Dirac-Like Wave Equations for Particles of Nonzero Rest Mass, and Their Quantization

J. S. LOMONT
International Business Machines Research Center, Yorktown Heights, New York

AND

H. E. MOSES*
Institute of Mathematical Sciences, New York University, New York, New York

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The basic algebraic structure of the Dirac equation for the electron is used as a model for wave equations for other particles of nonzero rest mass. Wave equations of the form $(\gamma^\mu\nabla_\mu+m)\psi=0$, where the γ-matrices satisfy the usual Dirac anticommutation rules $[\gamma_\mu,\gamma_\nu]_+=2g_{\mu\nu}$ are then found for every positive integral and half-odd-integral spin. Wave equations of the above form describing multiple spin particles are also found. The improper transformations are given explicitly in their most general form, and quantization is performed. Finally, the vector meson field is treated as an example.

1. INTRODUCTION

In the past many formulations of higher spin ($s>\frac{1}{2}$) wave equations have been given. Among these, two broad classes might be considered to be more or less Dirac-like. The first class consists of those employing higher rank undors[1–4] (i.e., 4-spinors). The second class[5–11] consists of those derived by algebraic rather than tensor-undor techniques and having the form

$$(\gamma^\mu\nabla_\mu+m)\psi=0, \tag{1.1}$$

where the γ's are square coefficient matrices.

The equations to be derived in the present article fall in the second category, but differ (for spin $\neq\frac{1}{2}$) from the previously derived equations of this category in five important respects. (I) The coefficient matrices γ_μ of the earlier equations are irreducible, whereas the coefficient matrices of our equations are reducible. (II) The spin matrices of earlier equations are polynomials in the coefficient matrices whereas ours are not. (III) The coefficient matrices of the earlier equations satisfy minimal equations of degree >2 for spin $>\frac{1}{2}$, whereas all of our coefficient matrices satisfy a minimal equation of degree two. (IV) The coefficient matrices of earlier equations satisfy complicated commutation relations of degree >2 for spin $>\frac{1}{2}$, whereas all of our coefficient matrices satisfy the ordinary simple Dirac anticommutation relations of degree two. (V) The earlier equations have no subsidiary conditions, whereas our single spin equations have, in a sense. Our subsidiary conditions consist of putting some of the components of ψ equal to zero when the form of all the relevant matrices (i.e., the form of the representation) is suitably chosen.

The reason for referring to our equations as Dirac-like is of course that the coefficient matrices satisfy the Dirac anticommutation relations. The Klein equations[5,6,12] (which include the Duffin-Kemmer equations), on the other hand, abandon the Dirac anticommutation relations but retain the equation from Dirac theory which expresses the spin matrices as bilinear functions of the coefficient matrices. The Pauli-Fierz equations when written in the form (1.1) with irreducible coefficient matrices, have neither of these properties.

The algebraic technique which will be employed here is a straightforward extension of the technique which was employed[13] in deriving Dirac-like wave equations for massless particles.

The wave equation for a particle of spin s has the form[14]

$$(\gamma_\mu\nabla^\mu+m)\psi=0, \tag{1.1}$$

where m is the mass of the particle, the γ's are $8s\times 8s$ square matrices satisfying the usual Dirac relations

$$[\gamma_\mu,\gamma_\nu]_+=2g_{\mu\nu}, \tag{1.2}$$

and ψ is an $8s$-component wave function in which $2s-1$ components are put equal to zero. The spin $\frac{1}{2}$ equation is the usual Dirac equation. As will be seen, this is a rather degenerate case.

The wave equations described above are covariant under the proper orthochronous Lorentz group and also under time-reversal transformations. They are not, however (except for $s=\frac{1}{2}$), covariant under space inversion transformations. Equations invariant under

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[14] Greek indices run from 0 to 3, Latin indices from 1 to 3 $\hbar=c=1$, $-g_{00}=g_{11}=g_{22}=g_{33}=1$, $x^0=t$, $x^1=x$, $x^2=y$, $x^3=z$, $\nabla_\mu=\partial/\partial x^\mu$, t and $\dagger$ mean transpose and Hermitian conjugate, respectively.

space inversion can be obtained from those of type (1.1) by simply doubling the number of components of ψ. These equations are $16s$-dimensional (except for $s=\frac{1}{2}$) and again have the form (1.1) where the γ_μ matrices again satisfy (1.2).

Finally, it should be mentioned that this approach also yields multiple spin equations of Dirac-like form.

2. GENERAL THEORY

We shall begin by assuming that our wave equation has the form

$$(\gamma_\mu \nabla^\mu + m)\psi = 0, \tag{2.1}$$

where

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \tag{2.2}$$

Furthermore, we shall assume that under the infinitesimal Lorentz transformation

$$\delta x^\lambda = \omega^{\lambda\mu} x_\mu, \quad \omega^{\lambda\mu} = -\omega^{\mu\lambda}, \tag{2.3}$$

the wave function ψ undergoes the transformation

$$\delta\psi = -\tfrac{1}{2}\omega^{\mu\nu} K_{\mu\nu}\psi, \tag{2.5}$$

$$K_{\mu\nu} = -K_{\nu\mu}, \tag{2.6}$$

where the infinitesimal transformations $K_{\mu\nu}$ satisfy the usual Lorentz group condition

$$[K_{\kappa\lambda}, K_{\mu\nu}]_- = g_{\kappa\mu}K_{\lambda\nu} + g_{\lambda\nu}K_{\kappa\mu} - g_{\kappa\nu}K_{\lambda\mu} - g_{\lambda\mu}K_{\kappa\nu}, \tag{2.7}$$

and the usual covariance condition[15]

$$[\gamma_\lambda, K_{\mu\nu}]_- = g_{\lambda\nu}\gamma_\mu - g_{\lambda\mu}\gamma_\nu. \tag{2.8}$$

The covariance of the wave equation (2.1) is now easily shown.

$$\begin{aligned}\delta[(\gamma_\lambda\nabla^\lambda + m)\psi] &= \gamma_\lambda(\delta\nabla^\lambda)\psi + (\gamma_\lambda\nabla^\lambda + m)\delta\psi \\ &= \gamma_\lambda\omega^{\lambda\mu}\nabla_\mu\psi - \tfrac{1}{2}(\gamma_\lambda\nabla^\lambda + m)\omega^{\mu\nu}K_{\mu\nu}\psi \\ &= \omega^{\lambda\mu}\gamma_\lambda\nabla_\mu\psi - \tfrac{1}{2}\omega^{\mu\nu}K_{\mu\nu}(\gamma_\lambda\nabla^\lambda + m)\psi \\ &\quad - \tfrac{1}{2}\omega^{\mu\nu}\gamma_\mu\nabla_\nu\psi + \tfrac{1}{2}\omega^{\mu\nu}\gamma_\nu\nabla_\mu\psi \\ &= 0\end{aligned} \tag{2.9}$$

Equations (2.1) and (2.2) assure us that ψ satisfies the Klein-Gordon equation as in the usual Dirac theory.

Now we shall find all possible matrix forms of the γ's and K's by finding all irreudible representations of the γ's and K's. If we let

$$M_{\mu\nu} \equiv K_{\mu\nu} + \tfrac{1}{4}[\gamma_\mu, \gamma_\nu]_-, \tag{2.10}$$

we see that

$$[M_{\kappa\lambda}, M_{\mu\nu}]_- = g_{\kappa\mu}M_{\lambda\nu} + g_{\lambda\nu}M_{\kappa\mu} - g_{\kappa\nu}M_{\lambda\mu} - g_{\lambda\mu}M_{\kappa\nu}, \tag{2.11}$$

and

$$[\gamma_\lambda, M_{\mu\nu}]_- = 0. \tag{2.12}$$

Consequently if

$$\gamma_\lambda \to D(\gamma_\lambda), \quad M_{\mu\nu} \to D(M_{\mu\nu})$$

is a finite-dimensional irreducible representation we see from (2.2), (2.11), and (2.12) that

$$D(\gamma_\lambda) = I^{(m,n)} \otimes \gamma_\lambda, \quad (m, n = 0, \tfrac{1}{2}, 1, \tfrac{3}{2} \cdots) \tag{2.13}$$

$$D(M_{\mu\nu}) = D^{(m,n)}(M_{\mu\nu}) \otimes I^{(4)}, \tag{2.14}$$

where γ_λ on the right side of (2.13) is the usual 4×4 Dirac γ_λ, $I^{(m,n)}$ is the $(2m+1)(2n+1)$-dimensional unit matrix, $I^{(4)}$ is the 4-dimensional unit matrix, and $D^{(m,n)}(M_{\mu\nu})$ is a $(2m+1)(2n+1)$-dimensional irreducible representation $\Delta_{m,n}$ of the Lie algebra of the homogeneous Lorentz group Let us call the representation of the γ's and K's defined by (2.13) and (2.14) $\Gamma_{m,n}$. The dimension of $\Gamma_{m,n}$ is $4(2m+1)(2n+1)$.

Let $\mathfrak{L}$ be the Lorentz Lie algebra generated by the K's, and let $\Gamma_{m,n}{}^{(s)}$ be the representation of $\mathfrak{L}$ "subduced" by restricting $\Gamma_{m,n}$ to $\mathfrak{L}$ We shall now decompose $\Gamma_{m,n}{}^{(s)}$ into a direct sum of irreducible representations $\Delta_{m',n'}$ of $\mathfrak{L}$ Let

$$\mathbf{K} = (K_{23}, K_{31}, K_{12}), \quad \mathbf{M} = (M_{23}, M_{31}, M_{12}), \tag{2.15}$$

$$\mathfrak{K} = (K_{01}, K_{02}, K_{03}), \quad \mathfrak{M} = (M_{01}, M_{02}, M_{03}), \tag{2.16}$$

$$C_1 = \mathbf{K}^2 - \mathfrak{K}^2, \quad D_1 = \mathbf{M}^2 - \mathfrak{M}^2, \tag{2.17}$$

$$C_2 = \mathbf{K} \cdot \mathfrak{K}, \quad D_2 = \mathbf{M} \cdot \mathfrak{M}. \tag{2.18}$$

It is then a straightforward matter to show that

$$(C_1 - D_1 + \tfrac{3}{2})^2 - 2(C_1 - D_1 + \tfrac{3}{2}) + D_1 \pm 2iD_2 = 0, \tag{2.19}$$

$$\begin{aligned}(D_2 - C_2)^2 \pm (i/2)(D_2 - C_2)& \\ + \tfrac{3}{16} - \tfrac{1}{4}D_1 \mp (i/2)D_2 &= 0.\end{aligned} \tag{2.20}$$

Since D_1 and D_2 commute with the M's and γ's they are represented by scalar matrices in $\Gamma_{m,n}$:

$$D_1 = -2\{m(m+1) + n(n+1)\}I, \tag{2.21}$$

$$D_2 = i\{m(m+1) - n(n+1)\}I, \tag{2.22}$$

where I is the $4(2m+1)(2n+1)$-dimensional unit matrix. Consequently (2.19) and (2.20) are simply equations for the eigenvalues of C_1 and C_2. The eigenvalues of C_1 and C_2 are as follows.

$$\begin{gathered} C_1 \\ -2\{(m+\tfrac{1}{2})(m+\tfrac{3}{2}) + n(n+1)\} \\ -2\{(m-\tfrac{1}{2})(m+\tfrac{1}{2}) + n(n+1)\} \\ -2\{m(m+1) + (n+\tfrac{1}{2})(n+\tfrac{3}{2})\} \\ -2\{m(m+1) + (n-\tfrac{1}{2})(n+\tfrac{1}{2})\} \end{gathered}$$

$$\begin{gathered} C_2 \\ i\{(m+\tfrac{1}{2})(m+\tfrac{3}{2}) - n(n+1)\} \\ i\{(m-\tfrac{1}{2})(m+\tfrac{1}{2}) - n(n+1)\} \\ i\{m(m+1) - (n+\tfrac{1}{2})(n+\tfrac{3}{2})\} \\ i\{m(m+1) - (n-\tfrac{1}{2})(n+\tfrac{1}{2})\} \end{gathered}$$

Since every irreducible representation of $\mathfrak{L}$ is characterized by a pair of eigenvalues of C_1 and C_2 we see

[15] The essential difference between our approach and Klein's is that instead of (2.2), Klein assumes $[\gamma_\mu, \gamma_\nu]_- = -4K_{\mu\nu}$ (see reference 5).

from the preceding table that

$$\Gamma_{m,n}{}^{(s)}=\Delta_{m+\frac{1}{2},n}\oplus\Delta_{m-\frac{1}{2},n}\oplus\Delta_{m,n+\frac{1}{2}}\oplus\Delta_{m,n-\frac{1}{2}}. \quad (2.23)$$

The infinitesimal operators K_{23}, K_{31}, and K_{12} generate a sub-algebra $\mathfrak{R}$ of $\mathfrak{L}$ which is the Lie algebra of the 3-dimensional rotation group. Let us denote the $(2j+1)$-dimensional representation of $\mathfrak{R}$ by Δ_j. Then if $\Delta_{m,n}{}^{(s)}$ is the representation of $\mathfrak{R}$ subduced by the representation $\Delta_{m,n}$ of $\mathfrak{L}$, we see from the Clebsch-Gordan expansion

$$\Delta_{m,n}{}^{(s)}=\sum_{j=|m-n|}^{m+n}\Delta_j, \quad (2.24)$$

together with (2.23), (2.5), and (2.1), that the wave equation (2.1) will in general describe multiple spin particles.

We want to show now that the irreducible representation $\Gamma_{m,n}$ can be chosen in a form such that the subduced representations of both $\mathfrak{L}$ and $\mathfrak{R}$ are completely reduced and such that

$$\mathbf{K}^\dagger=-\mathbf{K}, \quad \text{or} \quad K_{\mu\nu}{}^\dagger=-K^{\mu\nu}. \quad (2.25)$$

$$\mathfrak{K}^\dagger=\mathfrak{K}, \quad (2.26)$$

$$\gamma_\mu{}^\dagger=\gamma^\mu. \quad (2.27)$$

We shall subsequently refer to this representation as the C.R. (completely reduced) form of $\Gamma_{m,n}$ or as the C.R. representation.

One form of $\Gamma_{m,n}$ is given by (2.13) and (2.14) where $D^{(m,n)}(M_{\mu\nu})$ can be chosen as follows:

$$D^{(m,n)}(\mathbf{M})=-i[\boldsymbol{\sigma}^{(m)}\otimes I^{(n)}+I^{(m)}\otimes\boldsymbol{\sigma}^{(n)}], \quad (2.28)$$

$$D^{(m,n)}(\mathfrak{M})=-[\boldsymbol{\sigma}^{(m)}\otimes I^{(n)}-I^{(m)}\otimes\boldsymbol{\sigma}^{(n)}], \quad (2.29)$$

where $\boldsymbol{\sigma}^{(m)}$ is the irreducible $(2m+1)$-dimensional spin vector matrix, and $I^{(m)}$ is the $(2m+1)$-dimensional unit matrix. Since $\boldsymbol{\sigma}^{(m)}$ is Hermitian we see that $D^{(m,n)}(\mathbf{M})$ is skew Hermitian and $D^{(m,n)}(\mathfrak{M})$ is Hermitian. Since the 4×4 Dirac γ's can be chosen to satisfy (2.27) we see that $D(\gamma_\lambda)$ given by (2.13) can also. Thus $\Gamma_{m,n}$ in the unreduced form (2.13), (2.14) can be chosen to satisfy (2.25)–(2.27). Let us consider now the $K_{\mu\nu}$ in reduced form. The irreducible components of $\mathfrak{K}$ are of the form (2.29) (with different indices) and, since the spin matrices are Hermitian, satisfy (2.26). Also since the irreducible components of $\mathbf{K}$ are $-i$ times spin matrices, $\mathbf{K}$ satisfies (2.25). The question now is whether the transformed γ's satisfy (2.27). Since the $K_{\mu\nu}$'s are all either Hermitian or skew Hermitian in both the reduced and unreduced form, it follows that the transformation matrix U can be chosen to be unitary. Since

$$(U\gamma_\mu U^{-1})^\dagger=U\gamma_\mu{}^\dagger U^{-1}=U\gamma^\mu U^{-1}. \quad (2.30)$$

It follows that (2.27) holds in the transformed representation.

3. TIME REVERSAL

We shall now show that there is a $4(2m+1)(2n+1)$-dimensional, unitary, time reversal matrix T such that

$$TT^*=(-1)^{2m+2n+1}I, \quad (3.1)$$

$$T^{-1}\gamma^\mu T=\gamma_\mu{}^*, \quad (3.2)$$

$$T^{-1}\mathbf{K}T=\mathbf{K}^*, \quad (3.3)$$

$$T^{-1}\mathfrak{K}T=-\mathfrak{K}^*,$$

$$\text{or} \quad T^{-1}K_{\mu\nu}T=K^{\mu\nu *}, \quad (3.4)$$

where the C.R. form of $\Gamma_{m,n}$ is used. Equation (3.1)[16] assures that two time reversals (where $\psi'=T\psi^*$) return a state to its original value. The numbers m, n characterize the irreducible representation $\Gamma_{m,n}$. Equation (3.2) assures the covariance of the wave equation.

To prove the existence of T we first observe that the two sets of matrices $\{\gamma_\lambda,\mathbf{K},\mathfrak{K}\}$ and $\{\gamma^\lambda,\mathbf{K}^*,-\mathfrak{K}^*\}$ satisfy the same commutation relations. Since they also have the same Casimir operators they must be equivalent. Again since the K's and γ's are either Hermitian or skew-Hermitian the matrix T transforming one set into the other can be chosen to be unitary. If N denotes a generic element of the set $\{\gamma_\lambda,\mathbf{K},\mathfrak{K}\}$, then

$$T^{-1}NT=\epsilon(N)N^*, \quad \epsilon=\pm1, \quad (3.5)$$

from which it follows that

$$[TT^*,N]_-=0. \quad (3.6)$$

By Schur's lemma

$$TT^*=\lambda I, \quad (3.7)$$

and since T is unitary and λ real[17]

$$\lambda=\pm1. \quad (3.8)$$

We shall now construct the operator T and show that $\lambda=(-1)^{2m+2n+1}$.

It will be convenient to introduce in addition to the C.R. representation, a representation of the $M_{\mu\nu}$'s and γ_λ's of the form (2.13)–(2.14) and (2.28)–(2.29), which we shall denote by primes, such that the matrices γ_λ' are real, γ_i' $(i=1, 2, 3)$ are symmetric and γ_0' is antisymmetric.

The operators M' are chosen so that M_1' and M_3' are imaginary and symmetric and M_2' is real and antisymmetric. Furthermore $\mathfrak{M}_1'$, $\mathfrak{M}_3'$ are real and symmetric and $\mathfrak{M}_2'$ is imaginary and antisymmetric. Since the N's are Hermitian or anti-Hermitian, a unitary operator V exists such that

$$VNV^{-1}=N'. \quad (3.9)$$

From (2.14) and the Clebsch-Gordan expansion (2.24), it is seen that the eigenvalues of iM_2 are all integers if both m and n are integers or half-integers,

[16] It should be made clear that (3.1) is a consequence of the other requirements on T and is not itself a requirement.

[17] On multiplying (3.7) on the left by T^{-1} and on the right by T, one has $T^*T=\lambda I$. By taking the complex conjugate of this and comparing with (3.7), one finds $\lambda^*=\lambda$.

and are all half-odd-integers if one of the pair m, n is an integer and the other a half-odd-integer. Consequently, we see that the eigenvalues of the square of the operator

$$D=e^{-\pi M_2'} \tag{3.10}$$

are all one in the first case and minus one in the second case. Hence

$$D^2=(-1)^{2m+2n}I. \tag{3.11}$$

Furthermore,

$$D^{-1}=e^{\pi M_2'},\quad D^*=D,\quad D^t=D^{-1}. \tag{3.12}$$

From the commutation rules for $M_{\mu\nu}'$ and the symmetry properties and M and $\mathfrak{M}$,

$$D^tM_{\mu\nu}'D=e^{\pi M_2'}M_{\mu\nu}'e^{-\pi M_2'}=-M_{\mu\nu}'^t=M'^{\mu\nu*}. \tag{3.13}$$

We maintain that the operator $\bar{T}$ given by

$$\bar{T}=V^{-1}D\gamma_1'\gamma_2'\gamma_3'V^{-1t} \tag{3.14}$$

satisfies (3.1)–(3.4) as can be proved by direct calculation, using the properties of D and the real γ' matrices.

We shall now show that the operator T is unique within a phase factor, i.e., if T_1 and T_2 are any two unitary operators satisfying (3.2)–(3.4), then $T_2=e^{i\phi}T_1$. From (3.2)–(3.4), it follows that

$$[T_1^{-1}T_2,N^*]_-=0. \tag{3.15}$$

Since N is an irreducible set,

$$T_2=\lambda T_1. \tag{3.16}$$

Since T_2 and T_1 are unitary, λ must be a phase factor. Hence

$$T_2=e^{i\phi}T_1, \tag{3.17}$$

where ϕ is arbitrary. Hence, generally

$$T=e^{i\phi}\bar{T}. \tag{3.18}$$

4. SPACE INVERSION

The theory as developed so far does not in general possess a space inversion operator. There are, however, some cases in which it does. These are the cases where $\Gamma_{m,n}$ is equivalent to $\Gamma_{m,n}^*$, i.e., $\Gamma_{m,n}=\Gamma_{m,n}^*$. A necessary and sufficient condition for these cases is $m=n$.

Under such circumstances ($m=n$) we can find a unitary, Hermitian space inversion matrix S such that

$$S^2=I, \tag{4.1}$$

$$SK_{\mu\nu}S=-K_{\mu\nu}^\dagger=K^{\mu\nu}, \tag{4.2}$$

$$S\gamma_\lambda S=-\gamma_\lambda^\dagger=-\gamma^\lambda, \tag{4.3}$$

and $\psi'=S\psi$ (where the C.R. form of $\Gamma_{m,n}$ is used). The first equation assures the return of ψ to its original value after two inversions, and the third assures the covariance of the wave equation.

To prove the existence of such an S we first observe that the two sets of matrices $\{K_{\mu\nu}^*,\gamma_\lambda^*\}$ and $\{-K_{\mu\nu}^\dagger,\ -\gamma_\lambda^\dagger\}$ satisfy the same commutation relations and have the same Casimir operators. They are therefore equivalent. Since $\Gamma_{m,n}=\Gamma_{m,n}^*$, it follows that the two sets $\{K_{\mu\nu},\gamma_\lambda\}$ and $\{-K_\mu{}^\dagger{}_\nu,\ -\gamma_\lambda^\dagger\}$ are equivalent. We again see that the transformation matrix can be chosen unitary. Hence (letting N denote a generic element of $\{K_{\mu\nu},\gamma_\lambda\}$) we see that there is a unitary matrix S' such that

$$S'NS'^{-1}=-N^\dagger. \tag{4.4}$$

From this it follows that

$$[S'^2,N]_-=0, \tag{4.5}$$

and hence that

$$S'^2=\lambda I. \tag{4.6}$$

Since S' is unitary $|\lambda|=1$. We then define S by

$$S=\lambda^{-\frac{1}{2}}S'. \tag{4.7}$$

We note that S is Hermitian.

For later purposes it will be useful to give an explicit construction for S.

We recall that if A and B are any two square d-dimensional matrices, then there exists a d^2-dimensional permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^{-1}A\quad B\mathcal{P}=B\otimes A. \tag{4.8}$$

It is important to note that $\mathcal{P}$ depends on d and on the ordering of the rows and columns of the Kronecker product but not on A and B.

We recall further that

$$\mathbf{M}'=-i(\boldsymbol{\sigma}^{(m)}\otimes I^{(m)}+I^{(m)}\otimes\boldsymbol{\sigma}^{(m)})\otimes I^{(4)}, \tag{4.9}$$

$$\mathfrak{M}'=-(\boldsymbol{\sigma}^{(m)}\otimes I^{(m)}-I^{(m)}\otimes\boldsymbol{\sigma}^{(m)})\otimes I^{(4)}. \tag{4.10}$$

We now let $\mathcal{P}^{(m)}$ be the $(2m+1)^2$ dimensional permutation matrix which transforms $\boldsymbol{\sigma}^{(m)}\otimes I^{(m)}$ into $I^{(m)}\otimes\boldsymbol{\sigma}^{(m)}$ and vice versa. Letting $\mathcal{P}'=\mathcal{P}^{(m)}\otimes I^{(4)}$ we have

$$\mathcal{P}'^{-1}\mathbf{M}'\mathcal{P}'=\mathbf{M}', \tag{4.11}$$

$$\mathcal{P}'^{-1}\mathfrak{M}'\mathcal{P}'=-\mathfrak{M}', \tag{4.12}$$

or

$$\mathcal{P}'^{-1}M_{\mu\nu}'\mathcal{P}'=M'^{\mu\nu}. \tag{4.13}$$

The space inversion operator is then given by

$$S=\pm iV^{-1}\mathcal{P}'\gamma_0'V. \tag{4.14}$$

It should be noted that $\mathcal{P}'$ is real, $\mathcal{P}'^2=I$, and $\mathcal{P}'$ commutes with γ_0'.

It can easily be shown that S as given by (4.14) satisfies (4.1)–(4.3). It can also be shown that if S' is another Hermitian matrix satisfying (4.1)–(4.3) then $S'=\pm S$.

In case $\Gamma_{m,n}\neq\Gamma_{m,n}^*$ we construct a new theory with twice the dimension of the original. Let

$$\tilde{K}_{\mu\nu}=\begin{pmatrix}K_{\mu\nu} & 0\\ 0 & K_{\mu\nu}^*\end{pmatrix}, \tag{4.15}$$

$$\tilde{\gamma}_\lambda=\begin{pmatrix}\gamma_\lambda & 0\\ 0 & \gamma_\lambda^*\end{pmatrix}, \tag{4.16}$$

and let $\tilde{\psi}$ be a wave function transforming under K

$$\delta\tilde{\psi}=-\tfrac{1}{2}\omega^{\mu\nu}\tilde{K}_{\mu\nu}\tilde{\psi}, \tag{4.17}$$

and satisfying

$$(\tilde{\gamma}_{\mu}\nabla^{\mu}+m)\tilde{\psi}=0. \tag{4.18}$$

We now show that there exists a unitary Hermitian space inversion matrix S such that

$$\tilde{S}^{2}=\tilde{I}, \tag{4.19}$$

$$\tilde{S}\tilde{K}_{\mu\nu}\tilde{S}=-\tilde{K}_{\mu\nu}{}^{\dagger}=\tilde{K}^{\mu\nu}, \tag{4.20}$$

$$\tilde{S}\tilde{\gamma}_{\lambda}\tilde{S}=-\tilde{\gamma}_{\lambda}{}^{\dagger}=-\tilde{\gamma}^{\lambda}, \tag{4.21}$$

where $\tilde{I}$ is the identy in the doubled space. Since the two sets $\{\tilde{K}_{\mu\nu},\tilde{\gamma}_{\lambda}\}$ and $\{-\tilde{K}_{\mu\nu}{}^{\dagger},\ -\tilde{\gamma}_{\lambda}{}^{\dagger}\}$ satisfy the same commutation relations and have equivalent Casimir operators they are equivalent. Hence there exists a unitary $\tilde{S}'$ such that

$$\tilde{S}'\tilde{N}\tilde{S}'^{-1}=-\tilde{N}^{\dagger}. \tag{4.22}$$

But $\Gamma_{m,n}\neq\Gamma_{m,n}{}^{*}$ implies $\{K_{\mu\nu},\gamma_{\lambda}\}$ is not equivalent to $\{-K_{\mu\nu}{}^{\dagger},\ -\gamma_{\lambda}{}^{\dagger}\}$ and hence

$$\tilde{S}'=\begin{pmatrix}0 & A_1\\ A_2 & 0\end{pmatrix}. \tag{4.23}$$

From (4.22)

$$A_1N^{*}=-N^{\dagger}A_1, \tag{4.24}$$

$$A_2N=-N^{*\dagger}A_2. \tag{4.25}$$

Furthermore, since the operators N are either Hermitian or anti-Hermitian we write

$$N^{\dagger}=\rho(N)N, \tag{4.26}$$

where $\rho(N)=\pm1$. From (4.24)–(4.26)

$$[A_2A_1,N^{*}]_{-}=0. \tag{4.27}$$

Hence

$$A_2A_1=\lambda I. \tag{4.28}$$

From the fact that $\tilde{S}'$ is unitary

$$A_2{}^{\dagger}=A_2{}^{-1},\quad A_1{}^{\dagger}=A_1{}^{-1}. \tag{4.29}$$

Hence

$$A_2=\lambda A_1{}^{\dagger}=\lambda A_1{}^{-1}, \tag{4.30}$$

and

$$\tilde{S}'^{2}=\lambda I. \tag{4.31}$$

From the fact that $\tilde{S}'$ is unitary, it follows that

$$\lambda=e^{-2i\theta}. \tag{4.32}$$

We define the space inversion operator by

$$\tilde{S}=e^{i\theta}\tilde{S}'=\begin{pmatrix}0 & e^{i\theta}A_1\\ e^{-i\theta}A_1{}^{\dagger} & 0\end{pmatrix}. \tag{4.33}$$

Clearly $\tilde{S}$ is Hermitian and unitary and satisfies (4.19)–(4.21).

It will be useful to write $\tilde{S}$ explicitly. We maintain that a matrix $\tilde{S}'$ which satisfies the requirements for $\tilde{S}$ is

$$\tilde{S}'=\begin{pmatrix}0 & iV^{-1}D\gamma_0{}'V^{*}\\ iV^{*-1}D^{t}\gamma_0{}'V & 0\end{pmatrix}. \tag{4.34}$$

This statement can be proved by straightforward calculation. Let us consider any other Hermitian matrix $\tilde{S}$ which satisfies (4.19)–(4.21). From the fact that

$$[\tilde{S}\tilde{S}',N]_{-}=0, \tag{4.35}$$

we see that $\tilde{S}$ must have the form

$$\tilde{S}=\begin{pmatrix}0 & e^{i\theta}A\\ e^{-i\theta}A^{\dagger} & 0\end{pmatrix}, \tag{4.36}$$

where

$$A=iV^{-1}D\gamma_0{}'V^{*}, \tag{4.37}$$

and θ is thus far arbitrary but will be fixed later.

5. TIME REVERSAL AND SPACE INVERSION

It will be useful to obtain relations between the time reversal and space inversion matrices.

Let us first consider the undoubled theory in which the representation is equivalent to its complex conjugate. On using (3.18) and (3.14) for T and (4.14) for S we can calculate explicitly

$$T^{-1}ST=S^{*}. \tag{5.1}$$

For the doubled theory we write

$$\tilde{T}=\begin{pmatrix}T & 0\\ 0 & T^{*}\end{pmatrix},\qquad \tilde{T}^{-1}=\begin{pmatrix}T^{-1} & 0\\ 0 & T^{*-1}\end{pmatrix}. \tag{5.2}$$

We should like to obtain

$$\tilde{T}^{-1}\tilde{S}\tilde{T}=\tilde{S}^{*}, \tag{5.3}$$

in order to have invariant Lagrangians. Toward this end we set the hitherto arbitrary angle θ equal to ϕ and find that on using (4.36) for $\tilde{S}$ (5.3) is satisfied.

6. BILINEAR COVARIANTS

Let us now consider theories possessing space and time inversion transformations which satisfy (5.1) or (5.3) and disregard the fact that the operators may have come from doubled theories.

By using the space inversion transformation matrix S we can now easily form bilinear covariants. Thus $\psi^{\dagger}S\psi$ is a regular scalar under the full homogeneous Lorentz group, and $\psi^{\dagger}S\gamma_{\mu}\psi$ is a 4-vector of the second kind[18] under the full homogeneous Lorentz group.

[18] See M. S. Watanabe, Phys. Rev. **84**, 1008 (1951) or J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, 1955) for a definition of the four kinds of tensors. See also J. S. Lomont, "Applications of Finite Groups" (Academic Press, Inc., New York, 1959), p. 325 no. 5.

Consequently, for the Lagrangian $\mathbf{L}$ we can take

$$\mathbf{L}=C'[\psi^\dagger S(\gamma^\mu\nabla_\mu+m)\psi+\text{complex conjugate}], \tag{6.1}$$

where C' is a real constant. As a current we have

$$j_\mu\sim(i\psi^\dagger S\gamma_\mu\psi). \tag{6.2}$$

The Lagrangian (6.1) leads to the covariant commutation relations

$$[\psi(x),\bar{\psi}(y)]=-iC'^{-1}(\gamma^\mu\nabla_\mu-m)\Delta(x-y), \tag{6.3}$$

$$[\psi(x),\psi(y)]=0, \tag{6.4}$$

where

$$\bar{\psi}=\psi^\dagger S. \tag{6.5}$$

The Hamiltonian

$$\mathcal{H}=-C'\int d^3x\,\bar{\psi}(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\psi \tag{6.6}$$

then satisfies the usual requirement

$$\dot{\psi}=i[\mathcal{H},\psi]_-. \tag{6.7}$$

7. INTERACTION WITH THE ELECTROMAGNETIC FIELD

Again we shall assume the existence of S and drop tildes. Let $\eta_{\mu\nu}=-\eta_{\nu\mu}$ be a set of matrices of the same dimension as ψ satisfying

$$\eta_{0j}^\dagger\equiv-\eta_{0j},\quad \eta_{ij}^\dagger=\eta_{ij}, \tag{7.1}$$

$$\eta_{\mu\nu}^\dagger=\eta^{\mu\nu}, \tag{7.2}$$

$$S\eta_{\mu\nu}S=\eta_{\mu\nu}^\dagger=\eta^{\mu\nu}, \tag{7.3}$$

$$T^{-1}\eta_{\mu\nu}T=-\eta^{\mu\nu *}, \tag{7.4}$$

$$[K_{\mu\nu},\eta_{\kappa\lambda}]_-=g_{\kappa\nu}\eta_{\lambda\mu}+g_{\lambda\mu}\eta_{\kappa\nu}-g_{\kappa\mu}\eta_{\lambda\nu}-g_{\lambda\nu}\eta_{\kappa\mu}, \tag{7.5}$$

$$\eta^{\mu\nu}\gamma_0=\gamma_0\eta_{\mu\nu}. \tag{7.6}$$

Equation (7.6) together with $[\gamma_0,\mathfrak{K}]_-=\boldsymbol{\gamma}$ and (2.8) leads to the additional relations

$$[\gamma_i,\eta_{jk}]_-=2\delta_{ik}\gamma_0\eta_{0j}-2\delta_{ij}\gamma_0\eta_{0k}, \tag{7.7}$$

$$[\gamma_i,\eta_{0j}]_-=2\gamma_0\eta_{ij}. \tag{7.8}$$

For the interacting field we can then take the Lagrangian $\mathbf{L}$ to be

$$C'^{-1}\mathbf{L}=\psi^\dagger S\{\gamma_\mu D^\mu+m+q_2\chi^{\mu\nu}\eta_{\mu\nu}\}\psi+\text{complex conjugate}, \tag{7.9}$$

where

$$D^\mu=\nabla^\mu-iq_1\phi^\mu, \tag{7.10}$$

q_1 and q_2 are real coupling constants, C' is a real constant, and ϕ^λ and $\chi^{\mu\nu}$ are the, respectively, potential and field strength of the electromagnetic field. The resulting current is

$$j_\lambda=j_\lambda^{(1)}+j_\lambda^{(2)}, \tag{7.11}$$

where

$$j_\lambda^{(1)}=(\partial L/\partial\phi^\lambda)=-2iq_1C'\psi^\dagger S\gamma_\lambda\psi, \tag{7.12}$$

$$j_\lambda^{(2)}=-\nabla^\kappa\frac{\partial L}{\partial(\nabla^\kappa\phi^\lambda)}=-4q_2C'\nabla^\kappa\psi^\dagger S\eta_{\kappa\lambda}\psi. \tag{7.13}$$

The equation for ψ is

$$[\gamma^\mu D_\mu+m+q_2\eta_{\mu\nu}\chi^{\mu\nu}]\psi=0. \tag{7.14}$$

8. CHARGE CONJUGATION

Again we shall consider theories with a space inversion matrix and drop tildes. We shall show that there exists a unitary matrix C, the charge conjugation matrix, such that

$$C^*C=I, \tag{8.1}$$

$$C^{-1}\gamma_\lambda C=\gamma_\lambda^*, \tag{8.2}$$

$$C^{-1}K_{\mu\nu}C=K_{\mu\nu}^*, \tag{8.3}$$

$$C^{-1}TC^{-1}=T^{-1}, \tag{8.4}$$

$$C^{-1}SC=\pm S^*, \tag{8.5}$$

where the C.R. representation is used again. If $\psi^{(C)}=C\psi^*$ then the above equations assure that when $q_2=0$, $\psi^{(C)}$ satisfies the wave equation (7.14) with q_1 replaced by $-q_1$ (if q_1 is real).

To prove the existence of C which satisfies (8.2) and (8.3) we note first that the two sets $\{\gamma_\lambda,K_{\mu\nu}\}$ and $\{\gamma_\lambda^*,K_{\mu\nu}^*\}$ are equivalent, so there exists a unitary C satisfying

$$C^{-1}NC=N^*, \tag{8.6}$$

and hence

$$[CC^*,N]_-=0. \tag{8.7}$$

If the set $\{\gamma_\lambda,K_{\mu\nu}\}$ is undoubled and hence irreducible then

$$CC^*=\lambda I, \tag{8.8}$$

where λ is real, and therefore ±1. To find λ we must go into the structure of C.

We shall now give C explicitly for both the doubled and undoubled theories. In the undoubled theory a unitary operator C satisfying (8.1)–(8.5) is given by

$$\bar{C}=V^{-1}\mathcal{P}'DV^*. \tag{8.9}$$

Any other unitary operator C which satisfies (8.6) is given by

$$C=e^{i\alpha}\bar{C}, \tag{8.10}$$

where α is arbitrary. This result is proved by using the fact $[C\bar{C}^{-1},N]_-=0$.

It is easily seen that (8.1)–(8.3), are satisfied. Also

$$C^{-1}SC=-S^*. \tag{8.5a}$$

To obtain (8.4) we evaluate $C^{-1}TC^{-1}$ using the explicit form of the operators

$$C^{-1}TC^{-1}=-e^{-2i(\phi-\alpha)}T^{-1}. \tag{8.11}$$

If we pick $\alpha=\phi-\pi/2$, then (8.4) follows.

In the doubled theory, every operator $\tilde{C}$ which satisfies (8.1)–(8.4) is of the form

$$\tilde{C}=e^{i\alpha}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \tag{8.2}$$

where $\alpha=(\pi/2)(2m+2n+1)$.

On using (4.36) and (4.37) for S and the fact that $D^{-1}=(-1)^{2m+2n}D$ we have for the doubled theory

$$C^{-1}SC=(-1)^{(2m+2n+1)}S^*. \tag{8.5b}$$

In order for q_2-terms in Eq. (7.12) to behave properly under charge conjugation we must require

$$C^*\eta_{\mu\nu}C=-\eta_{\mu\nu}^*. \tag{8.13}$$

9. SINGLE SPIN EQUATIONS

In this section we shall see how to select from the wave equations (2.1), corresponding to the representation (2.13), (2.14), equations which can describe particles of a single spin. Let us put

$$s=m+\tfrac{1}{2}, \tag{9.1}$$

$$n=0, \tag{9.2}$$

so the decomposition (2.23) becomes

$$\Gamma_{s-\frac{1}{2},0}{}^{(s)}=\Delta_{s,0}\oplus\Delta_{s-\frac{1}{2},\frac{1}{2}}\oplus\Delta_{s-1,0}. \tag{9.3}$$

We shall assume that $\mathfrak{R}$ is in completely reduced form with the blocks in the order (9.3) from top to bottom. Since the subduced representation $\Delta_{s-\frac{1}{2},\frac{1}{2}}{}^{(s)}$ of $\mathfrak{R}$ decomposes according to

$$\Delta_{s-\frac{1}{2},\frac{1}{2}}{}^{(s)}=\Delta_s\oplus\Delta_{s-1}, \tag{9.4}$$

we shall assume that the subduced representation of $\mathfrak{R}$ is completely reduced and that the blocks have the order Δ_s, Δ_s, Δ_{s-1}, Δ_{s-1}.

We shall now show that if the components of ψ transforming under $\Delta_{s-1,0}$ are put equal to zero that we have a single spin theory describing particles of spin s. To do this we shall consider the spin of engergy eigenfunctions in a rest state, so $\psi=u\exp(ip_0x^0)$, where

$$u=\begin{bmatrix}u_s\\ u_s'\\ u_{s-1}\\ 0\end{bmatrix}, \tag{9.5}$$

and

$$-i\gamma_0u=\epsilon u,\quad \epsilon=\pm1. \tag{9.6}$$

We shall now show that $u_{s-1}=0$. For this we first note that $[\gamma_0,\mathbf{K}]=0$, so that

$$\gamma_0=\begin{bmatrix}\lambda_{11}I^{(s)} & \lambda_{12}I^{(s)} & & \\ \lambda_{21}I^{(s)} & \lambda_{22}I^{(s)} & & \\ & & \lambda_{33}I^{(s-1)} & \lambda_{34}I^{(s-1)}\\ & & \lambda_{43}I^{(s-1)} & \lambda_{44}I^{(s-1)}\end{bmatrix}, \tag{9.7}$$

where the λ's are constants, the I's are unit matrices, and the omitted elements are zero. Thus

$$-i\gamma_0u=-i\begin{bmatrix}\lambda_{11}u_s+\lambda_{12}u_s'\\ \lambda_{21}u_s+\lambda_{22}u_s'\\ \lambda_{33}u_{s-1}\\ \lambda_{43}u_{s-1}\end{bmatrix}=\epsilon\begin{bmatrix}u_s\\ u_s'\\ u_{s-1}\\ 0\end{bmatrix}. \tag{9.8}$$

Consequently either $u_{s-1}=0$ or $\lambda_{43}=0$. If $\lambda_{43}=0$ then the set $\{\gamma_0,K_{\mu\nu}\}$ is reduced. Since $[\gamma_0,\mathbf{\mathfrak{K}}]_-=\boldsymbol{\gamma}$ it follows that $\boldsymbol{\gamma}$ is reduced, and hence that the entire set $\{\gamma_\lambda,K_{\mu\nu}\}$ is reduced. Since the set $\{\gamma_\lambda,K_{\mu\nu}\}$ is irreducible it follows that $u_{s-1}=0$, and hence that u describes particles of spin s.

This scheme of setting some components of ψ equal to zero is obviously covariant under proper, orthochronous, homogeneous Lorentz transformations. The covariance of the scheme under improper transformations is not so obvious and requires proof. From (2.17), (3.3), and (3.4) we find that

$$[T,C_1]_-=0. \tag{9.9}$$

Consequently T must have the same block form as the $K_{\mu\nu}$'s, and thus if the last $(2s-1)$ elements of ψ are zero so are the last $(2s-1)$ elements of $T\psi^*$.

Since the only undoubled theory with a space inversion transformation is the Dirac theory, $s=\frac{1}{2}$, which has no zero components, we shall consider only doubled theories $s>\frac{1}{2}$. From (2.17), and (4.20) we find that

$$[S,C_1]_-=0. \tag{9.10}$$

Since

$$S=\begin{pmatrix}0 & A\\ A^\dagger & 0\end{pmatrix},\quad C_1=\begin{pmatrix}C_1 & 0\\ 0 & C_1\end{pmatrix}, \tag{9.11}$$

we find that

$$[A,C_1]_-=0$$

from which it follows that A has the same block structure as $K_{\mu\nu}$. If

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where the last $2s-1$ components of both ψ_1 and ψ_2 are zero, then ψ_1' and ψ_2' given by

$$\psi'=S\psi=\begin{pmatrix}\psi_1'\\ \psi_2'\end{pmatrix}$$

have the same property.

Finally the simple form of the charge conjugation operator (8.2) assures us that the last $2s-1$ components remain zero under charge conjugation.

10. SINGLE SPIN COVARIANT COMMUTATION RULES

Let us consider a complete set of commuting observables consisting of $H=-i\gamma_0(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)$, $-i\mathbf{K}\cdot\boldsymbol{\nabla}$, $P=-i\boldsymbol{\nabla}$, and other Hermitian operators which can always be chosen so that they commute with the

operator $iS\gamma_0$. A simultaneous eigenstate of all these operators will be designated by

$$\psi(\mathbf{x}|\mathbf{p},\epsilon,\zeta), \quad (10.1)$$

where $\mathbf{p}$ is the momentum, ϵ is the sign of energy and ζ collectively denotes all the other eigenvalues. We shall separate out the $\mathbf{x}$-dependence in the usual way

$$\psi(\mathbf{x}|\mathbf{p},\epsilon,\zeta)=\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}u(\mathbf{p},\epsilon,\zeta), \quad (10.2)$$

where u is a column vector.

For many purposes one might be inclined to normalize u so that

$$u^\dagger(\mathbf{p},\epsilon,\zeta)u(\mathbf{p},\epsilon',\zeta')=\delta_{\epsilon\epsilon'}\,\delta_{\zeta\zeta'}. \quad (10.3)$$

This normalization leads to

$$\int\psi^\dagger(\mathbf{x}|\mathbf{p},\epsilon,\zeta)\psi(\mathbf{x}|\mathbf{p}',\epsilon',\zeta')d^3x=\delta(\mathbf{p}-\mathbf{p}')\delta_{\epsilon\epsilon'}\,\delta_{\zeta\zeta'}. \quad (10.4)$$

However, this normalization is not covariant. It can be shown that the following covariant normalization is possible for $\epsilon=\epsilon'$.

$$i\bar{u}(\mathbf{p},\epsilon,\zeta)\gamma_\mu u(\mathbf{p},\epsilon,\zeta')=\lambda(\epsilon,\zeta)p_\mu\delta_{\zeta\zeta'}, \quad (10.5)$$

where here

$$p_0=-\epsilon\omega(p)=-\epsilon(p^2+m^2)^{\frac{1}{2}}, \quad \bar{u}=u^\dagger S, \quad (10.5\text{a})$$

and $\lambda(\epsilon,\zeta)$ is an invariant function of the arguments which can have only the values $+1$ or -1. This function must be calculated in particular cases.

We have also an orthogonality relation

$$u^\dagger(\mathbf{p},\epsilon,\zeta)S\gamma_0 u(\mathbf{p},-\epsilon,\zeta')=0,$$

since $S\gamma_0$ commutes with H. Hence

$$i\bar{u}(\mathbf{p},\epsilon,\zeta)\gamma^0 u(\mathbf{p},\epsilon',\zeta')=\epsilon\lambda(\epsilon,\zeta)\omega(p)\delta_{\epsilon\epsilon'}\delta_{\zeta\zeta'}, \quad (10.6)$$

which implies the completeness relation

$$\sum_{\epsilon,\zeta}\frac{u(\mathbf{p},\epsilon,\zeta)\bar{u}(\mathbf{p},\epsilon,\zeta)}{\epsilon\lambda(\epsilon,\zeta)\omega(p)}=i\gamma^0. \quad (10.7)$$

Let $u^{(s)}(\mathbf{p},\epsilon,\zeta)$ and $\bar{u}^{(s)}(\mathbf{p},\epsilon,\zeta)$ be the single spin eigenfunctions (i.e., the last $2s-1$ components are zero). Then if $\psi^{(s)}(x)$ is a single spin field operator satisfying the equation $(\gamma^\mu\nabla_\mu+m)\psi^{(s)}(x)=0$, the following commutation rules are covariant.

$$[\psi^{(s)}(x),\bar{\psi}^{(s)}(y)]=C'^{-1}D^{(s)}(x-y), \quad (10.8)$$

where

$$D^{(s)}(x)=-\frac{1}{(2\pi)^3}\sum_{\epsilon,\zeta}\int\frac{u^{(s)}(\mathbf{p},\epsilon,\zeta)\bar{u}^{(s)}(\mathbf{p},\epsilon,\zeta)}{\epsilon\lambda(\epsilon,\zeta)\omega(p)}\times\exp(ip^\mu x_\mu)d^3p. \quad (10.9)$$

Let us define $u^{(s-1)}$ as the spin eigenfunctions orthogonal to $u^{(s)}$. These are the remaining eigenfunctions which span the space. We also define $D^{(s-1)}$ in a manner analogous to the definition of $D^{(s)}$ but use $u^{(s-1)}$ instead of $u^{(s)}$. Then

$$D^{(s)}(x)+D^{(s-1)}(x)=-i(\gamma^\mu\nabla_\mu-m)\Delta(x) \quad (10.10)$$

on using (10.7). This is expected from (6.3).

If in (10.9) the summation is taken only over a subset of the ζ's then we obtain a matrix function $D^{(s)}(x)$ which when substituted in (10.8) still yields covariant commutation relations.

We can show that $D^{(s)}$ is a causal function. Consider any $8s$-dimensional column vector $f(x)$. It can always be split up in the following way:

$$f(x)=f^{(s)}(x)+f^{(s-1)}(x), \quad (10.11)$$

where $f^{(s)}(x)$ and $f^{(s-1)}(x)$ are superpositions of the $u^{(s)}$ and $u^{(s-1)}$ eigenfunctions, respectively. Since the operator $D^{(s)}$ transforms the $u^{(s-1)}$ eigenfunctions to zero, we have

$$\int D^{(s)}(x-x')f(x')dx'=\int D^{(s)}(x-x')f^{(s)}(x')dx' =\int D(x-x')f^{(s)}(x')dx'. \quad (10.12)$$

Since D is a causal operator when it acts on $f^{(s)}(x)$, the causality of $D^{(s)}$ follows.

11. EXAMPLE: $s=1$

A set of matrices γ_λ, $K_{\mu\nu}$, and T satisfying the desired relations is given in Appendix I. To obtain a single spin equation we set $\psi_8=0$.

Since the Hamiltonian

$$H=-i\gamma_0(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m), \quad (11.1)$$

$-i\nabla$, and $-i\mathbf{K}\cdot\boldsymbol{\nabla}$ all commute, we can choose a complete orthogonal set of vectors which are simultaneously eigenvectors of H, $-i\boldsymbol{\nabla}$ and $-i\mathbf{K}\cdot\boldsymbol{\nabla}$. Thus $\psi=u\exp(ip^\mu x_\mu)$, where

$$\mathbf{K}\cdot\mathbf{p}u=-ip\zeta u \quad (\zeta=\pm1,\text{ or }0), \quad (11.2)$$

$p=|\mathbf{p}|$ and

$$H(\mathbf{p})u=\epsilon\omega u, \quad \epsilon=\pm1, \quad (11.3)$$

and $\omega=(p^2+m^2)^{\frac{1}{2}}$. To be specific let us append some indices to u so $u=u(\mathbf{p},\epsilon,\zeta)$. Also, for convenience let us put

$$[u_4(\mathbf{p},\epsilon,\zeta),u_5(\mathbf{p},\epsilon,\zeta),u_6(\mathbf{p},\epsilon,\zeta)]=\mathbf{e}(\mathbf{p},\epsilon,\zeta). \quad (11.4)$$

Then equation (11.2) implies that

$$\mathbf{p}\times\mathbf{e}(\mathbf{p},\epsilon,\zeta)=-ip\zeta\mathbf{e}(\mathbf{p},\epsilon,\zeta). \quad (11.5)$$

Since $\mathbf{e}(\mathbf{p},\epsilon,\zeta)$ does not depend on ϵ, the ϵ can be dropped. In the Appendix an explicit form of $\mathbf{e}(\mathbf{p},\zeta)$ is

presented which satisfies the following conditions:

$$\mathbf{e}^*(\mathbf{p},\zeta)\cdot\mathbf{e}(\mathbf{p},\zeta)=1, \tag{11.6}$$

$$\mathbf{e}(\mathbf{p},0)=p^{-1}\mathbf{p}, \tag{11.7}$$

$$\mathbf{e}^*(\mathbf{p},\zeta)=\mathbf{e}(\mathbf{p},-\zeta), \tag{11.8}$$

$$\mathbf{e}(-\mathbf{p},-\zeta)=-\mathbf{e}(\mathbf{p},\zeta). \tag{11.9}$$

Also it follows that

$$\mathbf{e}^*(\mathbf{p},\eta)\times\mathbf{e}(\mathbf{p},\eta)=i\eta\mathbf{e}(\mathbf{p},0),\quad \eta=\pm1. \tag{11.10}$$

The eighth row of the wave equation says

$$\frac{\partial\psi_4}{\partial x}+\frac{\partial\psi_5}{\partial y}+\frac{\partial\psi_6}{\partial z}+\frac{\partial\psi_7}{\partial t}=0. \tag{11.11}$$

Consequently, since $p^0=\epsilon\omega$,

$$u_7(\mathbf{p},\epsilon,\zeta)=(\epsilon/\omega)\mathbf{p}\cdot\mathbf{e}(\mathbf{p},\zeta). \tag{11.12}$$

If we put

$$(\psi_1,\psi_2,\psi_3)=\boldsymbol{\psi},\quad (u_1,u_2,u_3)=\mathbf{u},\quad (\psi_4,\psi_5,\psi_6)=\mathbf{A},$$

and

$$\psi_7=V,$$

we find from the first three lines of the wave equation that

$$-m\boldsymbol{\psi}=\frac{\partial\mathbf{A}}{\partial t}+\nabla V+i\nabla\times\mathbf{A}. \tag{11.13}$$

Consequently

$$\begin{aligned}-m\mathbf{u}(\mathbf{p},\epsilon,\zeta)&=-i\epsilon\omega\mathbf{e}(\mathbf{p},\zeta)+(i\epsilon/\omega)\mathbf{p}\mathbf{p}\cdot\mathbf{e}(\mathbf{p},\zeta)-\mathbf{p}\times\mathbf{e}(\mathbf{p},\zeta)\\&=-i\epsilon(\omega-\epsilon\zeta p)\mathbf{e}(\mathbf{p},\zeta)+(i\epsilon/\omega)\mathbf{p}\mathbf{p}\cdot\mathbf{e}(\mathbf{p},\zeta).\end{aligned}$$

Hence

$$u(\mathbf{p},\epsilon,\zeta)=\begin{Bmatrix}(i\epsilon/m)(\omega-\epsilon p\zeta)\mathbf{e}(\mathbf{p}\cdot\zeta)-(i\epsilon/m\omega)\mathbf{p}\mathbf{p}\cdot\mathbf{e}(\mathbf{p},\zeta)\\ \mathbf{e}(\mathbf{p},\zeta)\\ (\epsilon/\omega)\mathbf{p}\cdot\mathbf{e}(\mathbf{p},\zeta)\\ 0\end{Bmatrix}. \tag{11.14}$$

Instead of normalizing u, we have normalized $\mathbf{e}$. The explicit forms for $\mathbf{e}$ and u are in the Appendix.

In addition to the bilinear covariants discussed earlier, we can also form other bilinear covariants. Let

$$b=iI\nabla_0-2i\mathfrak{M}\cdot\nabla, \tag{11.15}$$

let Γ be any 8-dimensional matrix which commutes with b,

$$[b,\Gamma]_-=0, \tag{11.16}$$

and let

$$J=\psi^\dagger\beta b\Gamma\psi,\quad \beta=-i\gamma_0. \tag{11.17}$$

Then from the relations

$$[b,\mathbf{K}]_-=-2i\mathfrak{M}\times\nabla, \tag{11.18}$$

and

$$[b,\mathfrak{K}]_+=-iI\nabla-b\gamma_0\boldsymbol{\gamma}+2i\mathfrak{M}\nabla_0, \tag{11.19}$$

which hold only for the present case, we find that under the infinitesimal Lorentz transformation (2.3)

$$\delta J=\tfrac{1}{2}\omega^{\mu\nu}\psi^\dagger\beta b[K_{\mu\nu},\Gamma]_-\psi. \tag{11.20}$$

Putting $\Gamma=I$ we find that $\psi^\dagger\beta b\psi$ is a scalar and putting $\Gamma=\gamma_\lambda$ we find that $\psi^\dagger\beta b\gamma_\lambda\psi$ is a 4-vector. Under time reversal $\psi^\dagger\beta b\psi\rightarrow(\psi^\dagger\beta b\psi)^*$. Consequently $\mathrm{Re}(\psi^\dagger\beta b\psi)$ is a regular scalar. Thus we can put

$$2C'\mathbf{L}=(\beta b\psi)^\dagger(\gamma^\mu\nabla_\mu+m)\psi+\{(\gamma^\mu\nabla_\mu+m)\psi\}^\dagger\beta b\psi. \tag{11.21}$$

This Lagrangian leads to the correct equation of motion if we cancel b from the Euler equations. The usual gauge invariance techniques for deriving currents leads to

$$j_\mu\sim\mathrm{Re}\psi^\dagger\beta b\gamma_\mu\psi. \tag{11.22}$$

This Lagrangian, however, is not satisfactory for introducing the electromagnetic interaction in the usual way because it leads to a second order equation.

Incidently, in the doubled theory the current (11.22) is proportional to the current $i\psi^\dagger S\gamma_\mu\psi$ where S is the space inversion operator.

A special case of the undoubled theory corresponds to neutral vector meson theory, while a special case of the doubled theory corresponds to the charged vector meson theory.[19,20]

To obtain the neutral vector meson theory from the undoubled equations we require $\mathbf{A}=(\psi_4,\psi_5,\psi_6)$ and $V=\psi_7$ to be real. Then we take [letting $\boldsymbol{\psi}=(\psi_1,\psi_2,\psi_3)$]

$$\mathbf{E}=\mathrm{Re}\boldsymbol{\psi},\quad\text{and}\quad \mathbf{H}=-\mathrm{Im}\boldsymbol{\psi}. \tag{11.24}$$

We shall now write the Dirac-like equations explicitly and separate real and imaginary parts.

The first three equations are

$$-m\mathbf{E}=(\partial\mathbf{A}/\partial t)+\nabla V,\quad m\mathbf{H}=\nabla\times\mathbf{A}. \tag{11.25}$$

The second three equations are

$$\begin{aligned}-m\mathbf{A}&=-(\partial\mathbf{E}/\partial t)+\nabla\times\mathbf{H},\\0&=(\partial\mathbf{H}/\partial t)+\nabla\times\mathbf{E}.\end{aligned} \tag{11.26}$$

The seventh equations yield

$$-mV=\nabla\cdot\mathbf{E},\quad 0=\nabla\cdot\mathbf{H}. \tag{11.27}$$

Finally the eighth equation is the Lorentz condition:

$$0=\nabla\cdot\mathbf{A}+\partial V/\partial t. \tag{11.28}$$

On writing

$$\mathbf{E}=(F^{01},F^{02},F^{03}),\quad \mathbf{H}=(F^{23},F^{31},F^{12}), \tag{11.29}$$

Eqs. (11.25) are equivalent to

$$mF_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu, \tag{11.30}$$

where $A^0=V$.

The first of Eqs. (11.26) and (11.27) can be written

$$\nabla^\mu F_{\mu\nu}=mA_\nu, \tag{11.31}$$

while (11.28) is

$$\nabla^\mu A_\mu=0. \tag{11.32}$$

From (11.30), (11.31), and (11.32) one obtains

$$(\nabla^\mu\nabla_\mu-m^2)A_\nu=0 \tag{11.33}$$

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[19] We should like to thank Dr. M. S. Watanabe for suggesting this identification.

[20] A special case of the theory could equally well be identified with the ordinary pseudovector meson theory.

as required. One could, of course, have obtained (11.33) directly from the Dirac type equation in the usual way by noting that all components of ψ must obey the Gordon-Klein equation.

The remaining set of equations, namely the second set of (11.26) and (11.25) can be written as the set

$$\nabla_\mu F_{\nu\kappa}+\nabla_\nu F_{\kappa\mu}+\nabla_\kappa F_{\mu\nu}=0. \tag{11.34}$$

We shall now introduce the doubled theory in order to have a space inversion operator S as in the Appendix.

In the doubled theory let $\tilde{\psi}$ be written

$$\tilde{\psi}=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{11.35}$$

where ψ_1 and ψ_2 are eight component column vectors which satisfy

$$(\gamma^\mu\nabla_\mu+m)\psi_1=0, \quad (\gamma^{\mu*}\nabla_\mu+m)\psi_2=0, \tag{11.36}$$

and where the last component of each column vector is required to be zero. In analogy to the undoubled theory let us write

$$\psi_1=\begin{bmatrix}\boldsymbol{\psi}_1\\ \mathbf{A}_1\\ V_1\\ 0\end{bmatrix}, \quad \psi_2=\begin{bmatrix}\boldsymbol{\psi}_2\\ \mathbf{A}_2\\ V_2\\ 0\end{bmatrix}. \tag{11.37}$$

To obtain the neutral vector meson theories we require solutions of (11.36) such that

$$\mathbf{A}_1=\mathbf{A}_2=\mathbf{A}, \quad V_1=V_2=V. \tag{11.38}$$

where $\mathbf{A}$ and V are real. We can show

$$\boldsymbol{\psi}_2=\boldsymbol{\psi}_1^*. \tag{11.39}$$

Hence

$$\mathbf{E}=(\boldsymbol{\psi}_2+\boldsymbol{\psi}_1)/2, \quad \mathbf{H}=(\boldsymbol{\psi}_2-\boldsymbol{\psi}_1)/2i. \tag{11.40}$$

In the case of the doubled neutral vector meson theory, we can show that the current $j_\mu\sim i\psi^*S\gamma_\mu\psi\equiv 0$.

To obtain the charged vector meson field we must use the doubled theory as above. But now we require A and V of (11.38) to be complex. The vectors E and H are *defined* by (11.40) and satisfy the same equations in terms of A and V (11.25)–(11.34) as in the neutral theory. In contrast to the neutral theory $j_\mu\neq 0$, in general.

We can now quantize the theory. For the charged meson field we have

$$\begin{aligned}\psi_1(\mathbf{x},t)=C''\int d^3p\Bigg\{\Bigg[\left(\frac{\omega}{m}\right)^{\frac{1}{2}}u(\mathbf{p},+,0)a(\mathbf{p},+,0)\\ +\sum_{\zeta=\pm1}\left(\frac{m}{\omega}\right)^{\frac{1}{2}}u(p,+,\zeta)a(\mathbf{p},+,\zeta)\Bigg]\exp(ip_\mu x^\mu)\\ +\Bigg[\left(\frac{\omega}{m}\right)^{\frac{1}{2}}u(\mathbf{p},-,0)a^\dagger(\mathbf{p},-,0)+\sum_{\zeta=\pm1}\left(\frac{m}{\omega}\right)^{\frac{1}{2}}\\ \times u(\mathbf{p},-,-\zeta)a^\dagger(\mathbf{p},-,\zeta)\Bigg]\exp(-ip_\mu x^\mu)\Bigg\},\end{aligned} \tag{11.40}$$

while for the neutral meson field

$$\begin{aligned}\psi(x,t)=C'''\int d^3p\Bigg\{\Bigg[\left(\frac{\omega}{m}\right)^{\frac{1}{2}}u(\mathbf{p},+,0)a(\mathbf{p},0)\\ +\sum_{\zeta=\pm1}\left(\frac{m}{\omega}\right)^{\frac{1}{2}}u(\mathbf{p},+,\zeta)a(\mathbf{p},\zeta)\Bigg]\exp(ip_\mu x^\mu)\\ +\Bigg[\left(\frac{\omega}{m}\right)^{\frac{1}{2}}u(\mathbf{p},-,0)a^\dagger(\mathbf{p},0)+\sum_{\zeta=\pm1}\left(\frac{m}{\omega}\right)^{\frac{1}{2}}\\ \times u(\mathbf{p},-,-\zeta)a^\dagger(\mathbf{p},\zeta)\Bigg]\exp(-ip_\mu x^\mu)\Bigg\},\end{aligned} \tag{11.41}$$

where

$$[a(\mathbf{p},\epsilon,\zeta),a(\mathbf{p}',\epsilon',\zeta')]_-=0, \tag{11.42}$$

$$[a(\mathbf{p},\epsilon,\zeta),a^\dagger(\mathbf{p}',\epsilon',\zeta'^\dagger)]_-=\delta_{\epsilon\epsilon'}\delta_{\zeta\zeta'}\delta(\mathbf{p}-\mathbf{p}'), \tag{11.43}$$

$$[a(\mathbf{p},\zeta),a(\mathbf{p}',\zeta')]_-=0, \tag{11.44}$$

$$[a(\mathbf{p},\zeta),a^\dagger(\mathbf{p}',\zeta')]_-=\delta_{\zeta\zeta'}\delta(\mathbf{p}-\mathbf{p}'). \tag{11.45}$$

APPENDIX I

$$\gamma_0=\begin{bmatrix}0&0&0&-1&0&0&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&-1&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&-1&0\end{bmatrix}, \quad \gamma_1=\begin{bmatrix}0&0&0&0&0&0&1&0\\0&0&0&0&0&-i&0&0\\0&0&0&0&i&0&0&0\\0&0&0&0&0&0&0&1\\0&0&-i&0&0&0&0&0\\0&i&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\end{bmatrix},$$

$$\gamma_2=\begin{bmatrix}0&0&0&0&0&i&0&0\\0&0&0&0&0&0&1&0\\0&0&0&-i&0&0&0&0\\0&0&i&0&0&0&0&0\\0&0&0&0&0&0&0&1\\-i&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\end{bmatrix}, \quad \gamma_3=\begin{bmatrix}0&0&0&0&-i&0&0&0\\0&0&0&i&0&0&0&0\\0&0&0&0&0&0&1&0\\0&-i&0&0&0&0&0&0\\i&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix},$$

$$K_{23}=\begin{bmatrix} 0&0&0&&&&&\\ 0&0&-1&&&&&\\ 0&1&0&&&&&\\ &&&0&0&0&&\\ &&&0&0&-1&&\\ &&&0&1&0&&\\ &&&&&&0&\\ &&&&&&&0 \end{bmatrix},\qquad K_{31}=\begin{bmatrix} 0&0&1&&&&&\\ 0&0&0&&&&&\\ -1&0&0&&&&&\\ &&&0&0&1&&\\ &&&0&0&0&&\\ &&&-1&0&0&&\\ &&&&&&0&\\ &&&&&&&0 \end{bmatrix},$$

$$K_{12}=\begin{bmatrix} 0&-1&0&&&&&\\ 1&0&0&&&&&\\ 0&0&0&&&&&\\ &&&0&-1&0&&\\ &&&1&0&0&&\\ &&&0&0&0&&\\ &&&&&&0&\\ &&&&&&&0 \end{bmatrix},\qquad K_{01}=\begin{bmatrix} 0&0&0&&&&&\\ 0&0&-i&&&&&\\ 0&i&0&&&&&\\ &&&0&0&0&-1&\\ &&&0&0&0&0&\\ &&&0&0&0&0&\\ &&&-1&0&0&0&\\ &&&&&&&0 \end{bmatrix},$$

$$K_{02}=\begin{bmatrix} 0&0&i&&&&&\\ 0&0&0&&&&&\\ -i&0&0&&&&&\\ &&&0&0&0&0&\\ &&&0&0&0&-1&\\ &&&0&0&0&0&\\ &&&0&-1&0&0&\\ &&&&&&&0 \end{bmatrix},\qquad K_{03}=\begin{bmatrix} 0&-i&0&&&&&\\ i&0&0&&&&&\\ 0&0&0&&&&&\\ &&&0&0&0&0&\\ &&&0&0&0&0&\\ &&&0&0&0&-1&\\ &&&0&0&-1&0&\\ &&&&&&&0 \end{bmatrix},$$

$$T=\begin{bmatrix} 1&&&&&&&\\ &1&&&&&&\\ &&1&&&&&\\ &&&-1&&&&\\ &&&&-1&&&\\ &&&&&-1&&\\ &&&&&&1&\\ &&&&&&&-1 \end{bmatrix},\qquad \tilde{S}=\begin{bmatrix} 0_8 & \begin{matrix} -1&&&&&&&\\ &-1&&&&&&\\ &&-1&&&&&\\ &&&-1&&&&\\ &&&&-1&&&\\ &&&&&-1&&\\ &&&&&&1&\\ &&&&&&&1 \end{matrix} \\ \begin{matrix} -1&&&&&&&\\ &-1&&&&&&\\ &&-1&&&&&\\ &&&-1&&&&\\ &&&&-1&&&\\ &&&&&-1&&\\ &&&&&&1&\\ &&&&&&&1 \end{matrix} & 0_8 \end{bmatrix},$$

$$\tilde{C}=\begin{pmatrix} 0_8 & I_8 \\ I_8 & 0_8 \end{pmatrix},\qquad \tilde{S}_1=\begin{bmatrix} 0_8 & \begin{matrix} -1&&&&&&&\\ &1&&&&&&\\ &&1&&&&&\\ &&&-1&&&&\\ &&&&1&&&\\ &&&&&1&&\\ &&&&&&1&\\ &&&&&&&1 \end{matrix} \\ \begin{matrix} -1&&&&&&&\\ &1&&&&&&\\ &&1&&&&&\\ &&&-1&&&&\\ &&&&1&&&\\ &&&&&1&&\\ &&&&&&1&\\ &&&&&&&1 \end{matrix} & 0_8 \end{bmatrix},$$

$$\tilde{S}_2=\begin{bmatrix} 0_8 & \begin{matrix} 1 &&&&&&& \\ & -1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& -1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} \\ \begin{matrix} 1 &&&&&&& \\ & -1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& -1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} & 0_8 \end{bmatrix}, \quad \tilde{S}_3=\begin{bmatrix} 0_8 & \begin{matrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && -1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& -1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} \\ \begin{matrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && -1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& -1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} & 0_8 \end{bmatrix},$$

$$\mathbf{e}(\mathbf{p},0)=\mathbf{p}/p,$$

$$\left.\begin{aligned} e_1(\mathbf{p},\zeta)&=\frac{pp_3-i\zeta p_1p_2}{p[2(p_1^2+p_3^2)]^{\frac{1}{2}}}, \\ e_2(\mathbf{p},\zeta)&=\frac{i\zeta}{p}\left(\frac{p_1^2+p_3^2}{2}\right)^{\frac{1}{2}}, \\ e_3(\mathbf{p},\zeta)&=\frac{-(pp_1+i\zeta p_3p_2)}{p[2(p_1^2+p_3^2)]^{\frac{1}{2}}}, \end{aligned}\right\}\zeta=\pm 1,$$

$$p=|\mathbf{p}|.$$

$$u(\mathbf{p},\epsilon,0)=\begin{bmatrix} i\epsilon mp_1/\omega p \\ i\epsilon mp_2/\omega p \\ i\epsilon mp_3/\omega p \\ p_1/p \\ p_2/p \\ p_3/p \\ \epsilon p/\omega \\ 0 \end{bmatrix},$$

$$u(\mathbf{p},\epsilon,\zeta)=\begin{bmatrix} (i\epsilon/m)(\omega-\epsilon\zeta p)e_1(\mathbf{p},\zeta) \\ (i\epsilon/m)(\omega-\epsilon\zeta p)e_2(\mathbf{p},\zeta) \\ (i\epsilon/m)(\omega-\epsilon\zeta p)e_3(\mathbf{p},\zeta) \\ e_1(\mathbf{p},\zeta) \\ e_2(\mathbf{p},\zeta) \\ e_3(\mathbf{p},\zeta) \\ 0 \\ 0 \end{bmatrix}, \quad \zeta=\pm 1.$$

APPENDIX II. LIST OF IMPROPER TRANSFORMATIONS

$T=e^{i\phi}V^{-1}D\gamma_1'\gamma_2'\gamma_3'V^{-1t}$,

$S=\pm iV^{-1}\mathcal{P}'\gamma_0'V$ for undoubled theory,

$S=\begin{pmatrix} 0 & e^{i\phi}A \\ e^{-i\phi}A^\dagger & 0 \end{pmatrix}$ for doubled theory,

where

$A=iV^{-1}D\gamma_0'V^*$,

$C=-ie^{i\phi}V^{-1}\mathcal{P}'DV^*$ for undoubled theory.

$C=\exp[i(\pi/2)(2m+2n+1)]\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

for doubled theory.

The operators D, $\mathcal{P}'$, γ_μ', V, are defined in Sec. 3.