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Validity of the Theory of Double Stream Amplification

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Misunderstandings have recently arisen concerning the validity of the original analysis of the interaction between interpenetrating ion streams to give double stream amplification. It is shown that none of the modes of propagation upon which criticism has been centered corresponds to that of double stream amplification. A direct theoretical proof of the validity of the theory is given.

1. INTRODUCTION

THE object of this paper is to clarify some misunderstandings which have arisen recently (Piddington[1,2]) concerning double stream amplification produced by the interaction between moving ion or electron streams. Only *amplification* is looked for (ω real), not *oscillation* which is indicated when there is exponential growth with time (at a point) and ω has a negative imaginary part; by convention negative (real) frequencies are disregarded.

As originally conceived by Haeff[3] and Pierce[4] double stream amplification can occur, for instance, when two equally dense (neutralized) streams of drifting electrons having nearly equal velocities $U\pm v (U\gg v)$ are excited by means of a constant rf source (ω real and positive).

For amplification an exponential increase with distance is required and it will be shown that *none* of the four modes considered previously[2] corresponds to this case (Sec. 3). One of the modes ($A2$) has ω mainly real and provides a good approximation to this case, but the mode criticized by Piddington ($B2$) has ω almost wholly imaginary.

A mode can be found (Sec. 4) having exponential variation with distance for constant amplitude excitation (ω real), and the dispersion curve for this mode is best shown on a three-dimensional graph (Fig. 2). It may indicate true growth, but Piddington[1] has pointed out the need to distinguish carefully between a wave growing in one direction and one decaying in the reverse direction. In Sec. 5 a direct theoretical proof will be given that this mode *does* represent amplification and not decay, and reference will also be made to the experimental evidence for closely related types of interaction.

The coupling problem is important since, if this wave is to be observable, it must be extracted from the streams (by some coupling mechanism). In Sec. 6 it is briefly pointed out that, while suitable coupling mechanisms have been used in the laboratory, the position regarding possible astrophysical occurrences is still not clear.

2. THE DISPERSION EQUATION

If steady-state solutions are expressed as sums of terms of the form $\exp(i\omega t-ikz)$, then it is agreed generally that the dispersion equation connecting ω and k is:

$$\frac{\omega_0^2}{[\omega-(U+v)k]^2}+\frac{\omega_0^2}{[\omega-(U-v)k]^2}=1, \qquad (1)$$

where

$$\omega_0=[ne^2/m\epsilon_0]^{\frac{1}{2}}, \text{ plasma frequency.} \qquad (2)$$

Here, the real part of ω is the true frequency seen by an observer at any point. The imaginary part of ω represents exponential increase or decrease with time (instability). The real part of k represents sinusoidal variation with distance. The imaginary part of k corresponds to exponential growth or decay with

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[1] J. H. Piddington, Phys. Rev. **101**, 14 (1956).
[2] J. H. Piddington, Phil. Mag. **3**, 1241 (1958).
[3] A. V. Haeff, Proc. Inst. Radio Engrs. **37**, 4 (1949).
[4] J. R. Pierce, Proc. Inst. Radio Engrs. **37**, 980 (1949).

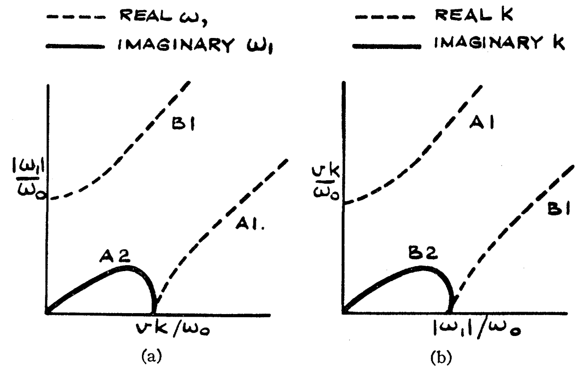

FIG. 1. Solutions of the dispersion equation for real and imaginary values of k. (a) real values of k. (b) real values of ω_1.

distance at any instant. It has been pointed out by Piddington[1] that care is needed to distinguish between a spatially growing wave and one decaying in the opposite direction.

It is convenient for purposes of analysis to follow the original workers[3,4] and to express the dispersion equation in terms of the new variable:

$$\omega - Uk = \omega_1. \tag{3}$$

Now this can be approximately (i.e., nonrelativistically) interpreted[1] as a change of axes to a frame moving with the mean velocity U of the two streams, so that the ions appear to move in opposite directions with velocities $\pm v$. However, if this interpretation is used, great care must be taken when considering the physical meaning of any results, since the original observer is not in this moving frame. In the moving frame, the apparent (Doppler shifted) frequency ω_1 is approximately given by Eq. (3), while the propagation constant remains approximately the same.

Alternatively, one can merely take this[3,4] as a (strictly accurate) change of variable without attaching any physical significance to it. The dispersion equation then takes the simple form[1–4]

$$\frac{\omega_0^2}{[\omega_1 - vk]^2} + \frac{\omega_0^2}{[\omega_1 + vk]^2} = 1. \tag{4}$$

3. THE MODES CONSIDERED BY PIDDINGTON

3.1 Solutions of the Dispersion Equation

Haeff and Pierce used this change of variable mainly so that an analytical solution of the dispersion equation can be obtained readily, thus:

$$\omega_1/\omega_0 = \{(vk/\omega_0)^2 + 1 \pm [4(vk/\omega_0)^2 + 1]^{\frac{1}{2}}\}^{\frac{1}{2}}. \tag{5}$$

Solutions for real values of $k > \sqrt{2}\omega_0/v$ are all real (curves $A1$, $B1$) but for $k < \sqrt{2}\omega_0/v$ imaginary values of ω_1/ω_0 are possible (curve $A2$). These curves are plotted in Fig. 1(a).

For real values of $\omega_1/\omega_0 > \sqrt{2}$ there are only real values of k (curves $A1$, $B1$ once more), and for $\omega_1/\omega_0 < \sqrt{2}$ imaginary values of k (curve $B2$) are possible. These curves are plotted in Fig. 1(b). The curves are labelled according to Piddington.[2]

3.2 Curves $A1$, $B1$

Curves $A1$, $B1$ are both for ω_1 real, k real and hence, from Eq. (3), ω real; there is no question of exponential growth or instability for either of these two waves. It is as well to note that it may prove difficult to excite either of the waves separately. There may in fact normally be an interference phenomenon between them, as originally suggested by Haeff, and just as on a single stream of ions.

If a single beam of ions is excited by means of a cavity, as for example in a klystron amplifier, the two possible space charge waves are excited equally. There is then a beat wavelength for the total standing wave and klystron type amplification is possible depending on the coupling. It is only recently[5] that more sophisticated devices have been made, in which it is possible for only one of the space charge waves to be excited.

3.3 Curve $A2$

This curve has k real, ω_1 imaginary and hence ω complex (although mainly real). Taking the point of maximum interaction on the curve, $\omega_1 = -i\omega_0/2$, $k = \sqrt{3}\omega_0/2v$ so that $\omega = (U/v)\sqrt{3}\omega_0/2 - i\omega_0/2$. The true frequency is therefore $(U/v)\sqrt{3}\omega_0/2$, the frequency near which Haeff[3] and Pierce[4] also find maximum gain. There is no exponential variation with distance and the imaginary part of ω gives the rate of increase of the rf wave with time, namely $31.5v/U$ db per rf cycle. This is not very large for $U \gg v$.

This type of interaction could be set up (albeit in a rather artificial way) by increasing the excitation level of the input (ω complex) at just the same rate as the gain mechanism down the stream, namely $31.5v/U$ db per wavelength (or per cycle), so that the level at any instant is constant along the stream, but there is an overall growth with time. The wavelength is about the same as that found by Haeff and Pierce. The gain mechanism is that discussed by Haeff and Pierce, but the excitation is obviously slightly different, since they take ω real, not complex, and find k complex, not imaginary. Nonetheless, the curve $A2$ does give an approximation to the case of double stream amplification and it is in fact much closer to that case than any of the other curves $A1$, $B1$, and especially $B2$. It must be noted, however, that it does not correspond exactly to that case.

3.4 An Alternative Way of Exciting the $A2$ Mode

An alternative way of setting up the $A2$ mode has been suggested, by means of a uniform sinusoidal excitation along the whole length of the stream. The

[5] S. Saito, IRE Trans. on Electron Devices **ED5**, 264 (1958).

level should then grow at the same rate as before. Although it may appear rather artificial to consider applying an external rf signal in this way, components of noise excitation within the streams may be of the form required for growth. Unfortunately, it is not obvious what boundary conditions are required at either end of streams excited in this way, and whether such conditions can be met.

Buneman[6] has suggested a mechanism which appears to be somewhat similar to this, in which he has taken $U=v$ and streams of different plasma frequencies corresponding to electrons and hydrogen ions. He has shown that the growth of such noise fluctuations will be very rapid (94.2 db per rf cycle when the growth rate is maximum) and he discusses the implications of this in the plasma field.

3.5 Curve $B2$

Piddington,[2] in discussing the $B2$ waves, claims to have "shown (Piddington[1]) that previous demonstrations of growth of these waves were not valid." However, there have not in fact been any demonstrations of growth of the $B2$ waves in connection with the electron-wave tube, or double stream amplification, and *these waves do not correspond, even approximately, to those considered by Haeff and Pierce.*

For the $B2$ waves, ω_1 is real and k is purely imaginary (so that there is no wave motion), while ω is complex but substantially imaginary. Taking maximum apparent interaction (i.e., increase with distance) $\omega_1=\sqrt{3}\omega_0/2$, $k=i\omega_0/2v$ so that $\omega=\sqrt{3}\omega_0/2+i(U/v)\omega_0/2$. Thus the frequency is very low [real part of $\omega<\sqrt{2}\omega_0$, while Haeff and Pierce take $U\gg v$ and find maximum interaction around $\omega\simeq(U/v)\omega_0$, a much higher frequency] and the rate of growth with distance is very rapid. Even for a quite large velocity difference of 20% it is more than 300 db in the mean distance travelled by an electron per rf cycle. However, this very rapid increase with distance is obtained only by having an equally rapid decay with time.

It therefore seems quite likely that this type of interaction does not represent a true gain mechanism and that, as Piddington[1] has suggested, the fact that there is no wavelength (k has no real part) may indicate attenuation rather than growth. In any case it does not correspond, even approximately, to the case of double stream amplification.

3.6 Summary

To sum up, it can be seen that none of the four curves considered by Piddington,[2] $A1$, $A2$, $B1$, $B2$, (which all have ω_1 either purely real or purely imaginary) corresponds to the case of double stream amplification, since that case has ω real, k complex and hence ω_1 complex. The closest approximation is given by $A2$, not by $B2$ as suggested by Piddington.

[6] O. Buneman, Phys. Rev. Letters **1**, 8 (1958).

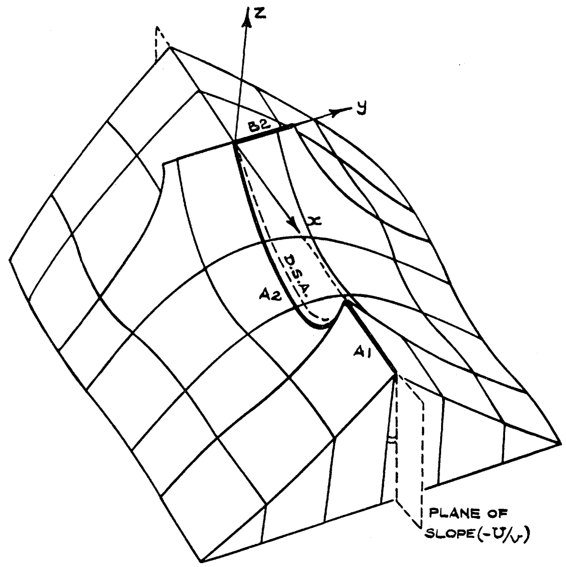

FIG. 2. The imaginary part of solutions of the dispersion equation for complex values of k.

4. DOUBLE STREAM AMPLIFICATION CURVE

The reason that the previous curves were introduced by Haeff and Pierce was a purely mathematical one. They used curve $A2$ merely as a useful normalized approximation from which to derive the actual values of the parameters of the double stream amplifier in any particular case.

It is useful to see the relationships between these various curves in the complex k plane. Since the parameter of interest is that showing the exponential increase with distance, it is useful to plot the imaginary part of $\omega-Uk$ (that is, of ω_1) obtained from Eq. (5) against complex values of k, to obtain a three-dimensional graph as shown in Fig. 2. The x and y axes are the real and imaginary parts of vk/ω_0 (which is, of course, proportional to k) and the z axis is the imaginary part of ω_1/ω_0 (which is, of course, proportional to ω_1 or $\omega-Uk$).

Since the equation is biquadratic, there are four sheets in the complex plane. These are merely reflections of each other in the $0xy$ and $0yz$ planes however, and, for simplicity, only the sheet of interest has been shown. (Curve $B1$ lies along the x axis on a reflected sheet.) For curves $A1$ and $A2$, k is real and hence these curves (of which, it must be remembered, only the imaginary parts of ω_1/ω_0 are plotted) lie in the real k plane, $0xz$. For curve $B2$, k is imaginary and so this curve lies along the imaginary k axis, $0y$.

For double stream amplification (D.S.A.), ω is real; thus, of course, the imaginary part of ω is zero.

$$\begin{aligned} 0&=I(\omega)\\ &=I(\omega_1+Uk) \text{ by (3)}\\ &=I(\omega_1)+(U/v)I(vk)\\ &=\omega_0[z+(U/v)y], \end{aligned}$$

remembering that the y and z axes are the imaginary parts of vk/ω_0 and ω_1/ω_0, respectively.

The curve therefore lies in the plane

$$z=-(U/v)y \tag{6}$$

through the x axis of slope $(-U/v)$. It can be seen from Fig. 2 that, as already stated, $A2$ provides a good approximately to the double stream amplification (D.S.A.) curve, particularly for large values of U/v, while $B2$ does not correspond even approximately.

5. VERIFICATION OF THE VALIDITY OF THE THEORY

5.1 True Amplification

Although the misunderstanding as to which curves are involved has now been cleared up, Piddington's[1] original point, that care is required to distinguish between a spatially growing wave and one decaying in the opposite direction, still remains. A complex solution of the $\omega-k$ dispersion equation does not necessarily signify amplification. True amplification can be verified in two ways: experimentally and theoretically.

5.2 Experimental Verification

There has been very little controlled experimental data on double stream amplification itself beyond the original work of Haeff and Pierce.[7] However, the theory of travelling wave interaction between an electron stream and a nearly synchronous rf wave (as in the travelling wave tube) is so similar in most fundamental respects to the theory of interaction between two electron streams that any basic criticism of either should apply equally to the other. Conversely, if the theory of either can be shown to agree closely with experimental data, such criticism becomes suspect.

Now there is in fact a wealth of controlled experimental evidence (not to mention working tubes throughout the world), which has been available for some time, and which shows in very great detail that the waves grow as predicted by the theory, in the case of travelling wave amplification. An extensive bibliography covering most aspects of the early experimental and theoretical work was given by Kompfner,[8] and this was supplemented in a review article by Hutter.[9] An excellent and concise statement of the position at that time was given in a letter by Cutler.[10] Recent work has experimentally verified detailed aspects of travelling wave tube theory such as noise,[11,12] backward wave amplification,[13] and various particular applications.[14–17] Other related types of interaction are also well documented.[18,19] It can therefore be said that the principles fundamental to both types of interaction have already been established on a firm experimental basis.

5.3 Theoretical Verification

It is, nevertheless, more satisfying if it can be shown on a rigorous formal basis that there is true amplification, and it must be confessed that this has proved surprisingly difficult. However, Buneman,[20] following Piddington's original criticism, has recently derived a condition for true amplification. He points out that a criterion for genuine amplification is the ability of the electrons and the rf field to balance a positive conductance in a matched probe circuit (i.e., to deliver power to an external load) while propagating a wave growing towards such a probe; hence he derives the required condition. If the dispersion equation obtained by matching admittances between the rf field and the electrons is $f(\omega,k)=0$, the condition for true amplification of an apparently growing wave is that $\partial f(\omega,k)/\partial k$ must have a negative imaginary part, when ω is real and positive, and k has a positive imaginary part. A detailed discussion is presented in Buneman's report[20] but for completeness a brief derivation is given in the Appendix.

Buneman himself has verified both O and M type travelling wave amplification,[20] and the double stream amplifier can be checked for a particular case of interest as follows. The dispersion Eq. (1), in the form obtained by matching admittances between the electrons and the rf fields is

$$f(\omega,k)=\epsilon_0\omega k\left(\frac{\omega_0^2}{[\omega-(U+v)k]}+\frac{\omega_0^2}{[\omega-(U-v)k\lambda]}-1\right)=0. \tag{7}$$

Hence

$$\partial f(\omega,k)/\partial k =2\epsilon_0\omega k\left(\frac{\omega_0^2(U+v)}{[\omega-(U+v)k]^3}+\frac{\omega_0^2(U-v)}{[\omega-(U-v)k]^3}\right)+\epsilon_0\omega\left(\frac{\omega_0^2}{[\omega-(U+v)k]^2}+\frac{\omega_0^2}{[\omega-(U-v)k]^2}\right). \tag{8}$$

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[7] G. A. Bernashevsky, Z. S. Voronov, T. I. Iziumova, and A. S. Tchernov, *Proceedings of the Symposium on Electronic Waveguides* (Polytechnic Institute, Brooklyn, 1958), p. 249.

[8] R. Kompfner, *Reports on Progress in Physics* (The Physical Society, London, 1952), Vol. 15, p. 275.

[9] R. G. E. Hutter, *Advances in Electronics and Electron Physics*, edited by L. Marton (Academic Press, New York, 1954), Vol. 6, p. 372.

[10] C. C. Cutler, Proc. Inst. Radio Engrs. **39**, 914 (1951).

[11] L. D. Buchmiller, R. W. DeGrasse, and G. Wade, IRE Trans. on Electron Devices **ED4**, 234 (1957).

[12] D. A. Watkins, Proc. Inst. Radio Engrs. **40**, 65 (1952).

[13] M. R. Currie and D. C. Forster, IRE Trans. on Electron Devices **ED4**, 24 (1957).

[14] N. Rynn, IRE Trans. on Electron Devices **ED4**, 172 (1957).

[15] L. W. Holmboe and M. Ettenberg, IRE Trans. on Electron Devices **ED4**, 78 (1957).

[16] H. R. Johnson, IRE Trans. on Electron Devices **PGED-4**, 15 (1953).

[17] C. B. Crumly, IRE Trans. on Electron Devices **ED3**, 62 (1956).

[18] P. K. Tien and L. M. Field, Proc. Inst. Radio Engrs. **40**, 688 (1952).

[19] T. G. Mihran, IRE Trans. on Electron Devices **ED3**, 32 (1956).

[20] O. Buneman, Stanford Technical Report No. 385-2, 1958 (unpublished).

The solution for optimum gain given by Pierce[4] and Haeff[3] is:

$$\omega=(U/v)\sqrt{3}\omega_0/2;\quad k\simeq\sqrt{3}\omega_0/2v;\quad \omega-Uk\simeq-i\omega_0/2.$$

It can be shown that both the approximations are very good—to second order in (v/U), in fact. Then $[\omega-(U\pm v)k]^3\simeq i\omega_0^3$ and so, from Eq. (8),

$$\partial f(\omega,k)/\partial k\simeq-3\epsilon_0\omega_0(U/v)^2 i. \tag{9}$$

This has a negative imaginary part and the solution therefore represents true amplification.

It has thus been demonstrated quite formally that the apparent growth with distance does represent true gain, and that double stream amplification is possible.

6. THE COUPLING PROBLEM

This demonstration does, however, emphasize the importance of a matched coupling (Buneman's probe[20]). If the amplified rf waves are to be observable, they must be extracted from the streams. In the laboratory, this presents no great difficulty at normal frequencies (a helix surrounding the streams has been used), although the necessity of having a slow wave structure limits the otherwise almost unbounded frequency range of the double stream amplifier to that of conventional travelling wave amplifiers. Indeed, it has been found that, since a physical structure must be used for coupling at the ends of the device in any case, it is more efficient to have it between as well, and have travelling wave tube interaction along the whole length.

However, in the ion streams with which astrophysics deals there are certainly no conveniently placed helices, and there is still room for argument as to whether any match to free space is possible. A gradual transition at the edge of the streams as the ions become more tenuous, or even a sharp discontinuity, may suffice, but the question is by no means settled.

APPENDIX. THE CONDITION FOR TRUE AMPLIFICATION

The rf voltages and currents may be Fourier-analyzed along the beam. If

$$V(z)=\int_{-\infty}^{\infty}V(k)\exp(-ikz)dk, \tag{1A}$$

and $Y_{\mathrm{rf}}(k)$ and $Y_s(k)$ are the Fourier admittances of the rf wave and the streams, respectively, then by matching Fourier currents $V(k)Y_{\mathrm{rf}}$ and $V(k)Y_s$ the dispersion equation (1) is obtained in the form

$$f\equiv Y_{\mathrm{rf}}+Y_s=0. \tag{2A}$$

If a small probe is now introduced at the point $z=0$ say, to couple power out of (or into) the system, the probe current I_p at the point $z=0$ is a delta function whose transform is simply $I_p/2\pi$. In this case matching the currents (Kirchoff's Law) gives

$$I_p/2\pi+V(k)(Y_{\mathrm{rf}}+Y_s)=0,$$

or

$$V(k)=-(I_p/2\pi)/f,$$

which by Fourier synthesis, Eq. (1A), gives

$$V(z)=-(I_p/2\pi)\int_{-\infty}^{\infty}dk\exp(-ikz)/f. \tag{3A}$$

For $z<0$, i.e., upstream of the probe, the path of integration can be closed by a large semicircle around the top half-plane according to Jordan's lemma, and the integral is then equal to $2\pi i$ times the sum of the residues of the enclosed poles. Now the poles of $1/f$ are the zeros of f, that is to say they occur at the values k_n which are solutions of the dispersion equation $f=0$ in the absence of the probe; the residues are therefore $1/(\partial f/\partial k)$. Hence letting z tend to zero

$$V_p/I_p=-\sum_{k_n}i/(\partial f/\partial k). \tag{4A}$$

Now if a wave which increases exponentially with distance towards $z=0$ (k_n in top half-plane, $z<0$) is a truly amplifying wave, it must be able to deliver real power to the probe, whereas if it is decaying in the reverse direction it merely represents a reactive transfer of power from the probe. Hence the condition for true amplification is that the streams and the rf wave must be able to balance a positive resistance in the probe circuit while propagating a wave which increases towards the probe, and from Eq. (4A) it can be seen that only growing waves k_n for which $\partial f/\partial k$ has a negative imaginary part contribute towards balancing a positive resistance in the probe circuit. This is the condition for true amplification.