depolarization of muons in nuclear emulsion at 25 000 gauss.

The experimental information available at present is still quite consistent with a value of $\xi=-0.9$. More accurate measurements are necessary to remove this ten percent uncertainty in the value of ξ. In view of the success which the V-A theory has had, the most likely explanation seems to be that the muons do depolarize even at this large field. Those depolarization mechanisms which have been treated adequately do not give rise to any appreciable depolarization at 25 000 gauss. However, the nature of depolarization mechanisms is not understood sufficiently to rule out the possibility that substantial depolarization occurs. In fact, what experimental evidence exists supports this conclusion.

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Low-Energy Pion Phenomena*

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The relation between low-energy pion-nucleon scattering and pion photoproduction is examined. Correct extrapolation to threshold of both the π^+ and π^- photoproduction data gives agreement with theory. A recent new method for analyzing the scattering data is applied giving $a_1=0.178$, $a_3=-0.087$, and reasonable agreement with the Panofsky ratio $P=1.5$ is obtained. An inner Coulomb correction to the scattering data helps to improve this agreement. The possibility of detecting a $\pi-\pi$ interaction by low-energy pion scattering is examined. A new dispersion relation connects the s- and p-wave phase shifts at low energies; this relation excludes some well-known sets of phase-shift curves.

I. INTRODUCTION AND SUMMARY

THE violation of the well-known connection between low-energy pion scattering and threshold pion photoproduction via the Panofsky ratio have given some stimulus to theoretical studies of these low-energy phenomena.

In 1958 the situation was clarified by Cini et al.,[1] who asserted that the data were in agreement with a Panofsky ratio $P=1.5$ and a threshold π^-/π^+ ratio $r=1.3$. This agreement was achieved by two steps:

(i) Following a suggestion of Bernardini,[2] the extrapolation of the π^+ photoproduction cross section to threshold was improved by allowing for the retardation term. This *increased* the threshold value.

(ii) It was suggested that the pion scattering crossing relations gave a new plot for the scattering phase shifts. This led to the very low value $a_1-a_3=0.24$ (in units $\hbar=c=\mu=1$) where a_1, a_3 are the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ scattering lengths.

A brief survey of the data and of these arguments is given in Sec. II below. Comments on this scheme include the following:

(a) Beneventano et al.[3] asserted that the increased threshold value for π^+ photoproduction was now in disagreement with the threshold photoproduction measurements of Adamovič et al.[4] (using $\gamma+D$) if we wished to retain $r=1.3$. We show in Sec. II that on using the correct extrapolation for both the π^+ and π^- photoproduction data, and using the correct values of Adamovič's results, this difficulty disappears. For this extrapolation we use the dispersion relation of Chew, Goldberger, Low, and Nambu.[5]

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[1] M. Cini, R. Gatto, E. L. Goldwasser, and M. A. Ruderman, Nuovo cimento **10**, 242 (1958).

[2] G. Bernardini, Suppl. Nuovo cimento **1**, 104 (1955).

[3] M. Beneventano, G. Bernardini, G. Stoppini, and L. Tau, Nuovo cimento **10**, 1109 (1958).

[4] See *1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958), p. 49.

[5] G. F. Chew, M. L. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **104**, 1345 (1956).

(b) The value $a_1-a_3=0.24$ is in disagreement with the usual plots of the experimental scattering data. Chiu and Lomon[6] get $a_1-a_3=0.28$. Another recent analysis of the low-energy scattering data by Barnes et al.[7] gives s-wave phase-shift values

$$\begin{aligned}\alpha_1&=0.205k-0.09k^3+0.018k^5,\\ \alpha_3&=-0.115k,\end{aligned}\tag{1}$$

where k is the pion c.m. momentum. We estimate that the statistical errors in the scattering lengths in this analysis are given by

$$a_1=0.205\pm0.005,\quad a_3=-0.115\pm0.003.$$

This gives $a_1-a_3=0.320\pm0.006$ and would require a great increase in the Panofsky ratio over the value $P=1.5$.

We wish to emphasize that the arguments of Cini et al.[1] imply that the conventional expansion of *either* of the s-wave phase shifts in powers of k is not a good approximation when energies greater than 50 Mev are involved. In Sec. VI recent values of both the phase shifts are analyzed by the new method. This gives

$$a_1=0.178\pm0.005,\quad a_3=-0.087\pm0.005,$$

so $a_1-a_3=0.265\pm0.007$. The value of a_3 has appreciably changed from the old value (-0.11).

Using these values of the scattering lengths, the correct threshold photoproduction values and a new Coulomb correction to the scattering lengths (see Sec. IV), it is clear that the low-energy data are consistent to within the experimental errors.

In Sec. III we examine whether the s-wave pion scattering data proves the existence of a direct pion-pion interaction. If we express the low-energy behavior of the $(T=\frac{1}{2})$ s-wave phase shift α_1 by the effective range formula

$$k\cot\alpha_1=(1/a_1)+\tfrac{1}{2}r_ek^2+\cdots,$$

then by (1) above we get $r_e\simeq5(\hbar/\mu c)$. This large value of the effective range might be thought to imply that the interaction between the s-wave pion and the nucleon has a radial extent of at least $\hbar/\mu c=1.4\times10^{-13}$ cm. The usual picture of the s-wave interaction being due to virtual nucleon-pair formation necessarily gives a very short range (a radial extent $\lesssim0.2\times10^{-13}$ cm). A pion-pion interaction could give a further scattering of the incident pion by the pion cloud which constitutes the outer part of the nucleon. Such an interaction could give values of r_e as large as 5 units.

It is shown in Sec. III that we cannot infer the existence of a pion-pion interaction in this way. When the scattering length a is small, the presence of a small velocity dependent part in a short-range potential can also lead to a large value of r_e. Another method which sometimes can determine the range of an interaction—Wigner's causality condition—is also invalid in the present case.

In Sec. IV we make small corrections to the scattering lengths to allow for Coulomb effects which are usually ignored. Two facts are relevant, (i) the electrostatic potential is not zero inside the nucleon (i.e., $r\leqslant1$) as is usually assumed, (ii) the proton's charge is spread over the nucleon's volume. Taking these facts into account, the corrected scattering lengths represent the strictly mesonic scattering. On comparing the scattering data with photoproduction via the Panofsky ratio, the value of (a_1-a_3) as deduced from scattering must be reduced by 0.02. This is because the values of a_1 and a_3 quoted above do not include these new Coulomb crorections.

In earlier analyses certain Coulomb interference terms were allowed for, and as a result, the final values of the s-wave phase shifts varied[7] with an arbitrary

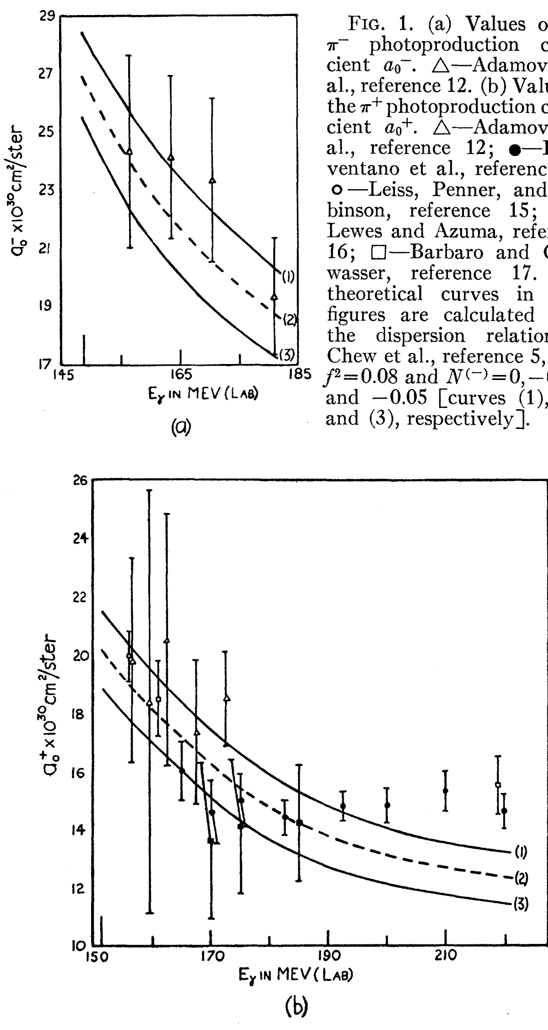

FIG. 1. (a) Values of the π^- photoproduction coefficient a_0^-. △—Adamovič et al., reference 12. (b) Values of the π^+ photoproduction coefficient a_0^+. △—Adamovič et al., reference 12; ●—Beneventano et al., reference 14; o—Leiss, Penner, and Robinson, reference 15; ■—Lewes and Azuma, reference 16; □—Barbaro and Goldwasser, reference 17. The theoretical curves in both figures are calculated from the dispersion relation of Chew et al., reference 5, with $f^2=0.08$ and $N^{(-)}=0,-0.025$ and -0.05 [curves (1), (2), and (3), respectively].

[6] H-Y Chiu and E. Lomon, Ann. Phys. (New York) **6**, 50 (1959). See also the report of the 1958 CERN Conference, reference 4, for other results.

[7] S. W. Barnes, B. Rose, G. Giacomelli, J. Ring, K. Miyahe, and K. Kinsey, Phys. Rev. **117**, 226 (1960); S. W. Barnes, W. H. Winet, K. Miyahe and K. Kinsey, **117**, 238 (1960). We are much indebted to Professor Barnes for sending us the manuscripts.

radius parameter r_0. Our method removes this undesirable dependence on r_0.

In Sec. V we give a new dispersion theory relation connecting the low-energy behavior of the s-wave and p-wave phase shifts. This relation suggests that the energy dependence of the set of s-wave and p-wave phase shifts proposed by Chiu and Lomon,[6] and of the set proposed by Barnes et al.,[7] is inadmissable. The new dispersion relation gives some support to the new set of phase-shift curves which we give in Sec. VI.

II. THE LOW-ENERGY DATA

(a) Panofsky Ratio

Recent measurements of the ratio

$$P=\frac{\sigma(\pi^-+p\rightarrow\pi^0+n)}{\sigma(\pi^-+p\rightarrow\gamma+n)},$$

for negative pions captured in hydrogen are $P=1.50\pm0.15$ (Cassels et al.[8]), 1.87 ± 0.10 (Marshall et al.[9]), and 1.46 ± 0.10 (Koller and Sachs[10]). Here we shall assume that the value is 1.5 ± 0.1.

(b) Photoproduction Data

The cross sections for the processes $\gamma+p\rightarrow\pi^++n$, $\gamma+n\rightarrow\pi^-+p$ are written in the form[11]

$$\frac{d\sigma}{d\Omega}=\frac{k\omega^*}{(1+\omega^*/M)^2}(a_0+a_1\cos\theta+a_2\cos^2\theta). \tag{2}$$

θ is the angle of the pion in the center-of-mass system, k is its center-of-mass energy and (ω^*+M) is the *total* energy in the c.m. system (M = nucleon mass; μ = pion mass).

The experimental values for the above coefficients $a_0^\pm$ (which refer to the $\pi^\pm$ cases) are plotted in Fig. 1. The a_0^- values are due to Adamovič et al.[12] and are deduced from the photoproduction in deuterium using the theoretical corrections of Baldin[13] to give the equivalent free nucleon values. The a_0^+ values are due to various workers as indicated.[12,14–17]

[8] J. Cassels et al., Proc. Phys. Soc. (London) **A70**, 405 (1957).
[9] J. Fischer, R. March, and L. Marshall, Phys. Rev. **109**, 533 (1958).
[10] E. L. Koller and A. M. Sachs, Bull. Am. Phys. Soc. 4, 24 (1958). See also A. W. Merrison et. al., Proc. Phys. Soc. (London) **73**, 545 (1959).
[11] See A. J. Lazarus, W. K. H. Panofsky, and F. R. Tangherlini, Phys. Rev. **113**, 1330 (1959). Our definition of a_0, a_1, a_2 agrees *at theshold* with that of M. Beneventano, G. Bernardini, D. Carlson-Lee, and L. Tau, Nuovo cimento **4**, 323 (1956). For photon energies, up to 220 Mev the two phase space factors differ by no more than a few percent.
[12] M. Adamovič et al., Dubna, 1959 (to be published). We are indebted to Professor Adamovič for a copy of their paper.
[13] A. M. Baldin, Nuovo cimento **8**, 569 (1958).
[14] M. Beneventano et al., Nuovo cimento **4**, 323 (1956).
[15] J. E. Leiss, S. Penner, and C. S. Robinson, Ninth International Conference on High-Energy Physics, Kiev, 1959 (unpublished).
[16] Lewis and Azuma, reference 15.
[17] G. Barbaro and E. L. Goldwasser, references 4 and 15.

The theoretical curves in Fig. 1 are the results of Chew et al.[5] obtained by using the fixed momentum transfer dispersion relation and the predominance of the $(\frac{3}{2},\frac{3}{2})$ pion-nucleon interaction. $N^{(-)}$ is a somewhat complicated function about which we can only say that it is believed to be small. It will be seen in the next section that our argument is fairly well independent of the precise value we use for $N^{(-)}$.

The sharp drop in the theoretical curves above threshold comes mainly from the interference of the retardation term in photoproduction with the main s-wave terms.[18] The retardation term vanishes at threshold, leaving the gauge invariant and the recoil terms. The older extrapolations to threshold ignored this drop and, therefore, appreciably underestimated the threshold values of $a_0^\pm$. It should be emphasized that *both* the a_0^+ and a_0^- curves show this drop.

(c) π^-/π^+ Ratio

The threshold value

$$r=\frac{\sigma(\gamma+n\rightarrow\pi^-+p)}{\sigma(\gamma+p\rightarrow\pi^++n)}$$

should, by the same theoretical argument,[5] be

$$r=\left(\frac{1+(g_p+g_n)(\mu/2M)+1.15\,N^{(-)}}{1-(g_p+g_n)(\mu/2M)+1.15\,N^{(-)}}\right)^2, \tag{3}$$

where $g_p=2.79$, $g_n=-1.91$ are the nucleon magnetic moments in nuclear magnetons. Putting $N^{(-)}=0$ gives $r=1.30$. Changing $N^{(-)}$ to 0.2 would give $r=1.25$, but it would also increase the absolute value of a_0^+ by 45%. Therefore the best procedure is to use the experimental points in Fig. 1 to suggest a range of values for $N^{(-)}$, and then use (3) to give the ratio r, which is comparatively insensitive to $N^{(-)}$.

At the energies we consider, the values of $a_0^\pm$ are proportional to the pion-nucleon coupling constant. For simplicity, and to give reasonable agreement with pion scattering we choose $f^2=0.08$. Theoretical curves for a_0^+ and a_0^- with $N^{(-)}=0$, -0.025 and -0.05 are shown in Fig. 1. The experimental points for γ-ray energies below 190 Mev are consistent with something lying between the $N^{(-)}=0$ and $N^{(-)}=-0.05$ curve. This suggests that suitable threshold values are

$$\begin{aligned} a_0^+&=(20.2\pm1.5)\ 10^{-30}\ \mathrm{cm^2/sr},\\ a_0^-&=(26.9\pm2.0)\ 10^{-30}\ \mathrm{cm^2/sr}. \end{aligned} \tag{4}$$

This also gives the threshold ratio $r=a_0^-/a_0^+=1.33$.

We do not discuss why the π^+ experimental points at 200 Mev and above deviate from the dispersion relation values.

[18] This was pointed out by G. Bernardini, Suppl. Nuovo cimento **1**, 104 (1955).

(d) Charge Exchange Scattering

Using the threshold value for $\gamma+n\rightarrow\pi^-+p$ given by this value of a_0^-, and following the usual argument via detailed balancing, and the Panofsky ratio $P=1.5$, we deduce the cross section for charge exchange scattering $\pi^-+p\rightarrow\pi^0+n$ for zero energy π^-. Assuming charge independence this requires[1]

$$a_1-a_3=0.245\pm0.01 \text{ (in units } \hbar=c=\mu=1), \quad (5)$$

where a_1 and a_3 are the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ pion-nucleon s-wave scattering lengths.

It should be noted that in this way we have obtained a value for the zero energy charge exchange cross section which is consistent with all the low-energy photoproduction data; it is also consistent with the ratio $r=1.3$. Further, the analysis is almost independent of the value we assume for f^2.

(e) Pion Scattering

Recent values of the scattering lengths deduced from low-energy $\pi^\pm$ scattering by protons are[19]

$$a_3=-0.110\pm0.004, \quad a_1=0.173\pm0.011. \quad (6)$$

This gives $a_1-a_3=0.283$. Thus for consistency with (c) above it would be necessary to increase the product a_0^-P [which is proportional to $(a_1-a_3)^2$] by 40%. We could either require $P=2.1$ or require $a_0^-=37.10^{-30}$ cm²/sr, or we could make some combination of these changes. We believe that such a 40% increase in the value of Pa_0^- is definitely ruled out by the experimental results given in (a) and (c) above.

Also, recent scattering results appear further to increase the discrepancy. A detailed analysis by Barnes et al.[7] of the scattering data below 150 Mev, including π^- results at 30 Mev and 40 Mev, suggests the values (1) above. The resulting value of (a_1-a_3) is quite inconsistent with the photoproduction data (5).

These values (1) and (6) are obtained by fitting the s-wave phase shifts α_1 and α_3 by formulas of the type

$$\alpha_1/k=a_i+b_ik^2+c_ik^4+\cdots, \quad (i=1,3) \quad (7)$$

where a_i are the scattering lengths and b_i and c_i are constants (k is the pion c.m. momentum). Cini et al.[1] by using the crossing relation are led to conjecture that the extrapolation formula should be

$$\frac{\alpha_1-\alpha_3}{k}=\frac{1}{1+\omega^*/M}(a'\omega^*+b'\omega^{*3}), \quad (8)$$

where a', b' are constants and (ω^*+M) is the total c.m. energy. They determined a' and b' from charge exchange scattering data by putting $(\alpha_1-\alpha_3)/k$ equal to 0.27 at 30 Mev and again at 150 Mev. This gave $a_1-a_3=0.24$ which is within the range of values required in (d) above.

[19] See the 1958 CERN Conference (reference 4).

It is, however, clear that this suggestion of Cini et al. necessitates an appreciable change from the accepted values of *both* α_1 and α_3. As an example we take the phase shifts at 35.75 Mev and 98 Mev based on the analysis[7] of recent accurate experiments.[20] The values are $\alpha_1=0.114$ (35.75 Mev), $\alpha_1=0.134$ (98 Mev), and[21] $\alpha_3=-0.114\,k$ (at 35.75 Mev and 98 Mev). Fitting (8) to these gives $a_1-a_3=0.29$ (±0.02).

This is unacceptable and we must in fact re-examine the energy dependence of both phase shifts. In Sec. VI below we discuss briefly a phase-shift analysis of the new type using the best recent data. It gives $a_1-a_3=0.265\pm0.007$. A further correction to the scattering lengths, which is discussed in Sec. IV, reduces this value by 0.02. The agreement with the photoproduction results is now satisfactory.

III. PION-PION INTERACTION

The data of Barnes et al.[7] can be fitted with the curves (1) above. We now consider whether on the basis of these values, any simple deductions can be made about the nature of the interaction between the pion and the real nucleon at low energies. The real nucleon is thought of as a core where many-pion and nucleon-pair processes can occur, and around the core is a pion cloud. We might expect that if the s-wave pion-nucleon interaction is entirely due to the nucleon-pair process then it is a short-range interaction. If, however, a pion-pion interaction exists, the s-wave pion could interact with a pion in the nucleon cloud, and this interaction could have a radial extent of the order $(\hbar/\mu c)$.

Effective Range Theory

First, we consider a simple model in which the pion scattering (with given isotopic spin T) at very low energies is described by a Schrödinger equation in which we insert an effective potential having the appropriate range and depth. If we assume also that the potential is velocity independent, the usual effective range expansion is valid at low energies, i.e., the phase shifts α obey

$$k\cot\alpha=(1/a)+\tfrac{1}{2}r_ek^2+\cdots, \quad (9)$$

where r_e is a measure of the range of the potential and a is the scattering length. Also

$$\tfrac{1}{2}r_e=\int_0^\infty \{[\bar{u}(r)]^2-[u(r)]^2\}dr, \quad (10)$$

where $u(r)$ is the actual s-wave solution, and

$$\bar{u}(r)=1+r/a.$$

[20] The values at 98 Mev are due to D. N. Edwards, S. G. F. Frank, and J. R. Holt [Proc. Phys. Soc. (London) **73**, 856 (1959)]. They are included in Barnes et al. analysis.

[21] This value of α_3 is used by Barnes et al. (reference 7). See also the 1958 CERN Conference Report, reference 4, p. 42, where a similar value is accepted.

The solution $u(r)$ is normalized so that $u \rightarrow \bar{u}$ as $r \rightarrow \infty$.

At low energies (9) gives

$$\alpha/k = a - \tfrac{1}{2}a^2 r_e k^2 + \cdots \quad (\text{for } |a| \ll 1). \tag{11}$$

Thus, from (1) we might deduce that the effective range in the isotopic state $T=\frac{1}{2}$ is of the order $r_e \simeq 5$ units. If this argument were valid, it would prove that a pion-pion force was present.[22] It can be seen from (10) that a potential which spreads over a radius of the order of unity could indeed give $r_e \simeq 5$.

Similar deductions could be made for π^- scattering without using charge independence. In this case our model has a potential $V(r)$ which includes an imaginary part to produce the inelastic scattering. The new scattering length a_- is complex, the imaginary part giving a measure of the total absorption of the π^- mesons by the proton. The imaginary and real parts of the potential are (at zero energy) related in much the same way. The same effective range formulas (9) and (10) can be deduced, but now α, a, and r_e are complex. Again this analysis of the data of Barnes et al. would show (if it were valid) that a π–π force is present.

Velocity Dependent Forces

The difficulty of using the above method to detect effects of a π–π interaction is that simple short-range velocity dependent interactions also may produce a strong dependence of α/k on k^2. This effect is particularly important when the scattering length is small (in terms of the natural unit of length).

From Schrödinger's equation we deduce that if the potential varies with the kinetic energy, the effective range formula becomes

$$k \cot\alpha = (1/a) + k^2 \int_0^\infty (\bar{u}\bar{u}_0 - uu_0)dr + \frac{2\mu}{\hbar^2}\int_0^\infty (V - V_0)uu_0 dr, \tag{12}$$

where a is the scattering length. $V(r)$ and $V_0(r)$ are the scattering potentials at energies $E=\hbar^2k^2/2\mu$ and $E=0$, respectively. $u(r)$, $u_0(r)$ are the corresponding solutions of the Schrödinger equation. Also $\bar{u} = \sin(kr+\alpha)/\sin\alpha$, $\bar{u}_0 = 1 + r/a$.

We obtain a general idea of the contribution of the last term in (12) by considering a square well potential V_0. The radius of the well b and its depth $(-V_0)$ are related by $(-V_0) \simeq 3\hbar^2 a/2\mu b^3$ provided $|a|/b$ is not large. Let V be a square well of the same radius b, having depth $(-V) = (-V_0)(\omega/\mu)$ where ω is the pion's energy.

The effect will be most marked for an attractive potential. Considering the $T=\frac{1}{2}$ state and remembering that $0 < a \ll 1$, a rough calculation shows that the last term in (12) is approximately

$$(a/b^2)(1+b/a)^2(\omega - \mu)/\mu.$$

For low energies this becomes $(k^2/2\mu^2)(a/b^2)(1+b/a)^2$. For a short-range force we put $b \simeq a$ giving $2k^2/a\mu^2$. On account of the small value of a this is a large term, therefore, unless we are sure that the potential is independent of velocity, we cannot deduce from the α_1 phase-shift values in (1) above that the range of the s-wave pion-nucleon interaction is large.

(For nucleon-nucleon scattering any velocity dependence of the potential does not have such a noticeable effect on the effective range formula because the scattering lengths in that case are large.)

Causality Condition

The range of the s-wave pion nucleon interaction has also been determined[23] by using the causality condition. For nonrelativistic potential scattering it is easy to deduce that the s-wave phase shift obeys[24]

$$d\alpha/dk > -R,$$

where the potential vanishes outside a sphere of radius R. By extending this method to relativistic potential scattering, Goebel, Karplus, and Ruderman[23] deduce from the old data that for low-energy s-wave pion scattering $R > 0.11(\hbar/\mu c)$.

Unfortunately, the method which is used is based on a form of the scattering matrix $S(k)$ which has not been shown to be valid when the effective scattering potential is velocity dependent. Therefore, we cannot regard as firm any conclusion which is drawn by applying the causality condition argument to the data in Eq. (1).

IV. COULOMB CORRECTIONS

The phase shifts for $\pi^\pm$ scattering are obtained by first removing the pure Coulomb scattering from the observed differential cross sections. This is done by the method of van Hove[25] on incorporating the relativistic and magnetic moment corrections of Solmitz.[26] From the remaining differential cross section we deduce the "observed" phase shifts α^0. These phase shifts give the nuclear scattering phase shifts α^N plus a small correction term.[27] van Hove[25] deduces the nuclear phase shift α^N from the observed phase shift α^0 by assuming that there is a sphere of radius r_0, such that for $r < r_0$ the Coulomb interaction can be ignored, and for $r > r_0$ the nuclear interaction can be ignored. He joins the nuclear wave function, which is valid for $r < r_0$, to the Coulomb wave function which is valid for $r > r_0$. The small correction $(\alpha^N - \alpha^0)$ arises from the interference between

[22] Notice that the nucleon core radius is $\simeq 0.2$ unit.

[23] C. J. Goebel, R. Karplus, and M. A. Ruderman, Phys. Rev. **100**, 240 (1955).

[24] E. P. Wigner, Phys. Rev. **98**, 145 (1955).

[25] L. van Hove, Phys. Rev. **88**, 1358 (1952).

[26] F. T. Solmitz, Phys. Rev. **94**, 1799 (1954).

[27] For example, at 41.5 Mev, $\alpha_3{}^N - \alpha_3{}^0 = -0.005$ radian (using $r_0 = 1$).

the Coulomb scattering in the region $r>r_0$ and the nuclear scattering. We could call it an *outer* Coulomb correction.

It is customary to take $r_0=\hbar/\mu c$, (i.e., $r_0=1$), on the grounds that this is roughly the size of the nucleon. However, at low energies (e.g., 30 or 40 Mev) the nuclear phase shifts α_N obtained in this way[7] vary by small but appreciable amounts as r_0 is changed from 1.0 to 0.5. We, therefore, examine the Coulomb correction at low energies in more detail. Our results show how the greater part of this variation with r_0 can be removed; they also lead to a further correction to the scattering lengths.

Inner Coulomb Correction

For $r<r_0$ the Coulomb field does not vanish. At $r=r_0$ the Coulomb potential in the π^++p case is 1 Mev, and it rises as r decreases. If the proton's charge were concentrated at $r=0$, the potential would be 2 Mev at $r=\frac{1}{2}$. As the proton's charge is spread out,[28] the potential is less than 2 Mev at $r=\frac{1}{2}$, and it reaches a finite value[29] of a few Mev at $r=0$. (We ignore the structure of the pion.)

In van Hove's method this inner Coulomb potential is lumped together with the purely nuclear interaction. For π^++p it gives a small extra repulsion and for π^-+p a small extra attraction. Thus, in each case it enhances slightly the magnitude of the scattering length over that due to the pure pion-nucleon interaction alone. These latter values—which we may call the *strictly nuclear values*—are those which obey the charge independence relations.

We now estimate the size of these effects. Consider first π^++p scattering. We only require the scattering length, so it is sufficient to work at low energies, where the scattering can be described by a Schrödinger equation (van Hove's analysis also uses the Schrödinger equation),

$$(d^2u/dr^2)+[k^2-(2\mu/\hbar^2)V(r)]u=0. \tag{13}$$

We assume that at very low energies a potential can adequately represent the nuclear scattering.

The solution of (13) for $r<r_0$ can be joined on to the Coulomb solution for $r>r_0$ by using the method of Landau and Smorodinsky.[30] This gives

$$\frac{1}{a}=\frac{1}{R_0}\left\{\ln\left(\frac{r_0}{R_0}\right)+2C\right\}+\frac{f_0}{1-f_0}\left(\frac{1}{r_0}+\frac{1}{R_0}\right),$$

where $R_0=\hbar^2/2\mu e^2\simeq 70$ units and

$$f_0=\frac{r_0}{u(r_0)}\frac{du}{dr}\bigg|_{r=r_0}.$$

($C=0.57\cdots$ is Euler's constant.) A small change $\Delta V(r)$ in the potential in the region $r<r_0$ gives rise to a change Δf_0 in f_0

$$\Delta f_0=\frac{r_0}{u^2(r_0)}\frac{2\mu}{\hbar^2}\int_0^{r_0}\Delta V(r_0)u^2(r)dr. \tag{14}$$

For weak scattering $f_0\simeq 1$, and the related change in the scattering length is $\Delta a=-r_0\Delta f_0$. When $f_0\simeq 1$, a fairly good approximation to (14) is

$$\Delta a\simeq -\frac{2\mu}{\hbar^2}\int_0^{r_0}\Delta V(r)r^2dr. \tag{15}$$

Corrected Scattering Lengths

Using (15) we evaluate the inner Coulomb correction for π^++p by giving $\Delta V(r)$ the constant value $\Delta V=+1.5$ Mev. This gives a contribution to the scattering length $\Delta a_+=-0.007$ unit. The value we have used for ΔV seems reasonable, bearing in mind that (a) the Coulomb potential stays finite as $r\to 0$, (b) the large values of r are heavily weighted in (15).

We must *subtract* Δa_+ from the usually quoted values of a_3 [such as (6) above] to get the strictly nuclear (or mesonic) scattering length. Reducing r_0 will reduce the magnitude of Δa_+. In fact, the variation of Δa_+ with r_0 will cancel out the variation of van Hove's correction with r_0. He gets, to the first order in e^2/k, (using units $\hbar=c=\mu=1$)

$$\alpha_3{}^0-\alpha_3{}^N=\frac{e^2}{k}[C+\ln(2kr_0)-\mathrm{Ci}(2kr_0)\cos(2\alpha_3{}^N)+\mathrm{si}(2kr_0)\sin(2\alpha_3{}^N)]. \tag{16}$$

Here

$$\mathrm{Ci}(x)=-\int_x^\infty\frac{\cos x}{x}dx,\quad \mathrm{si}(x)=-\int_x^\infty\frac{\sin x}{x}dx.$$

In the range $0.5<kr_0\leqslant 1$, Eq. (16) gives approximately

$$(1/k)(\alpha_3{}^0-\alpha_3{}^N)\simeq e^2r_0{}^2=0.007\times r_0{}^2. \tag{17}$$

Hence the two Coulomb corrections happen to cancel[31] for $r_0=1$. Using (14) or (15) and the above rough method of estimating Δa_+, it is clear that the variation of Δa_+ with r_0 will almost cancel the variation with r_0 of van Hove's correction.

The inner Coulomb correction for π^-+p scattering is made by using a complex potential $V(r)$ in (13) to describe the nuclear scattering. This gives a complex

[28] See, for example, W. H. K. Panofsky's report in the 1958 CERN Conference, reference 4. An exponential charge distribution with a rms radius of 0.8×10^{-13} cm fits the electron scattering data.

[29] If the charge e were spread uniformly over a sphere of radius r_0, the central potential would be $3e/2r_0$. For the actual charge distribution the central potential is greater.

[30] See J. M. Blatt and J. D. Jackson, Revs. Modern Phys. **22**, 77 (1950) for an account of this method. It has been modified slightly here to deal with weak scattering (i.e., $f_0\simeq 1$).

[31] We could choose other (artificial) radial distributions of the proton's charge which would give Δa_+ a different magnitude from van Hove's correction. However, the variations with r_0 will remain equal.

scattering length a_-. Re a_- is then the scattering length for the elastic scattering. Im a_- gives the s-wave reaction cross section (i.e., the charge exchange cross section) $\sigma_r=(4\pi/k)\,\mathrm{Im}\,a_-$. Re$V(r)$ can be a short-range velocity independent potential, but Im $V(r)$ must be proportional to v_0 the velocity of the neutral pion in charge exchange scattering. Re $V(r)$ and Im $V(r)$ can then be chosen to give the correct behavior of the π^- wave function in low-energy scattering.

The inner Coulomb potential gives a real ΔV which is opposite in sign to that used for π^+ scattering. By (15) the inner Coulomb correction for elastic scattering is $\Delta a_-\simeq 0.007$. Equation (15) gives zero inner Coulomb correction for charge exchange scattering (because ΔV is real). A more careful examination of the correction for charge exchange scattering can be made with the aid of Eq. (14). If we assume that the charge exchange process is of short range (for example, if it only occurs for $r<0.3$) then the ratio Im $[u(r)]^2/\mathrm{Re}\,[u(r)]^2$ is almost constant for those values of r for which the integrand is appreciable. Therefore, to a good first approximation, the inner Coulomb correction to charge exchange scattering is zero. van Hove finds that the observed s-wave charge exchange amplitude is

$$(\sqrt{2}/3)[\exp(2i\alpha_3{}^N)-\exp(2i\alpha_1{}^N)](1+O(e^2)].$$

Therefore, his (outer) Coulomb correction is also negligible for charge exchange scattering.

From the scattering data and charge independence we infer the charge exchange cross section using $(a_1-a_3)=\frac{3}{2}(\mathrm{Re}\,a_--a_+)$. From this value we should *subtract* $\Delta(a_1-a_3)=\frac{3}{2}(\Delta a_--\Delta a_+)=0.02$, before comparing with the value of (a_1-a_3) deduced from the photoproduction data and the Panofsky ratio. (At present this order of magnitude is just greater than the limits imposed by the experimental errors.)

V. DISPERSION RELATION FOR LOW-ENERGY PHASE SHIFTS

We derive two dispersion relations which relate the s- and p-wave phase shifts at low energies; these are of value in selecting or rejecting certain sets of phase shifts which have been proposed.

The pion-nucleon forward scattering dispersion relations are usually written as follows. Let $D_\pm(\omega_L)$ be the real parts of the forward scattering amplitude in the lab system, and ω_L, k_L the pion (lab) energy and momentum. Putting $\hbar=c=1$ we have[32]

$$\begin{aligned}D_\pm(\omega_L)-D_\pm(\mu)\\ &=\pm\tfrac{1}{2}(\mu-\omega_L)[D_-(\mu)-D_+(\mu)]-\frac{2f^2k^2}{(\mu^2/2M)\mp\omega_L}\\ &\quad+\frac{k_L{}^2}{4\pi^2}\int_\mu^\infty\frac{d\omega'}{k'}\left(\frac{\sigma_+(\omega')}{\omega'\mp\omega_L}+\frac{\sigma_-(\omega')}{\omega'\pm\omega_L}\right).\end{aligned}\quad(18)$$

[32] For details and references see J. Hamilton, Phys. Rev. **110**, 1134 (1958).

f^2 is the coupling constant and $\sigma_\pm$ are the total cross sections. The right-hand side of (18) vanishes when $\omega_L=\mu$ and for small k_L it behaves as $k_L{}^2$. We shall evaluate the coefficient of $k_L{}^2$.

It is desirable to express our results in terms of the center-of-mass scattering amplitudes $D_\pm{}^B$, and the center-of-mass momentum k. To order k^2 (with $\mu=1$)

$$D_\pm{}^B(\omega)=D_\pm(\omega_L)[(1+1/M)^{-1}-k^2/2M].$$

Equation (18) gives an expansion of the form

$$D_\pm{}^B(\omega)=D_\pm{}^B(\mu)+C^\pm k^2+\cdots,$$

where we have to find the constants $C^\pm$.

First we notice that the right-hand side of (18) is not very sensitive to the values we use for the scattering lengths a_1 and a_3. The first term contributes $\mp\frac{1}{6}(a_1-a_3)\times(1+1/M)^2k^2$. Hence changing (a_1-a_3) by as much as 0.03 will only alter $C^\pm$ by 0.005. The scattering lengths also enter the dispersion integrals in (18) via the values we assume for $\sigma_\pm(\omega)$ at very low pion energies. For example, in this way, a change of 0.03 in a_3 will change C^+ by 0.003 and C^- by a lesser amount. Below, we give the results for two sets of scattering lengths; they are almost identical.

We have estimated the contribution to the dispersion integral arising from the fact that $k\sigma(\omega)\to$ const as $k\to 0$. This singularity in $\sigma_-(\omega)$ at threshold is due to the inelastic processes $\pi^-+p\to\pi^0+n$, $\pi^-+p\to\gamma+n$. For charge exchange scattering $k\sigma(\omega)$ must be continued into the unphysical region as far as the pion energy $\mu-\Delta m$ where $\Delta m=3.3$ Mev. For the radiative process the continuation into the unphysical region extends down to the neutron pole.[32] In the resulting integrals we assume a smooth continuation of $k\sigma(\omega)$ into the unphysical region, and then take the principal value at $\omega=\mu$. This has the consequence that a negligible error[33] is introduced if we neglect the rise in $\sigma_-(\omega)$ with $(1/k)$ as $\omega\to\mu$ and take $\sigma_-(\omega)\to$ constant as $\omega\to\mu$, provided we also neglect the *extra* contributions to the dispersion integral coming from the unphysical region $\omega<\mu$ (i.e., the neutron pole is the only unphysical region term included).

The coupling constant f^2 appears explicitly in (18) and contributes to $C^\pm k^2$ in the form

$$\pm\frac{(1+1/M)}{(1\mp 1/2M)}2f^2k^2.\quad(19)$$

A change of 0.01 in f^2 therefore gives a contribution which should not be ignored.

Phase-Shift Relation

Using $f^2=0.08$, $D_+{}^B(\mu)=-0.105$, $D_-{}^B(\mu)=0.078$ (which with charge independence are equivalent to

[33] This calculation is more readily done with the unsubtracted dispersion relation.

Orear's values[34] $a_1=0.165$, $a_3=-0.105$) we get

$$D_+{}^B=a_3+0.317k^2+\cdots,$$
$$D_-{}^B=D_B{}^-(\mu)-0.020k^2+\cdots. \tag{20}$$

Assuming charge independence this gives for $T=\frac{1}{2}$

$$D_1{}^B=a_1-0.188k^2+\cdots. \tag{20a}$$

Alternatively with $f^2=0.08$, $a_1=0.178$, $a_3=-0.087$, we get (assuming charge independence)

$$D_+{}^B=a_3+0.322k^2+\cdots,$$
$$D_1{}^B=a_1-0.197k^2+\cdots. \tag{21}$$

For low energies we write the phase shifts in the form

$$\alpha_1/k=a_1+c_1k^2,\quad \alpha_3=a_3+c_3k^2,$$
$$\alpha_{11}=c_{11}k^3,\quad \alpha_{13}=c_{13}k^3,$$
$$\alpha_{31}=c_{31}k^3,\quad \alpha_{33}=c_{33}k^3.$$

Working[35] to the nearest 0.005 we find (21) gives (with $f^2=0.08$)

$$c_1+c_{11}+2c_{13}=-0.195,$$
$$c_3+c_{31}+2c_{33}=0.32. \tag{22}$$
$$\text{(in units } \hbar=\mu=c=1).$$

As an example of the use of (22) we apply it to the phase-shift values of Barnes et al.[7] First we note that these authors have used $f^2=0.087$ (in their Chew-Low plot for α_{33}, etc.). We adjust (22) to this value of f^2, giving

$$D_+{}^B=a_3+0.34k^2,\quad D_1{}^B=a_1-0.23k^2. \tag{23}$$

The p-wave phase shifts which fit the low-energy scattering data are given by[36] $c_{11}=-0.02$, $c_{13}=-0.07$, $c_{31}=-0.04$. Also

$$(k^3/\omega^*)\cdot\tfrac{4}{3}f^2\cot\alpha_{33}=1-\omega^*/2.17 \quad\text{so}\quad c_{33}=0.22.$$

Relation (23) now predicts $c_1=-0.07$, $c_3=-0.06$. If we use $f^2=0.08$ and keep the same values of c_{11}, c_{13}, c_{31}, we require $c_1=-0.04$, $c_3=-0.06$.

We deduce that (i) the p-wave phase shifts α_{31}, α_{33} of Barnes et al.[7] are noticeably inconsistent with their straight line plot for α_3 ($\alpha_3/k=-0.114$). (ii) their values of α_{11}, α_{13} are somewhat inconsistent with their suggested value for α_1 ($\alpha_1/k=0.205-0.09k^2+\cdots$).

Another example is provided by the Chiu-Lomon phase shifts.[6] These authors suggest $c_{11}=-0.014$, $c_{13}=+0.018$, $c_{31}=-0.018$, $c_{33}=0.200$ with $f^2=0.08$. Inserting in (22) gives $c_1=-0.22$, $c_3=-0.06$. This large value of $|c_1|$ is in disagreement with the values these authors use for α_1. The value of c_3 is inconsistent with their straight line plot for α_3.

The values of Barnes et al. are less in disagreement with relation (22) than those of Chiu and Lomon.[37] Neither set is consistent with the straight line plot for α_3 (i.e., $c_3=0$). The phase-shift curves derived in Sec. VI are in reasonable agreement with relation (22).

VI. S-WAVE PHASE-SHIFT CURVES

Chew, Low, Goldberger, and Nambu[9] by using the finite momentum transfer dispersion relations, expanding in terms of μ/M and keeping only terms up to order μ/M find that the s-wave phase shifts obey the relations

$$\frac{\sin 2\alpha_1-\sin 2\alpha_3}{2k\omega^*}\,\frac{1+\omega^*/M}{1+\mu/M}=f(\omega^{*2}),$$
$$\frac{\sin 2\alpha_1+2\sin 2\alpha_3}{2k}\,\frac{1+\omega^*/M}{1+\mu/M}=g(\omega^{*2}). \tag{24}$$

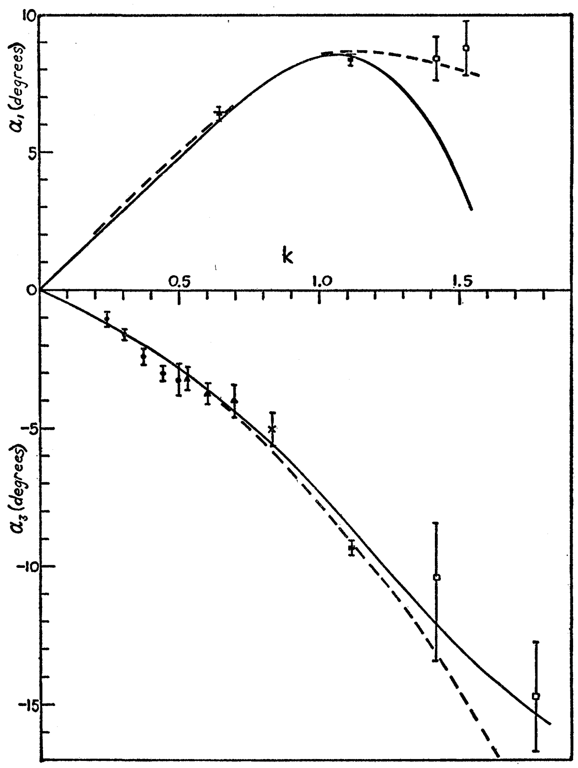

FIG. 2. A plot of the s-wave phase shifts α_1 and α_3 against c.m. momentum. ●—Fischer and Jenkins, reference 38; ▲—from data computed by Barnes et al., reference 7; ✕—Bodansky, Sachs, and Steinberger, reference 40; □—Chiu and Lomon, reference 6; ■—Edwards et al., reference 41. (The bar through the 35-Mev α_1 point is the new mean value of α_1 which is consistent with the broken line for α_3. The corresponding value at 98 Mev is shown by the bar above the point; it is close to the α_1 broken line.)

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[34] J. Orear, Nuovo cimento 4, 856 (1956).

[35] Terms $-\frac{2}{3}a_1{}^3k^2$, $-\frac{2}{3}a_3{}^3k^2$ are negligible to the order of accuracy used in Eq. (22).

[36] These are satisfactory for the 35-Mev phase shifts.

[37] A. Kanazawa, T. Sakuma, and S. Furui, Progr. Theoret. Phys. (Kyoto) 21, 856 (1959), have also found that the set of Chiu and Lomon do not agree with a dispersion relation. They use an integrated dispersion relation whereas (22) comes from a differentiated dispersion relation. *Note added in proof.*—It has been brought to our notice that D. A. Geffen, Phys. Rev. **112**, 1370 (1958), has used a relation like ours to show that α_1 and α_3 cannot both be straight lines.

f and g are functions to be determined and (ω^*+M) is the total energy in the c.m. system. It is hoped that, at least for low energies, f and g can be expressed in convergent power series about $\omega^*=1$, so that

$$\begin{aligned} f(\omega^{*2})&=(a_1-a_3)-b(\omega^{*2}-1)+d(\omega^{*2}-1)^2+\cdots,\\ g(\omega^{*2})&=(a_1+2a_3)-c(\omega^{*2}-1)+e(\omega^{*2}-1)^2+\cdots, \end{aligned} \tag{25}$$

where b, c, d, e, $\cdots$ are constants.

In Fig. 2 we show the results of two solutions of this type. The experimental points used in making the first fit are, for π^+: up to 24 Mev, Fischer and Jenkins[38]; 24.8 Mev, Miller and Ring[39]; 31.5 and 41.5 Mev, Barnes et al.[7]; 58 Mev, Bodansky, Sachs, and Steinberger.[40] For π^-: 35.75 Mev, Barnes et al.[7]; 98 Mev, Edwards, Frank, and Holt.[41]

For the second solution we have in addition used other points at higher energies (the 98-Mev value of Edwards et al.,[41] plus results from the analysis of Chiu and Lomon[6] and the data of Ashkin et al.[42] at 150, 170, and 220 Mev). For convenience and ease of comparison with other authors, the phase-shift values we have used contain van Hove's Coulomb correction, but they do not as yet contain the inner Coulomb correction of Sec. IV. We can make that correction at the end of the analysis.

We assumed Barnes et al.[7] values for the p-wave phase shifts and then found solutions of the type (24), (25) which gave the best fit to the scattering data. [As Barnes values of α_1 were computed using $\alpha_3=-0.115k$, we have adjusted the α_1 values so as to keep $(2\alpha_1+\alpha_3)$ constant while using our solid curve for α_3. Changing to the dashed curve for α_3 improves the agreement between the α_1 values and the curve.]

[38] G. E. Fischer and E. W. Jenkins, Phys. Rev. **116**, 749 (1959).
[39] Miller and Ring, Rochester University (to be published).
[40] D. Bodansky, A. M. Sachs, and J. Steinberger, Phys. Rev. **93**, 1367 (1954).
[41] We are indebted to these authors for permission to quote this result. It was not available when the first solution was found.
[42] J. Ashkin, J. P. Blaser, F. Feiner, and M. O. Stern, Phys. Rev. **101**, 1149 (1956).

Our first solution—shown by the solid curves—is obtained by using *four* constants: $a_1-a_3=0.255$, $a_1+2a_3=0$, $b=0.05$, $c=0.095$. The α_3 curve is also a fairly good fit to the results above 58 Mev; the α_1 curve obviously goes wrong above 98 Mev.

Our second solution—shown by the broken curves—is a fit to all the accurate data up to 170 Mev. Its uses *six* constants: $a_1-a_3=0.265$, $a_1+2a_3=0.005$, $b=0.065$, $c=0.120$, $d=0.008$, $e=0.010$. This solution clearly gives somewhat better agreement with the data than the first solution. Using the second solution, the scattering lengths and their estimated errors are

$$a_1=0.178\pm0.005,\quad a_3=-0.087\pm0.005. \tag{26}$$

Also, for small k^2 we have

$$\alpha_1/k=0.178-0.01k^2,\quad \alpha_3/k=-0.087-0.07k^2. \tag{27}$$

In (27) the coefficients (c_1 and c_3) of the terms in k^2 are in tolerably good agreement with the values, -0.04 and -0.06, respectively, which we deduced from the dispersion relation in Sec. V. This is further evidence in favor of our phase-shift solution.

Finally, we notice that (26) gives $a_1-a_3=0.265\pm0.007$. Now subtracting the inner Coulomb contribution 0.02 (see Sec. IV) we deduce *from the scattering data* that the strictly mesonic interaction gives $a_1-a_3=0.245\pm0.007$. This is the quantity which should be compared with the value (5) above ($a_1-a_3=0.245\pm0.01$) deduced from the *photoproduction threshold data*. All the low-energy pion data are, therefore, in good agreement with the charge independence hypothesis.