Higher Resonances in Pion-Nucleon Interactions*†

RONALD F. PEIERLS‡
Lawrence Radiation Laboratory, University of California, Berkeley, California

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The recent experiments on pion-nucleon scattering and photoproduction at energies up to about 1.2 Bev are examined from a phenomenological standpoint. The most useful information seems to come from the photoproduction angular distribution and polarization results. The data seem to imply the existence of two "resonances" in the $J=\frac{3}{2}$, odd parity and $J=\frac{5}{2}$, even parity states at photon energies of about 750 and 1100 Mev. These assignments satisfy several nontrivial consistency requirements. The same two states are also a consistent assignment for the observed scattering resonances at 615 Mev and 950 Mev. A qualitative model is proposed to explain these resonances as consequences of the 33 resonance acting in two-meson final states; their isotopic spin dependence seems to require some additional assumptions. Finally, the relation between the photoproduction and scattering phases in the presence of strong inelastic scattering is examined.

INTRODUCTION

OVER the past two years there has been a considerable amount of new experimental data on interactions of the pion-nucleon system at center-of-mass (c.m.) energies corresponding to pions incident on nucleons at rest, with kinetic energies from 300 Mev up to greater than 1 Bev. The reactions that have been at least partially studied include the following:

$$\pi+N \rightarrow \begin{cases} \pi+N \\ 2\pi+N \end{cases}$$

$$\gamma+N \rightarrow \begin{cases} \pi+N \\ 2\pi+N, \end{cases}$$

each in several charge states. The present state of the theory of strong interactions makes it impossible to calculate the expected cross sections on the basis of any formal theory, and so we have to use the qualitative features of the results to try to obtain information about the nature of the interactions.

Our understanding of the situation below about 300 Mev is satisfactory up to a point.[1] In this energy region all processes are dominated by the 33 resonance, whose existence can be related to the odd intrinsic parity of the pion and the requirements of charge independence, and whose shape and position can be fitted by the introduction of two parameters. However, its existence was established first on phenomenological grounds alone. If we assume one state to dominate the scattering, the phase shift is known from the energy dependence of the cross section, because elastic scattering is the only process allowed. The elastic-scattering angular distribution determines the angular momentum and parity; the charge dependence fixes the isotopic spin. The photoproduction results can be analyzed similarly, and the photoproduction matrix element must have a phase equal to the scattering phase shift in the corresponding state. Thus we are provided with a consistency check on the interpretation of the scattering. This is not quite the chronological order in which the analysis was carried out,[2] but would probably be the most satisfactory argument for the existence of the 33 resonance if all the present data were available.

In this paper we try to find a similar phenomenological program to cover the higher energy range in which many experiments have been done that suggest, at first glance, the existence of more resonances. In Sec. I we outline the experimental data. Section II contains a discussion of the general problem of analysis of photoproduction results. In Sec. III we give the results of the analysis for the two new resonances. Section IV contains a discussion of the scattering, and a possible qualitative explanation of the existence of the higher resonances. In Sec. V we consider the question of the isotopic spin dependence which is not covered by the model. Section VI contains a discussion of the relation between the photoproduction and scattering phases in the presence of strong inelastic scattering.

I. EXPERIMENTAL RESULTS

Figure 1 shows the energy dependence of the total cross sections for π^+ and π^- incident on protons, and for the two photoproduction processes

$$\gamma+p \rightarrow \begin{cases} \pi^++n \\ \pi^0+p. \end{cases}$$

It should be emphasized that all the results mentioned in this paper are only approximate representations of the experimental data, some of which, particularly

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† Based in part on a thesis submitted to the Graduate Faculty of Cornell University in June, 1959, in partial fulfillment of the requirements for the Ph.D. degree.
‡ Present address: Department of Mathematical Physics, The University, Birmingham, England.
[1] Geoffrey F. Chew, in Handbuch der Physik (Springer-Verlag, Berlin, to be published), Vol. 43, issued as a University of California Radiation Report UCRL-1957-45 (1957). Geoffrey F. Chew, University of California Radiation Laboratory Report UCRL-8670, January, 1959 (unpublished).
[2] See, for example, H. A. Bethe and F. deHoffman, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1956), Vol. II.

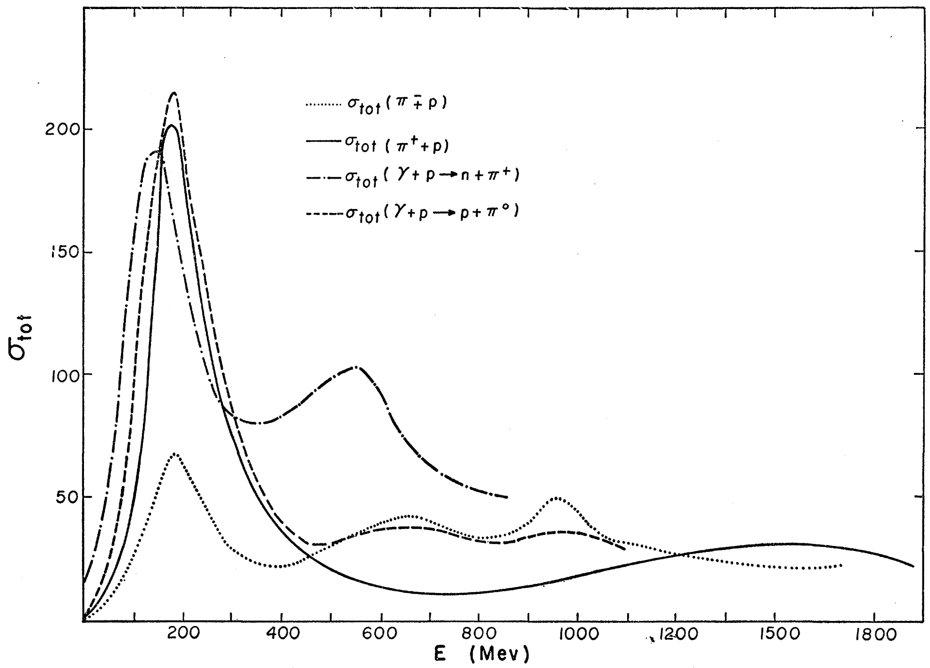

FIG. 1. Total cross sections for π^+ and π^- incident on protons, and for the two charged photoproduction processes $\gamma+p\to n+\pi^+$ and $\gamma+p\to p+\pi^0$. Approximate representation of a variety of experimental results quoted in the text. The energy E, is the laboratory kinetic energy of the incident pion or, in the photoproduction cases, of an incident pion needed to produce the same center-of-mass energy.

at higher energies, has poor statistical accuracy as yet. However, the qualitative behavior is probably correctly shown. Now let us consider the separate processes in more detail.

A. $\pi+p$ Cross Sections

Burrowes et al., have measured the absorption of π^- and π^+ by hydrogen at energies from about 470 to 1200 Mev, with the results shown in Fig. 1.[3] We note that the π^+ cross section falls off above the 33 resonance, has a minimum at about 650 Mev, and then steadily rises. Other experiments indicate a broad peak in this cross section at high energies with a maximum at about 1.5 Bev.[4] The π^- cross section shows two clear peaks at 600-650 Mev and at about 950 Mev. Using a bubble chamber, Crittenden et al., have measured the π^-+p cross sections in the same energy region.[5] The direct elastic-scattering cross section seems to vary with energy following the total cross section with maxima of about 20 mb at each peak. The inelastic cross section seems to show the second peak much more strongly than the first. The charge-exchange elastic cross section falls off steadily above the 33 resonance up to about 800 Mev, above which there are no experimental results as yet.

The differential elastic cross section throughout the energy range discussed above has a large peak in the forward direction, a minimum in the region of 90°, and a second peak in the backwards hemisphere.[6] The relative magnitudes and positions of these peaks and minima vary with energy.

B. Single-Pion Photoproduction

Work at Cornell University[7–9] and California Institute of Technology[10,11] indicates that the total cross section for positive-pion photoproduction is as shown in Fig. 1, showing a second maximum at an energy somewhat below that corresponding to the peak in π^- scattering, and apparently rising towards a third peak at 1100-Mev photon energy. The total cross section for production of π^0 has a small second peak at an energy corresponding to the scattering peak, and another small peak at 1100 Mev. The angular distributions of the photoproduced pions analyzed in powers of $\cos\theta$ are shown in Figs. 2 and 3, where

$$d\sigma^{+,0}/d\Omega=\sum_n A_n{}^{+,0}\cos^n\theta.$$

Again, these curves are only approximate; in particular, the statistical uncertainties in $A_3{}^0$ and $A_4{}^0$ are large.[9]

[3] H. C. Burrowes, D. O. Caldwell, D. H. Frisch, D. A. Hill, D. M. Ritson, R. A. Schluter, and M. A. Wahlig, Phys. Rev. Letters **2**, 119 (1959).

[4] R. Cool, O. Piccioni, and D. Clark, Phys. Rev. **103**, 1082 (1956).

[5] R. R. Crittenden, J. H. Scandrett, W. D. Shepherd, W. D. Walker, and J. Ballam, Phys. Rev. Letters **2**, 121 (1959).

[6] *1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958), p. 66.

[7] M. Heinberg, W. M. McClelland, F. Turkot, W. M. Woodward, R. R. Wilson, and D. M. Zipoy, Phys. Rev. **110**, 1211 (1958); J. W. DeWire, H. E. Jackson, and R. Littauer, Phys. Rev. **110**, 1208 (1958); P. C. Stein and K. C. Rogers, Phys. Rev. **110**, 1209 (1958).

[8] H. E. Jackson, Cornell University (private communication).

[9] K. Berkelman, Cornell University (private communication); F. Turkot, Cornell University (private communication).

[10] F. P. Dixon and R. L. Walker, Phys. Rev. Letters **1**, 142, 458 (1958).

[11] J. I. Vette, Phys. Rev. **111**, 622 (1958).

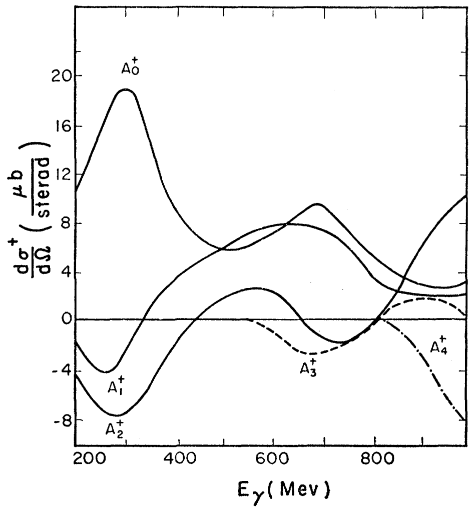

FIG. 2. Angular coefficients for the reaction $\gamma+p \to n+\pi^+$. Adapted from curves provided by Turkot.[9]

Measurements of the polarization of the recoil proton in π^0 production have been made by Stein[12] and by Connolly and Weill.[13] At 700-Mev photon energy, the polarization is found to be 0.59 ± 0.12 along the negative normal to the production plane, in the sense to be discussed below. At 550 Mev the polarization is smaller (0.35) in the same direction, while at 900 Mev it is very small and in the opposite direction, though in fact statistically consistent with zero.

C. Double-Pion Photoproduction

Double-pion photoproduction has been measured by Sellen et al.,[14] and by Bloch and Sands.[15] At present the statistics are not very extensive. The total cross section rises very sharply, starting somewhat above threshold, to a maximum at about 550-Mev photon energy. The differential cross sections are rather difficult to interpret since there are so many parameters involved; however there is some indication that in the reaction,

$$\gamma+p \to p+\pi^+ +\pi^-,$$

the Q values of the $(p\pi^+)$ pair are not distributed statistically, but favor values near 170 Mev.

II. THE PHENOMENOLOGICAL ANALYSIS OF PHOTOPRODUCTION

The most informative data is that from the single-pion photoproduction. Phenomenologically the transition matrix for this process may be written as a sum of contributions from all possible multipole transitions α, where α stands for the multipole order l_α, the total

[12] P. C. Stein, Phys. Rev. Letters **2**, 473 (1959).
[13] P. L. Connolly and R. Weill, Bull. Am. Phys. Soc. **4**, 23 (1959).
[14] J. M. Sellen, G. Cocconi, V. T. Cocconi, and E. L. Hart, Phys. Rev. **113**, 1323 (1959).
[15] M. Bloch and M. Sands, Phys. Rev. **108**, 1101 (1958).

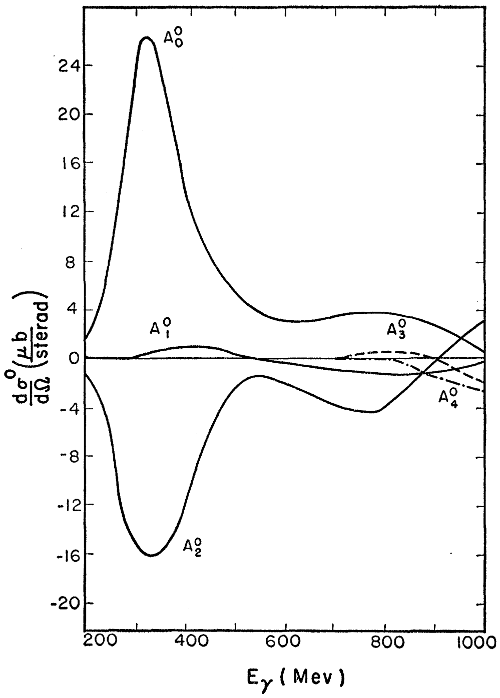

FIG. 3. Angular coefficients for the reaction $\gamma+p \to n+\pi^0$. Adapted from results prepared by Berkelman.[9] The results for $A_3{}^0$ and $A_4{}^0$ have still large statistical uncertainties.

angular momentum J_α, the parity ω_α, and the isotopic spin T_α. Each such contribution has an energy dependence specified by a complex amplitude,

$$M_\alpha e^{i\delta_\alpha},$$

where M_α, δ_α are real functions of energy. If we allow M_α to take both positive and negative values, we may restrict δ_α to lie between 0 and π. We consider pion photoproduction by unpolarized photons at an angle θ in the c.m. system. The differential cross section may then be written as

$$d\sigma/d\Omega = \sum_\alpha M_\alpha{}^2 f_\alpha(x) + \sum_{\alpha<\beta} M_\alpha M_\beta \cos(\delta_\alpha - \delta_\beta) f_{\alpha\beta}(x) \cdots \quad (1)$$

and the polarization of the recoil nucleon as $\mathbf{P}$, where

$$\mathbf{P}(d\sigma/d\Omega) = \sum_{\alpha>\beta} M_\alpha M_\beta \sin(\delta_\beta - \delta_\alpha) g_{\alpha\beta}(x) \sin\theta \hat{n} \quad (2)$$

in which $x=\cos\theta$, and $\hat{n}=(\mathbf{k}\times\mathbf{q})/|\mathbf{k}\times\mathbf{q}|$ is the unit vector normal to the production plane, where $\mathbf{k}$, $\mathbf{q}$ are the momenta of the incoming photon and outgoing pion, respectively, in the c.m. system.

We have to consider separate amplitudes for the production of π^+ and π^0 from protons. These are related by

$$M_\alpha{}^0 = (T_\alpha - 1)2^{-T_\alpha} M_\alpha{}^+; \quad T_\alpha = \tfrac{1}{2}, \tfrac{3}{2}, \quad (3)$$

TABLE I. Pion photoproduction angular distributions and polarizations.

Final state	α	β	e_1	m_1	M_1	e_2	E_1	m_2	M_2	e_3	E_2	m_3	R_1	R_0
		N_β	1	1	1	$\sqrt{3}$	1	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{\frac{3}{8}}$	$\sqrt{3}$	$\sqrt{\frac{3}{8}}$		
		N_α												
$s\frac{1}{2}$	e_1	1	1	$-2x$	$2x$	$2x$	$3x^2-1$	-2	2	$4(3x^2-1)$	$x(5x^2-3)$	$4x(5x^2-3)$	$-(1+x)$	0
$p\frac{1}{2}$	m_1	1	2	1	$1-3x^2$	$3x^2-1$	$2x$	$2x$	$x(3-5x^2)$	$4x(5x^2-3)$	$3x^2-1$	$2(1-3x^2)$	$1+x$	$1-x^2$
$p\frac{3}{2}$	M_1	1	1	$-3x$	$\frac{1}{2}(5-3x^2)$	$1-3x^2$	$-2x$	$2x(2-3x^2)$	$2x(3-2x^2)$	$4x(3-5x^2)$	$1-3x^2$	$-5+42x^2-45x^4$	$-(1+x)$	x^2-1
$p\frac{3}{2}$	e_2	$\sqrt{3}$	1	$3x$	0	$\frac{1}{2}(1+x^2)$	$2x(3x^2-2)$	$2x(2x^2-1)$	$2x(1-2x^2)$	$4x(1+x^2)$	$1-9x^2+10x^4$	$1-18x^2+25x^4$	$(1-2x)(1+x)$	$1-x^2$
$d\frac{3}{2}$	E_1	1	$-3x$	1	-4	$+2(1-3x^2)$	$\frac{1}{2}(5-3x^2)$	$3x^2-1$	$1-3x^2$	$5-42x^2+45x^4$	$2x(3-2x^2)$	$4x(5x^2-3)$	$(2-3x)(1+x)$	$3x(x^2-1)$
$d\frac{3}{2}$	m_2	$\sqrt{3}$	$3x$	-1	$2(3x^2-1)$	$4x^2$	0	$\frac{1}{2}(1+x^2)$	$-1+9x^2-10x^4$	$1-18x^2+25x^4$	$2x(2x^2-1)$	$4x(1+x^2)$	$x(1+x)$	$x(1-x^2)$
$d\frac{5}{2}$	M_2	$\sqrt{3}$	$2x$	$1-5x^2$	$2-x^2$	x^2	$5x$	$5x(1-2x^2)$	$\frac{1}{2}(1+6x^2-5x^4)$	$-1+18x^2-25x^4$	$2x(1-2x^2)$	$-19x+86x^3-75x^5$	$-x(1+x)$	$x(x^2-1)$
$d\frac{5}{2}$	e_3	$\sqrt{\frac{3}{8}}$	$8x$	$4(1-5x^2)$	$5x^2-1$	$3+x^2$	$5x(9x^2-5)$	$5x(1-5x^2)$	0	$5+6x^2+5x^4$	$19x-86x^3+75x^5$	$-22x+160x^3+150x^5$	$(3+8x-15x^2)(1+x)$	$8x(1-x^2)$
$f\frac{5}{2}$	E_2	$\sqrt{3}$	$1-5x^2$	$2x$	$-5x$	$5x(1-2x^2)$	x^2-2	x^2	$-6x^2$	$-3(1-18x^2+25x^4)$	$\frac{1}{2}(1+6x^2-5x^4)$	$1-18x^2+25x^4$	$(1+3x-5x^2)(1+x)$	$(1-5x^2)(1-x^2)$
$f\frac{5}{2}$	m_3	$\sqrt{\frac{3}{8}}$	$4(5x^2-1)$	$-8x$	$5x(9x^2-5)$	$5x(5x^2-1)$	$5x^2-1$	$-(3+x^2)$	$3(1-18x+25x^2)$	$6(5x^2-1)^2$	0	$5+6x^2+5x^4$	$(5x^2-1)(1+x)$	$(5x^2-1)(1-x^2)$
all	R_1		1	1	-1	$(1+2x)$	$-(3x+2)$	x	$-x$	$15x^2+8x-3$	$1-3x-5x^2$	$+(5x^2-1)$	$1+x$	$\cdots$
	R_0		1	$-x$	x	x	$3x^2-2$	$-x^2$	x^2	$7x^2-3$	$x(5x^2-4)$	$x(1-5x^2)$	$\cdots$	$\frac{1}{2}(1-x^2)$

where T_α is the isotopic spin corresponding to the state α. The angular momenta and parity associated with α determine the functions f_α, $f_{\alpha\beta}$, $g_{\alpha\beta}$. These can be calculated for any particular states,[16] and the results are shown in Table I for the first ten multipole transitions, as well as for two particular combinations of states which we shall discuss later. (The $\alpha\beta$ element in the table must be multiplied by the normalization factor $N_\alpha N_\beta$). The functions f and g are all polynomials in x with the following properties:

$$f_{\alpha\beta}(x)=(-1)^{\omega_\alpha-\omega_\beta}f_{\alpha\beta}(-x),$$
$$g_{\alpha\beta}(x)=-(-1)^{\omega_\alpha-\omega_\beta}g_{\alpha\beta}(-x), \tag{4}$$

$$\int_{-1}^{+1} f_{\alpha\beta}(x)dx=(2J_\alpha+1)\delta_{\alpha\beta}. \tag{5}$$

In principle, using these results and the corresponding ones for the production by polarized photons, from a complete knowledge of the experiments at a given energy, we could calculate the amounts and phases of each state present. However, at least at the present time, far too little information is available for this, so we must use the less satisfactory procedure of assuming various states to be present and compare the resulting predictions with the observed results.

To start with, we assume that, apart from the contributions from those states already known to be present from the low-energy data, where they give good quantitative agreement,[1] the behavior of the higher energy results is due to two new states only. These are labeled by B and C where the magnitudes $|M_B|$ and $|M_C|$ have maxima as functions of energy near the 700-Mev and 1100-Mev peaks in the total cross sections, respectively. We also assume that each of the peaks in the total cross section is due mainly to the single state to which it corresponds.

The states already known are the 33 resonance, labeled by A, and the contribution to charged-meson production alone, which comes from a component of the production matrix of the form[1]

$$\frac{ef}{(4k\omega_q)^{\frac{1}{2}}}\mathcal{J}\left(i\boldsymbol{\sigma}\cdot\hat{e}+\frac{i\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q})(\mathbf{q}\cdot\hat{e})}{k\omega_q(1-\beta\mathbf{k}\cdot\mathbf{q})}\right), \tag{6}$$

where e and f are the electron charge and renormalized pion coupling constant, ω_q is the pion total energy, $\hat{e}$ is the photon polarization direction, β is the pion center-of-mass velocity, $\boldsymbol{\sigma}$ is the nucleon spin operator, and $\mathcal{J}$ is a projection operator vanishing for neutral-pion production.

The first term in the bracket, denoted by S, corresponds to pure electric dipole, $J=\frac{1}{2}$ production. The second, denoted by R, includes contributions from all possible states, owing to the presence of the denominator. For the limiting cases $\beta=0$ and $\beta=1$, the contri-

[16] R. F. Peierls, thesis, Cornell University, 1959 (unpublished).

butions of this term to the angular distributions and polarizations are shown in Table I as R_0 and R_1. The exact form of this term and the connection with e, f is as predicted by lowest-order perturbation theory, or from the more successful approaches to the theory of low-energy photoproduction. In view of its success at these lower energies, we may hope that it will be at least an approximate representation of that part of the photoproduction which is not connected with the strong final-state interaction of pion and nucleon. But we should not expect very exact agreement at higher energies.

The so-called "retardation term" R, has two important effects. Firstly, it makes the angular distribution of π^+ production much more difficult to interpret in terms of only low powers of $\cos\theta$; in Fig. 2 the coefficients as shown can only be taken as approximate fits except at very low energies and very high energies (extremely relativistic pions) when the higher powers do not contribute. Secondly, and more important, it has an effect on the total cross section for π^+ photoproduction. Since R contains all states, the interference of R with any other state α will not vanish when integrated over angles.[17,18] Instead it will be proportional to

$$M_{R\alpha} M_\alpha \cos\delta_\alpha, \tag{7}$$

where $M_{R\alpha}$ is the component of R corresponding to the angular momentum state $(J_\alpha, l_\alpha, \omega_\alpha)$.

In addition to the quantum numbers J, l, ω, T, to be determined for the new states B and C, we must also determine the sign of the amplitudes. Let us fix $\lambda = M^+/|M^+|$. With the requirement that $|M_B|$, $|M_C|$ have maxima as stated above, these five parameters for each state determine the qualitative behavior of the cross sections. Table II gives the values of these parameters for the states A, B, and C. In the next section we discuss how these are identified for the new states B and C.

III. IDENTIFICATION OF THE STATES B AND C

The isotopic spin assignment is trivial, because the contribution of these states to the π^+ production is much larger than the contribution to the production of π^0 (we are assuming that the states are specific eigenstates of T and not mixtures). The next step is to assign l and J. To do this, we examine the angular distribution near each peak, where the contribution from the corresponding state should dominate that from the other states and the interference terms. For the production of π^0 near 750 Mev, the angular distribution is close to

$$5-3x^2,$$

TABLE II. Quantum numbers describing the second and third photoproduction resonances.

Level	l	J	ω	λ	T
A	1	$\frac{3}{2}$	$+$	$-$	$\frac{3}{2}$
B	1	$\frac{3}{2}$	$-$	$+$	$\frac{1}{2}$
C	2	$\frac{5}{2}$	$+$	$+$	$\frac{1}{2}$

which, from the table corresponds to a transition with $l=1$, $J=\frac{3}{2}$. This is the only assignment even approximately resembling the experimental result and, in fact, is quite close. Looking at the π^+ distribution at 1000 Mev (it would be simpler to analyze the π^0 results to which S and R do not contribute, but the results are not known well enough in this region), we see that A_0^+ is small, A_2^+ is large and positive, and A_4^+ is nearly as large and negative. Again there is only one assignment even approximately fitting this, and that is $l=2$, $J=\frac{5}{2}$; this would predict

$$1+6x^2-5x^4,$$

which is in quite good agreement.

Now we must try to determine the parities. The large polarization of the recoil protons observed at 700 Mev (for $x=0$) means, according to Eq. (4), that there is strong interference between states of opposite parity. If the states in question are A and B, then $\omega_B=-1$. We can rule out the possibility that C is the main contributor to this interference on several grounds. Firstly, there is no evidence of appreciable terms in x^4 in the angular distributions below 800 Mev, implying that at 700 Mev $|M_C|^2$ is much less than $|M_A|^2+|M_B|^2$. Secondly, we must explain the strange rise in A_2^+ in the region of 500 Mev. For $\omega_B=-1$ this comes from interference between B and S of the form $(3x^2-1)$ in this case. This gives us the additional information $\lambda_B=+1$, since $\delta_S=0\leqslant\delta_B\leqslant\pi$. If, on the other hand we have $\omega_B=+1$ and $\omega_C=-1$, then neither the interference of B with S (which is odd) or of C with S (which is isotropic) can explain this behavior. Some contribution to A_2^+ could come from the interference of C with R, but consideration of the requirements that this, together with the observed direction of the polarization, would impose on λ_B and λ_C would also imply a small value for A_1^+, which is clearly not the case. So we may conclude that ω_B equals -1.

There are two other checks on the consistency of this assignment, which gives $\lambda_B=+1$ and δ_B increasing as a function of energy. Firstly the sign of the polarization is correctly predicted (negative, with our definition of $\hat{n}$) by using the fact that the phase is greater than 90°, which follows from the large positive value of A_1^+ (note that it is not essential to use the relation with the scattering phase shift δ_{33}). Secondly, there is the effect of the interference of the retardation term in the total cross section.[17,18] With the assignment given above, the predicted effect is to add a term that

[17] G. F. Chew, Phys. Rev. **95**, 1669 (1954).

[18] A. M. Wetherell, California Institute of Technology, Pasadena, 1959 (unpublished). Wetherell uses essentially this argument in reverse, assuming the specific resonant forms for a_α and δ_α. The qualitative result does not depend on his assumptions.

has the effect of lowering the energy at which the maximum in the total cross section occurs, and of causing the total cross section to fall off more sharply above the maximum. The large polarization gives us some more information. The maximum possible polarization due to interference between the states A and B, with the above assignments is 80%, which is possible for $|M_A|=|M_B|$ and $\delta_A-\delta_B=90°$. The fact that the observed polarization averaged over an energy spread of about 70 Mev, over which $|M_A|$ and $|M_B|$ must vary appreciably, is so large means that the assumption about the small number of states present is probably correct. It also confirms the phase assignment above.

The final quantities to be determined are ω_C and λ_C. We observe that A_3^+ is appreciable already near 700 Mev and must presumably be due to interference of C with the dominant terms at this energy. This can only happen if we have $\omega_C=+1$. The sign of A_3^+ gives the result $\lambda_C=+1$. The polarization has been measured in the region where interference between B and C might contribute, but unfortunately it is small because the ratio of $|M_C|$ to $|M_B|$ is small at this energy, and in fact even the sign is uncertain. It is interesting to note that only the large size of the 33 resonance compared with B, together with the change from a $T=\frac{3}{2}$ to a $T=\frac{1}{2}$ state allows the energy at which $|M_A|=|M_B|$ to be close to the peak of B and the energy where $\delta_A-\delta_B=90°$.

This last quantum number λ_C as yet has no very good check on the consistency of its assigned value, and is the least well determined. But the remainder seem to be fairly uniquely established under the assumptions we have made. Since it would require far more experimental information to formally determine the assignments, it is clear that at any particular energy there must exist other possible combinations that would fit. However, these would probably involve many higher angular-momentum states with complicated cancellations, and to fit the whole range of energies would presumably require very unlikely energy dependence of the corresponding amplitudes.

It should be emphasized that the above analysis has considered the photoproduction data given as a closed system, making no assumption about the relationship to the scattering. In particular we have not had to assume anything about the behavior of the phases. It is interesting to note that the solution obtained does seem to have the sort of phase behavior associated with resonances, even though the usual theorem does not hold in this energy region. This is discussed in more detail in Sec. VI.

IV. SCATTERING RESULTS AND A POSSIBLE CAUSE OF THE RESONANCES

The results of the scattering experiments are more difficult to analyze as well as being much less complete as far as details of the differential cross sections are concerned. We may ask ourselves, however, whether the same two states, B and C, that seem sufficient to explain the photoproduction can also account for the scattering results in the corresponding energy region. At lower energies it was shown that, in fact, exactly the same states should be important in the two processes, with the addition of the terms S and R for photoproduction. Even though this may not be possible to prove formally for these higher energies, it is still to be expected that the same will apply, since the electromagnetic coupling is so weak that one would expect the final-state interactions to dominate the photoproduction. It is clear that the results cannot be explained entirely by the two $T=\frac{1}{2}$ resonances. First of all there is too much π^+ scattering to be explained by the tail of the 33 resonance. Then, the forward peak occurring in both π^+ and π^- scattering,[6] which seems to indicate the presence of a number of higher-angular-momentum states in both isotopic spin states. This is supported by the observations on the charge exchange scattering.[5] For a pure isotopic spin state we have

$$\frac{\sigma_{\text{tot}}(\pi^-+p\rightarrow\pi^0+n)}{\sigma_{\text{tot}}(\pi^-+p\rightarrow\pi^-+p)}=\begin{cases}\frac{1}{2}; & T=\frac{1}{2}\\ 2; & T=\frac{3}{2}.\end{cases}$$

The only way the observed result of about $\frac{1}{5}$ can be achieved is through some angular momentum state contributing to scattering in both isotopic spin states.

The states in question responsible for the forward peak in the angular distribution have been interpreted as a diffraction peak and tentatively related to the pion-pion interaction.[19,20] We shall return to this question later; for the moment, we observe that if indeed the higher angular momenta are important, then it is possible to understand why these particular states do not show up in the photoproduction. For in photoproduction the higher multipoles should be strongly suppressed. (This depends to some extent on the energy region in question. However, we have observed that for each of the states B and C there is very little contribution from the higher multipoles which lead to the same J values).

Hence, we may tentatively describe the scattering results at these energies as being indeed due to the same two states B and C as in photoproduction, which are responsible for the peaks in the π^- cross section, together with an "optical" absorption that varies smoothly with energy and occurs in both isotopic spin states. With this interpretation we can consider the total cross sections, elastic plus inelastic. The maximum total cross section coming from a state of

[19] C. Goebel, Phys. Rev. Letters **1**, 337 (1958); G. F. Chew and F. E. Low, Phys. Rev. **113**, 1690 (1959).
[20] W. R. Frazer and J. Fulco, Phys. Rev. Letters **2**, 365 (1959).

angular momentum J can be expressed as[21]

$$\sigma_{\text{total}} \leqslant \pi\lambda\!\!\!^{-2}(2J+1)(1+\mu), \quad 0\leqslant\mu\leqslant 1$$

where μ determines the contribution of the inelastic scattering to this maximum:

$$\sigma_{\text{inel}} = \tfrac{1}{2}\pi\lambda\!\!\!^{-2}(2J+1)(1-\mu^2).$$

At the peak of B, the total cross section is in the region of $8\pi\lambda\!\!\!^{-2}$. Bearing in mind that this must include overlap from other states such as A and C, both of which should be appreciable near the peak of B, as well as the states discussed above, it is clear that μ cannot be very large (if all the cross section were due to B alone we would have $\mu=1$). In other words, the inelastic scattering must be large in this state. Certainly the experiments indicate that the total inelastic scattering is indeed large.

This suggests a possible model for the existence of resonances B and C, if we assume that the peak in the cross section is due primarily to the inelastic scattering and only appears as a consequence in the elastic cross section as well. The inelastic scattering in this energy region is due mainly to the production of another meson, so we must ask the question: Can we understand the occurrence of peaks in the cross section for the process

$$\pi+N \rightarrow \pi+\pi+N \tag{1}$$

in the appropriate states at the appropriate energies? The data on reaction (1) are still fairly meager. Attempts have been made to analyze them in terms of the so-called "isobar model."[22] In this model it is assumed that one of the two final-state mesons and the final-state nucleon always emerge in such a way that they are in a relative resonant 33 state, while the remaining meson is produced in an s state relative to this "isobar." One can immediately see that this three-body system has angular momentum $J=\frac{3}{2}$ and odd parity, since the 33 state has even parity and the pion is pseudoscalar. This model was introduced in order to discuss the behavior of the differential cross sections at a given total energy in the center-of-mass system. The problem we are concerned with is slightly different—not the relative probabilities of different configurations of the system at a given total energy, but the relative probabilities of producing the system at different energies. Crudely speaking, what we shall do is to assume that these are given by the proportion of "strongly favored" configurations at the corresponding energies.

We assume that the mechanism by which a final state with two real mesons is reached may be extremely complicated but probably not too sharply energy-dependent. However, because of the possibility of one of the final-state mesons "rescattering" off the nucleon, if their relative energy lies in the range in which the 33 resonance is strong it can be seen that certain of the possible final states will be very strongly enhanced. The situation is somewhat analogous to the single photoproduction of pions, which may be qualitatively interpreted as the "rescattering" off the nucleon of an electromagnetically excited pion. In the case so far described, only one of the mesons interacts in this way. However, at higher energies both final state mesons might be able to interact separately. In this case both would be p-state mesons, and since the orbital angular momentum of each one must be parallel to the nucleon spin to produce an angular momentum of $\frac{3}{2}$ for the pair, they must also be parallel to one another. Thus the total angular momentum must be $\frac{5}{2}$. This argument treats the addition of the angular momentum classically, but essentially the same result is true if they are added quantum-mechanically—the $J=\frac{5}{2}$ state still dominates. The parity of such a configuration is even, and therefore it has the correct quantum numbers to describe level C.[23]

We have to be careful in describing the angular-momentum states of a three-body system. There are a great many different possible representations of the states of such a system, and such quantities as the relative orbital angular momentum of two of the three particles depend on which representation is used. In general, two orbital momenta completely specify the system, and one should talk about the "relative orbital angular momentum of particles 1 and 2" only when the other quantum number used to describe the system is the orbital angular momentum of particle 3 relative to the center-of-mass of particles 1 and 2. In other words, strictly speaking, we cannot really talk about a state having both mesons simultaneously in p states relative to the nucleon. However, if we examine the kinematics of the configurations at the energies concerned, we find that the motion of the mesons is relativistic, while that of the nucleon is comparatively small. In particular, for level C, as we shall see below, the important configurations will be those in which the nucleon is almost at rest, in which case the ambiguity does not arise. For level B we have already used a correct description.

The cross section may be written in the following form:

$$\sigma(\omega) = \textstyle\sum_\xi \sigma_0(\omega)\lambda(\xi)[\rho(\xi,\omega)/\rho(\omega)], \tag{2}$$

where $\sigma_0(\omega)$ would be the cross section if there were no enhancement due to the 33 resonance, ξ is some parameter specifying the configuration of the system, $\lambda(\xi)$ is the factor by which this particular configuration is enhanced, $\rho(\xi,\omega)$ is the density of final states with the particular configuration ξ, and $\rho(\omega)$ is the total density of the two-body final states at the energy ω. The

[21] See, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, New York, 1952), Chap. VIII.

[22] S. J. Lindenbaum and R. M. Sternheimer, Phys. Rev. **106**, 1107 (1957).

[23] For the third resonance this model is similar to that discussed by K. A. Brueckner, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics* (Interscience publishers New York, 1956), p. IV-12.

assumption we make is that in the energy region where $\lambda(\xi)$ and $\rho(\xi,\omega)/\rho(\omega)$ vary very rapidly, $\sigma_0(\omega)$ changes comparatively slowly, so that the position of the peak in the cross section is given roughly by the maximum of $[\lambda(\xi)\rho(\xi,\omega)/\rho(\omega)]$. Let $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ be the momenta of the two pions and the nucleon, respectively; then, after taking into account the requirement of momentum conservation and ignoring the trivial integrations, we can set

$$\rho(\omega,\xi)\propto\int dk_1dk_2d(\hat{k}_1,\hat{k}_2)\,k_1^2k_2^2\delta(\omega)\delta(\xi),$$

where $\delta(\omega)$ represents the restriction due to energy conservation and $\delta(\xi)$ represents the restriction imposed by specifying the configuration. The total density, $\rho(\omega)$ is given by the same expression except for the term $\delta(\xi)$. The enhancement factor $\lambda(\xi)$ is given by the shape of the 33 resonance and the energies of the pions relative to the nucleon. The exact evaluation of the integral would have to be done numerically, but since this is a very rough model, we can estimate the result at least to the accuracy to which it is meaningful.

Consider the case of level B. We may make a very rough estimate of the position of the peak if we assume the pions to be extremely relativistic and the nucleon to be extremely heavy. Then, the restriction on the configuration will be essentially the specification of the magnitude of k_1, say. The energy restriction becomes simply $k_1+k_2=E$, and we easily find that the maximum of $\rho(\xi,\omega)/\rho(\omega)$ occurs, for fixed k_1, at $E=5k_1/3$. Therefore the maximum cross section will be for this value of E corresponding to the resonance value of k_1. The same sort of calculation can be carried out taking into account first-order terms in the ratio of the pion mass to energy, though still assuming a static nucleon. The result is again similar—the ratio of $\rho(\xi,\omega)$ to $\rho(\omega)$ reaches its maximum at an energy at which the total energy of the two pions is slightly less than the assigned kinetic energy of the resonant pion. Let us assume that if we could do the very complicated exact calculation, the result would be approximately the same—the maximum will occur in the region where the two pions share the kinetic energy about equally. Having stated this condition, we can calculate the corresponding energy, taking the finite masses into account quite exactly. For the case in which we are interested, this predicts a peak in the region of 600 Mev. Clearly the agreement is as accurate as the model warrants.

For the case of the third resonance, C, if we fix the energy of both pions relative to the nucleon, then the total energy of the system is determined by the one remaining parameter, the angle between k_1 and k_2. Thus such a configuration can only occur for a limited range of energies. In the case when the two pions have the same energy, the calculation can easily be carried out exactly, and it turns out that the contribution to $\rho(\xi,\omega)/\rho(\omega)$ comes almost entirely from the largest values of ω allowed for the particular ξ. In this case, this would be the largest total energy corresponding to a given kinetic energy of the pions relative to the nucleon, which comes when the nucleon is at rest and the momenta of the two pions are equal and opposite. In this case the shape of the peak is mainly given by the factor $\lambda(\xi)$ or the shape of the 33 resonance. This gives a maximum in the region of 800 Mev. This is somewhat low; however we must remember that we have made the drastic approximation that the two pions must be in the same energy state relative to the nucleon. Removal of this restriction will presumably raise the position of the maximum, but the agreement is not too bad in view of the approximate nature of the model.

Hence we may say that the action of the 33 resonance in two-meson final states is a possible explanation of the observed resonances in the $(\frac{3}{2}-)$ and $(\frac{5}{2}+)$ states at the observed energies. If this explanation is correct, then we must conclude that the occurrence of these, at first sight very striking, features of the cross sections tells us nothing very new about the nature of the interaction.

V. ISOTOPIC SPIN DEPENDENCE OF THE INTERACTION

The mechanism proposed above makes no prediction of the isotopic spin to be associated with these resonances. In the case of the second resonance, where there is one "free" pion together with an "isobar" of isotopic spin $\frac{3}{2}$, either $T=\frac{3}{2}$ or $T=\frac{1}{2}$ for the whole system will be possible. In the case of the third resonance, an argument similar to that predicting the angular momentum would predict the strongest enhancement in the state $T=\frac{5}{2}$. However, this cannot be reached by any of the experiments with which we are concerned. The other two states are about equally likely to give the doubly resonant configuration.

Hence we must conclude that the observed predominance of the $T=\frac{1}{2}$ state is due to some other mechanism. Some support for this is given by the fact that the $T=\frac{1}{2}$ state shows signs of being the most important state also near the threshold for the production of a second pion,[24] when both must come out in s states, and an isobar cannot be formed. Another thing to be explained is the appreciable scattering in the higher angular-momentum states and both isotopic spin states, as discussed above.

To try to understand these facts, we can consider, as an example, the following picture. We consider the physical nucleon to be made up of a nucleon-like core surrounded by a meson cloud which has components with one, two, $\cdots$, etc. mesons. Then we can consider the absorption of a pion by the physical nucleon with the emission of two pions as occurring in one of two ways. The pion can be absorbed either through an interaction with the cloud, in which the incoming

[24] W. A. Perkins, J. C. Caris, R. W. Kenney, E. A. Knapp, and V. Perez-Mendez, Phys. Rev. Letters 3, 56 (1959); L. Rodberg, Phys. Rev. Letters 3, 58 (1959).

pion "knocks out" a single meson from the cloud, or else through an interaction in which the incident meson is absorbed by the core, releasing the two final-state mesons from the cloud. The former process we consider as giving rise to the diffraction-like absorption. This process will be dominated by the pion-pion interaction, and if this interaction has a resonance in the $T=1$ state, at the energy required to explain the nucleon form factor, we should expect it to contribute to both possible isotopic spin states of the two-pion-plus-nucleon system. Such a process might possibly be the cause of the observed peak in π^+ scattering at 1.5 Bev, with its effects spread over a very wide range of energies. This is essentially the mechanism proposed by Dyson and Takeda at one stage to explain the apparent single resonance at 800 Mev in pion-nucleon scattering,[25,26] before it was known that in fact this peak was due to the two separate resonances discussed above.

The absorption by the core can also be considered from the point of view of isotopic spin. Since the two final-state mesons at one stage were combined with a nucleonlike core to form the physical nucleon, their total isotopic spin must be either 0 or 1. In the former case only incident π^- can be absorbed (considering the initial nucleon to be a proton), and in the latter case the relative probability of absorbing π^- is twice that of absorbing π^+. If the probabilities for the two cases were equal, this would predict a ratio of 6:1 for the absorption of π^- over that of π^+, which is probably sufficient to explain the experiments. In addition there are reasons for expecting the $T=0$ component of the cloud, which is symmetric in the two mesons, to be more important, which strengthens the π^- absorption still more.

We have been talking in a rather pictorial way about the nature of these processes. Another way of looking at them is to say that they correspond to the two perturbation-theory diagrams shown in Fig. 4. Note that in the case of Fig. 4(b) we have only shown one of the three possible diagrams that could contribute: the one with the initial pion absorbed after the emission of both final-state pions. This can be supported on the grounds that in perturbation theory this is the one with the most important energy denominators. However, even if we do not wish to attach any great significance to this particular argument, the main point seems to be supported: It is reasonable to dissociate the problem of the isotopic spin dependence of the cross sections from that of the occurrence of the resonances. In fact it seems that an understanding of the former problem rather than the latter is more likely to yield information about the nature of the pion-nucleon interaction.

VI. RELATION BETWEEN PHOTOPRODUCTION AND SCATTERING PHASES

We consider here the problem of relating the photoproduction and scattering phases when there is inelastic

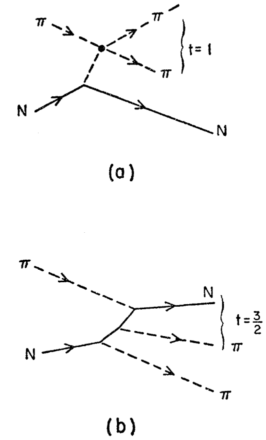

FIG. 4. Perturbation-theory diagrams representing the two types of pion absorption by a nucleon giving two pions: (a) absorption by the cloud, with the pion-pion interaction dominating; (b) absorption by the core, with the pion-nucleon interaction dominating the final state.

as well as elastic scattering. In this case, instead of the S matrix being diagonal as for pure elastic scattering (we ignore the electromagnetic coupling for the moment) there are many independent three-particle eigenstates having the same total energy, angular momentum, parity, and isotopic spin.

Consider a general S matrix describing a scattering process with state vector $|\psi\rangle$. We have

$$\langle\psi|\beta-\rangle=\textstyle\sum_\alpha S_{\beta\alpha}\langle\psi|\alpha+\rangle, \tag{1}$$

where $|\alpha+\rangle$ are a complete set of basis states with asymptotically ingoing boundary conditions, and $|\beta-\rangle$ are the corresponding outgoing states, with the time-reversed boundary conditions. With this definition (with correct choice of phase for $|\alpha+\rangle$; see below) the elements $S_{\alpha\beta}$ form a symmetric unitary matrix. Consider a unitary transformation U among the states $|\alpha+\rangle$:

$$|\alpha+\rangle'=\textstyle\sum_\gamma U_{\alpha\gamma}|\gamma+\rangle. \tag{2a}$$

The corresponding transformation on the states $|\beta-\rangle$ is

$$|\beta-\rangle'=\textstyle\sum_\epsilon U_{\beta\epsilon}^*|\epsilon-\rangle. \tag{2b}$$

In the representation defined by these transformed basis states, the new S matrix is

$$\begin{aligned}S'&=U^*SU^{-1}\\&=(U^\dagger)^TSU^\dagger,\end{aligned}$$

since U is unitary. Note that this preserves the symmetry and unitarity of S'. Consider in particular the transformation defined by

$$\begin{aligned}&U|1+\rangle=|1+\rangle\\&U|2+\rangle=e^{-i\eta}[1-|S_{11}|^2]^{-\frac{1}{2}}\sum_{\beta>1}S_{1\beta}^*|\beta+\rangle,\end{aligned} \tag{4}$$

the rest of the elements of U being chosen so as to make it unitary. Then we have

$$S_{\alpha\beta}'=\sum_{\gamma\epsilon}(U\dagger)_{\gamma\alpha}S_{\gamma\epsilon}U_{\epsilon\beta}\dagger,$$

[25] F. J. Dyson, Phys. Rev. 99, 1037 (1955).
[26] G. Takeda, Phys. Rev. 100, 440 (1955).

whence it follows, using Eq. (4) and the fact that U is unitary,

$$S_{1\beta}'=0. \quad \beta>2 \tag{5}$$

The unitarity of S' gives us the result

$$(S'^{\dagger}S')_{\alpha\beta}=\sum_\gamma S_{\alpha\gamma}'^{*}S_{\gamma\beta}'=\delta_{\alpha\beta}.$$

Thus, for $\alpha=1$, $\beta>2$, as a result of Eq. (5), only one term in the summation survives, and we have

$$S_{12}'^{*}S_{2\beta}'=0,$$

and therefore

$$S_{2\beta}'=0 \quad \beta>2 \tag{6}$$

(except for the case $S_{12}'=0$, but this corresponds to pure elastic scattering in channel $|1\rangle$, which we are not considering). Hence the elements

$$\begin{pmatrix} S_{11} & e^{i\eta}(1-|S_{11}|^2)^{\frac{1}{2}} \\ e^{i\eta}(1-|S_{11}|^2)^{\frac{1}{2}} & -e^{2i\eta}S_{11}^{*} \end{pmatrix}$$

form a unitary symmetric submatrix, so that channels $|1\rangle$ and $|2'\rangle$ are completely independent of the others. The fact that it is possible to construct such a channel $|2'\rangle$ which represents the entire effects of all other channels on the scattering in channel $|1\rangle$ can be easily understood physically by considering the wave function describing the system when there are particles incident in channel $|1\rangle$ alone. Using the completeness of the $|\alpha-\rangle$ and expressions (1) and (4), we have for this wave function:

$$\begin{aligned}|1+\rangle&=\sum_\alpha|\alpha-\rangle\langle\alpha-|1+\rangle\\&=\sum_\alpha\langle 1+|\alpha-\rangle^{*}|\alpha-\rangle\\&=S_{11}|1-\rangle+e^{i\eta}(1-|S_{11}|^2)^{\frac{1}{2}}|2'-\rangle.\end{aligned}$$

We have still to discuss the phase η. In general multiplying any state by a phase factor will transform S in such a way that it is still unitary and symmetric, but so that the transition amplitude $T=1-S$ is multiplied also by a phase factor using the relation between the unitary transformations (2a) and (2b) on the ingoing and outgoing states. The point is that these are only related by complex conjugation for sets of states satisfying a relation such as[27]

$$|\alpha-\rangle=(-)^M|\alpha+\rangle^{*},$$

where M is the z component of total angular momentum. For example, we have

$$[e^{i\eta}e^{i\mathbf{q}\cdot\mathbf{r}}]^{*}=e^{i\eta}e^{i(-\mathbf{q})\cdot\mathbf{r}}$$

for $\eta=0$ only. This condition determines the phase η, which we may define as $(\delta_1+\delta_2+\pi/2)$, so that S' becomes

$$\begin{pmatrix} \lambda e^{2i\delta_1} & i(1-\lambda^2)^{\frac{1}{2}}e^{i(\delta_1+\delta_2)} \\ i(1-\lambda^2)^{\frac{1}{2}}e^{i(\delta_1+\delta_2)} & \lambda e^{2i\delta_2} \end{pmatrix}.$$

[27] See, for example, M. Gell-Mann and K. M. Watson, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, 1954), Vol. IV, p. 267.

Suppose now we perturb the scattering by the addition of another channel $|3\rangle$ weakly coupled to the other two so that

$$|S_{33}|\cong 1, \quad |S_{13}|\ll 1, \quad |S_{23}|\ll 1,$$

and with its elastic phase shift equal to zero. We neglect the squares of the small terms. Then, for $\lambda=1$, the unitarity requirement on S shows that S_{13} and S_{23} can be written as[28]

$$S_{13}=i\rho_1e^{i\delta_1}, \quad S_{23}=i\rho_2e^{i\delta_2},$$

where ρ_1 and ρ_2 are real (though not necessarily positive). If $\lambda\neq 1$, then in general there is no such simple result following from unitarity, but the phases depend on the details of the scattering. If λ is nearly 1, then the phases will be close to those given above. However, we can say something about the special case $\lambda=0$. In this case the phases δ_1 and δ_2 are not separately determined, but only their sum, $\delta_1+\delta_2$, enters into the problem which is thus symmetric in the two channels. Now let us define new channels by the transformation

$$\begin{aligned}|A\rangle&=|1\rangle+|2\rangle\\|B\rangle&=|1\rangle-|2\rangle,\end{aligned}$$

so that S now becomes

$$\begin{bmatrix} e^{i\eta} & 0 & S_{A3} \\ 0 & -e^{-i\eta} & S_{B3} \\ S_{A3} & S_{B3} & S_{33} \end{bmatrix},$$

where η has been defined above as $\delta_1+\delta_2+\pi/2$. The unitarity requirement, as above, tells us that we may write

$$S_{A3}=iae^{i\eta/2}, \quad S_{B3}=-be^{i\eta/2},$$

where a and b are real. Hence it follows that we have

$$S_{13}=i(a+ib)e^{i\eta/2}, \quad S_{23}=i(a-ib)e^{i\eta/2},$$

or, in other words,

$$S_{13}=i\rho e^{i[(\eta/2)+\delta]}, \quad S_{23}=i\rho e^{i[(\eta/2)-\delta]},$$

where

$$\rho=|a+ib|=|a-ib|; \quad \tan\delta=b/a.$$

This is the extent of our information from unitarity. Since we have assumed the coupling of channel $|3\rangle$ to be very weak, the interaction will be dominated by the outgoing waves in the processes with incident waves in channel $|3\rangle$. Hence we may try the assumption that the magnitudes of the transition amplitudes from channel $|3\rangle$ are proportional to the elastic-scattering amplitudes corresponding to the final states. Thus we have

$$\tan\delta=\frac{b}{a}=\pm\frac{|1+e^{i\eta}|}{|1-e^{i\eta}|}=\pm\cot\eta/2,$$

$$\delta=\pm[(\pi/2)-\eta/2].$$

[28] E. Fermi, Suppl. Nuovo cimento **10**, 17 (1955).

Hence of the two channels $|1\rangle$ and $|2\rangle$, one will have the phase $\delta_1+\delta_2$, and the other phase will be $\pi/2$.

The application of this to the pion nucleon problem comes when we consider channel $|1\rangle$ to be the two-body, pion-plus-nucleon channel with the appropriate total quantum numbers, while $|2'\rangle$ as shown above can represent the inelastic scattering. The third channel represents the photon-plus-nucleon channel. If we have a resonance of the sort discussed in the last section, the elastic phase must increase through $\pi/2$ on passing through the resonance region, while the parameter λ starts at 1, decreases to 0, and then increases to 1 again. Hence the photoproduction phase starts off together with the scattering phase shift and finally rejoins it, but there is an intermediate region in which the phase is given by one of the two forms given above. The choice of $\delta_1+\delta_2$ seems to agree somewhat better with the experiments unless the phase shift δ_2 is small. The model as outlined in the previous section would predict a phase shift δ_2 of $\pi/2$ at all energies through the second resonance, and a phase shift increasing through π for the third. Perhaps further experiments on the scattering and photoproduction will enable a comparison to be made.

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Quantum Limitations on the Measurement of Gravitational Fields*

ASHER PERES AND NATHAN ROSEN
Department of Physics, Israel Institute of Technology, Haifa, Israel

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By means of the analogy that exists between the gravitational field, in the weak, quasi-static case, and the electromagnetic field, uncertainty relations are obtained for the average values of some of the Christoffel symbols measured in two domains, similar to those for the components of the quantized electromagnetic field. Furthermore, it is shown that there exists a limitation on the accuracy to which the average value of a single one of these Christoffel symbols can be measured. The existence of uncertainty relations provides an argument in support of the standpoint that the gravitational field must be quantized.

THERE has recently been some controversy about the necessity of quantizing the gravitational field.[1] It is therefore of interest to show that it must be subject to some uncertainty relations. This can be done on the basis of an analogy between the gravitational and the electromagnetic fields.

In the case of gravitation, the motion of a test particle is described by the equation of the geodesic[2]:

$$du^k/ds=-\Gamma^k{}_{00}(u^0)^2-2\Gamma^k{}_{0n}u^0u^n-\Gamma^k{}_{mn}u^mu^n.$$

For slow motion, $u^0\approx 1$, $u^k\approx 0$, so that the last term can be neglected. This equation then has a form analogous to the equation of motion of a particle of charge e and mass m acted upon by the Lorentz force in a given electromagnetic field

$$du^k/ds=-(e/m)(F^k{}_0u^0+F^k{}_nu^n).$$

Here $F^k{}_0=-E_k$ and $F^k{}_n=-B_m$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field vectors and (k, n, m) is a cyclic permutation of (1, 2, 3).

We see then, that there is a correspondence between the Christoffel symbols and the electromagnetic field components given by

$$F^k{}_0\to\Gamma^k{}_{00},\quad F^k{}_n\to 2\Gamma^k{}_{0n},$$

provided we also let $e\to m$.

In what follows, it will be assumed that the gravitational field is weak and quasi-static, and that we are using quasi-Galilean coordinates. The analogy can then be pursued further, because $-\Gamma^k{}_{00}$ and $-2\Gamma^k{}_{0n}$ ($\approx 2\Gamma^n{}_{0k}$) are produced by masses (multiplied by G,

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[1] P. G. Bergmann, *Summary of the Colloque International de Royaumont* (to be published).

[2] u^μ is the velocity four-vector of the particle. The velocity of light is taken as unity. Latin indices run from 1 to 3.