Eq. (19) for the total energies.