nuclear radiation by examining the spin resonance spectrum of ruby near the first cross-over, occurring near 2000 gauss. If lattice vacancies produced by nuclear radiations become associated with the chromium ions, the crystalline electric field will no longer have axial symmetry, so that the appropriate spin-Hamiltonian will contain a rhombic field term. This will have the effect of admixing the states $S_z=-\frac{1}{2}$ and $\frac{3}{2}$. This particular region of the energy level diagrams can be explored with frequencies near 5.75 kMc/sec and/or 17.25 kMc/sec. The advantage of using chromium rather than vanadium is that the results will not be complicated by hyperfine interactions.

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Surface Impedance of a Superconductor in a Magnetic Field*

G. Dresselhaus and M. Spiewak Dresselhaus†
Laboratory for Atomic and Solid State Physics and Department of Physics, Cornell University, Ithaca, New York
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Explicit expressions for the static magnetic field variation of the surface impedance in a superconductor are derived. Detailed consideration is given to the two limiting cases of the classical skin effect and of the extreme anomalous skin effect, with the static field either parallel or perpendicular to the rf current. The transport calculation is carried out for a two-fluid model with the supercurrent obeying the London equation and the normal current following a nonlocal relation. It is suggested that a comparison between theory and the experimental data yields information on the energy band structure of normal electrons in the superconducting state.

I. INTRODUCTION

EXPERIMENTAL investigations of the surface impedance of superconducting tin in a static magnetic field have been reported by Pippard at 9.4 kMc/sec [1] and by Spiewak at 1 kMc/sec.[2] In particular, the observation at 1 kMc/sec of a decrease in both the surface resistance, R, and the surface reactance, X, with increasing static magnetic field, H, has not been explained. Nor has it been understood why the relative signs of $[R(H)-R(0)]$ and $[X(H)-X(0)]$ are dependent on temperature, on the magnitude and orientation of H relative to the rf magnetic field, on crystalline orientation, and on the rf frequency. In this paper, an attempt is made to achieve an understanding of these experimental results and to consider the type of information that can be obtained on the energy band structure of the superconducting metal from studies of the surface impedance in a static magnetic field.

Studies of the anomalous skin effect in a magnetic field have provided a powerful tool for the investigation of the band structure in normal metals.[3] Important information has been obtained by cyclotron resonance experiments on the effective masses of the electrons and on the shape of the Fermi surface in Sn,[4,5] Cu,[4,6] and Bi.[7] Since the magnetic field dependence of the surface impedance differs as to whether the metal is in the normal or superconducting state, the information obtained through these studies is different for the two cases.

In this paper, the surface impedance of a superconductor in a magnetic field is calculated by extending the two-fluid model first introduced by Maxwell, Marcus, and Slater[8] to treat the zero field problem. Specular reflection boundary conditions are applied to the case of a superconductor in a magnetic field following the Serber treatment of the zero field superconducting surface impedance problem[9] and the Mattis and Dresselhaus treatment of the normal metal in a magnetic field.[10] There are two principal reasons for choosing this approach. Its relative simplicity permits an explicit determination of the field variation for both R and X. Since at frequencies, $\nu<10$ kMc/sec the electromagnetic propagation does not directly involve

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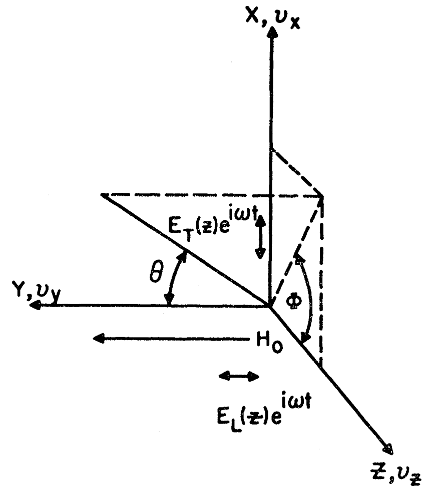

FIG. 1. The geometrical arrangement for surface impedance calculation. The xy plane is the surface of the metal which occupies the space $z>0$. The dc magnetic field H_0 is taken along the y direction and the two principal orientations for the applied rf electric field are: E_L along H_0 (longitudinal geometry) and E_T along the x axis (transverse geometry). The spherical coordinates used in velocity space are also indicated.

an energy gap, a two-fluid model may be expected to yield useful information for both $H=0$ and $H>0$. For example, Bardeen has reported a two-fluid model to be appropriate to the low-frequency limit of the BCS theory.[11]

The calculation of the magnetic field variation of R and X to terms in H^2 is presented in Sec. II. Explicit expressions are derived for the two limiting cases of the classical skin effect and of the extreme anomalous skin effect. The calculation is carried out for the static field parallel to the rf current (longitudinal field orientation) and perpendicular to it (transverse field orientation). The results are expressed in terms of a dimensionless parameter, (λ/δ), in which λ is the penetration depth for the static field in a superconductor and δ is the skin depth of the rf fields associated with the normal electron assembly. By introducing further assumptions which relate λ and δ to the temperature, a detailed comparison with the experimental results can be made. This analysis is to be presented elsewhere. Section III is devoted to a discussion of the results of Sec. II. Comments are made on some of the assumptions which are introduced into the theory of Sec. II. It is suggested that surface impedance studies provide a promising tool for studying the Fermi surface of normal electrons in a superconductor.

II. CALCULATION OF THE MAGNETIC FIELD VARIATION OF SURFACE IMPEDANCE

The surface of the superconductor is considered to be a flat plate in the xy plane with the z direction normal to the surface and extending into the superconductor. The dc magnetic field H_0 is taken to be parallel to the y axis in the plane of the sample surface, as shown in Fig. 1, and to have a spatial variation $H_0(z)$. In general, the rf electric field and current are arbitrarily directed with respect to H_0. The two cases of special interest, the longitudinal and transverse field orientations, are treated in detail.

Because of the high sensitivity required to observe the field dependence of the surface impedance, the most successful experimental technique in both longitudinal and transverse fields does not employ precisely the geometry of Fig. 1.[1,2] In order to attain the required sensitivity, a sufficiently large fraction of the microwave losses must be confined to the sample itself. This is accomplished by choosing the superconducting specimen in the form of a thin wire, the length of which determines the resonant frequency of an rf resonant cavity. The sample diameter, d, is large compared with both the rf skin depth, δ, and the superconducting penetration depth for the static field, λ. Since $(d/\lambda)>10^2$ and since at frequencies $\nu\approx 1$ kMc/sec $(d/\delta)>10^2$ also, the electrodynamic behavior of a wire sample in the absence of H_0 can be approximately described by that of a plane surface. In the experimental arrangement, the rf electric current j_1, is along the wire axis. In the longitudinal field experiments, H_0 is applied parallel to j_1, and the local static field at the surface of the superconductor is constant and is directed along the wire axis. Thus, the plane surface approximation is equally appropriate to the longitudinal field case.

The transverse field experiments are carried out with H_0 perpendicular to the wire axis. Because of demagnetization effects, the local static field in the cylindrical sample is directed parallel to the surface with a magnitude no longer a constant but depending on the polar angle defined by the radius vector and the applied static field. However, by using the London equations to describe the field penetration, it is possible to interpret the transverse field experiments on a cylindrical surface in terms of an equivalent plane surface, with an equivalent uniform static field applied along the y direction.[1,2] In this case, the rf electric field E_1 is in the xz plane. Experimental data are available for the transverse case at frequencies $\nu\cong 1$ kMc/sec and 9.4 kMc/sec while for the longitudinal case only at $\nu\cong 1$ kMc/sec.

The current in the superconductor is written as

$$\mathbf{j}=\mathbf{j}_n e^{i\omega t}+\mathbf{j}_{s0}+\mathbf{j}_{s1}e^{i\omega t}, \tag{1}$$

where the normal current $\mathbf{j}_n$ is obtained in the usual fashion by the solution of Boltzmann's equation. The dc supercurrent, denoted by $\mathbf{j}_{s0}$, is found from solution of the London equation

$$c\,\mathrm{curl}(\Lambda\mathbf{j}_{s0})=-\mathbf{H}_0. \tag{2a}$$

In the London theory, $\Lambda=4\pi\lambda^2/c^2$ is a tensor quantity, a fact which is for the moment ignored; thus, the

[11] J. Bardeen, Phys. Rev. Letters 1, 399 (1958).

anisotropic nature of the penetration depth, λ, is neglected in this calculation. The London constant, Λ, can be written in terms of the plasma frequency for the superconducting electrons, ω_s, with

$$\omega_s^2=4\pi N_s e^2/m=4\pi/\Lambda, \tag{3}$$

in which N_s is the concentration of superconducting electrons, and m is the free electron mass. In this calculation it is assumed that the superconducting electrons screen the static magnetic field without in any way impairing their ability to screen the rf fields. Thus, the rf supercurrent, $\mathbf{j}_{s1}$, also obeys the London equation,

$$c\,\mathrm{curl}(\Lambda\mathbf{j}_{s1})=-\mathbf{H}_1, \tag{2b}$$

in which $\mathbf{H}_1$ is the rf magnetic field.

The spatial variation of the dc and rf fields is obtained by combining the London equations (2a) and (2b) with the Maxwell equations for the dc and rf fields,

$$\text{dc};\ c\,\mathrm{curl}\mathbf{H}_0=4\pi(\mathbf{j}_{s0}+\mathbf{j}_{b0}), \tag{4a}$$

$$\text{rf};\ c\,\mathrm{curl}\mathbf{H}_1=\partial\mathbf{D}_1/\partial t+4\pi(\mathbf{j}_n+\mathbf{j}_{s1}+\mathbf{j}_{b1}), \tag{4b}$$

in which $\mathbf{j}_{b0}$ and $\mathbf{j}_{b1}$ describe the dc and rf surface currents, respectively, at $z=0$. (See Fig. 1.) By applying the specular reflection boundary condition for the electron trajectories, the semi-infinite superconductor can be replaced by an equivalent infinite superconducting medium with the boundary current sheets at the plane $z=0$ given by

$$\text{dc};\ j_{b0x}=-(c/2\pi)H_0(0)\delta(z), \tag{5a}$$

$$\text{rf};\ \mathbf{j}_{b1}=-(c/2\pi)[\mathbf{H}_1(0)\times\mathbf{k}]\delta(z), \tag{5b}$$

in which $\mathbf{k}$ is a unit vector in the z direction. Note that by symmetry the fields and currents depend only on z.

The specular reflection boundary condition is handled more conveniently in terms of the Fourier transforms

$$\mathbf{H}_q=\int_{-\infty}^{\infty}e^{iqz}\mathbf{H}(z)dz, \tag{6a}$$

and

$$\mathbf{H}(z)=(1/2\pi)\int_{-\infty}^{\infty}e^{-iqz}\mathbf{H}_q dq. \tag{6b}$$

In terms of Eqs. (6a) and (6b), the London equation becomes

$$\mathbf{H}_q=-ciq\Lambda(\mathbf{j}_{sq}\times\mathbf{k}). \tag{7}$$

Since Eqs. (4a) and (5a) do not involve the normal current $\mathbf{j}_n$, the Fourier transform of the static field is related to the dc supercurrent Fourier transform $\mathbf{j}_{s0q}$ by

$$ciqH_{0qy}=4\pi j_{s0qx}-2cH_0(0). \tag{8}$$

The solution of Eq. (8) is obtained by substitution of Eq. (7) for $\mathbf{j}_{s0q}$, yielding

$$H_{0qy}=\frac{2iqH_0(0)}{q^2+\lambda^{-2}}, \tag{9}$$

in which $H_0(0)$ is the magnitude of the static field at the surface $z=0$. From Eq. (6b), the spatial variation of H_0 is given by

$$H_0(z)/H_0(0)=(i/\pi)\int_{-\infty}^{\infty}\frac{qe^{-iqz}dq}{q^2+\lambda^{-2}}=\begin{cases} e^{-z/\lambda}; & z>0 \\ -e^{-|z|/\lambda}; & z<0\end{cases}. \tag{10}$$

Physically, the specular reflection of the electron trajectories at $z=0$ is equivalent to reversing the sign of the magnetic fields, but not of the electric fields in the extended medium $z<0$. [See Eqs. (8) and (9).]

The solution for the rf fields is more complicated, since $\mathbf{E}_1$ and $\mathbf{H}_1$ depend on $\mathbf{j}_n$. An explicit expression for $\mathbf{j}_n$ is found from the solution of the Boltzmann equation for the normal current distribution function $f(z)$. A convenient form for the Boltzmann equation can be written in terms of $\delta f(z)$, the departure of $f(z)$ from the equilibrium value $f_0(z)$,[12]

$$\begin{aligned}(1+i\omega\tau)\delta f(z)&+v\tau\sin\phi\sin\theta(\partial/\partial z)[\delta f(z)]\\ &+\omega_c\tau[H_0(z)/H_0(0)](\partial/\partial\phi)[\delta f(z)]\\ =&-ev\tau f_0'[E_x(z)\sin\theta\cos\phi\\ &+E_y(z)\cos\theta+E_z(z)\sin\theta\sin\phi]\\ =&-e\tau f_0'\mathbf{E}(z)\cdot\mathbf{v},\end{aligned} \tag{11}$$

in which v, θ, ϕ are the polar coordinates of velocity space shown in Fig. 1, $\omega_c=[eH_0(0)/m^*c]$, and $f_0'=(\partial f_0/\partial\epsilon)$, ϵ being the energy. The quantities m^* and τ of Eq. (11) refer to the effective mass and the relaxation time for the normal electrons in the superconducting state. In this treatment the presence of superconductivity is exhibited explicitly by allowing the magnetic field term of the Boltzmann equation [Eq. (1) of reference 10] to have a spatially dependent factor $[H_0(z)/H_0(0)]$. Thus, much of the formal development of the present problem parallels the solution of the longitudinal cyclotron resonance in the normal metal.[10]

In writing Eq. (11) the possible z dependence of the equilibrium distribution function f_0 has been neglected. The presence of a term in $[\partial f_0(z)/\partial z]$ could come about from local heating of the sample by the electromagnetic fields, thus giving rise to the possibility of temperature variations in the sample.

The Fourier transform of Eq. (11) can be written as

$$[1-iv\bar{\tau}q\sin\theta\sin\phi]\delta f_q+\frac{i\omega_c\bar{\tau}}{\pi}\frac{\partial}{\partial\phi}\int_{-\infty}^{\infty}\frac{q'\delta f_{q-q'}dq'}{q'^2+\lambda^{-2}}=-e\bar{\tau}f_0'\mathbf{E}_q\cdot\mathbf{v} \tag{12}$$

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[12] For simplicity, the subscript 1 on the rf currents and fields is omitted after Eq. (10).

in which $\bar{\tau}=\tau(1+i\omega\tau)^{-1}$ and δf_q and $\mathbf{E}_q$ are the Fourier transforms of δf and $\mathbf{E}$, respectively. The solution of Eq. (12) is obtained by iteration in terms of the function $\Omega(x)$,

$$\Omega(x)=[1-x\sin\theta\sin\phi]^{-1}, \tag{13}$$

and is to order $(\omega_c\bar{\tau})^2$

$$\delta f_q=-ef_0'\bar{\tau}\Omega(iv\bar{\tau}q)\Bigg((\mathbf{E}_q\cdot\mathbf{v})-\frac{i\omega_c\bar{\tau}}{\pi}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\frac{\partial}{\partial\phi}[\Omega(iv\bar{\tau}(q-q'))(\mathbf{E}_{q-q'}\cdot\mathbf{v})]$$
$$-\frac{(\omega_c\bar{\tau})^2}{\pi^2}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}\frac{\partial}{\partial\phi}\left\{\Omega(iv\bar{\tau}(q-q'))\frac{\partial}{\partial\phi}[\Omega(iv\bar{\tau}(q-q'-q''))(\mathbf{E}_{q-q'-q''}\cdot\mathbf{v})]\right\}+\mathcal{O}(\omega_c\bar{\tau})^3\Bigg). \tag{14}$$

For this calculation it is sufficient to evaluate the Fermi distribution function f_0 and its derivative f_0' at $T=0$°K.

The Fourier transform of the ith component of the normal current is given by

$$j_{nq,i}=2em^{*3}h^{-3}\int\int\int\delta f_q v_i v^2\sin\theta dv d\theta d\phi \tag{15}$$

in which v_i is the ith component of the velocity. In order to carry out the integrals explicitly, the effect of the magnetic field on $j_{nq,i}$ is treated as a perturbation, with $(\omega_c\bar{\tau})$ as the expansion parameter. Because of the smallness of the observed longitudinal field variation of the surface impedance, it is sufficient for this case to retain terms to $(\omega_c\bar{\tau})^2$. Observation of the surface impedance in a transverse static field yields a field dependence larger in magnitude, and in which contribution from terms of order >2 is important for at least part of the temperature range.[2] Since the calculation is carried out only to quadratic terms, the range of validity of the calculation (expressed in terms of reduced field[13]) is smaller for the transverse case than for the longitudinal case. Substitution of Eq. (14) into Eq. (15) yields

$$j_{nq,i}=\frac{3Ne^2\bar{\tau}}{4m^*}\sum_j\Bigg[K_{ij}(i\bar{l}q)E_{q,j}-\frac{i\omega_c\bar{\tau}}{\pi}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}L_{ij}(i\bar{l}q,i\bar{l}q')E_{q-q',j}$$
$$-\left(\frac{\omega_c\bar{\tau}}{\pi}\right)^2\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{ij}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')E_{q-q'-q'',j}+\mathcal{O}(\omega_c\bar{\tau})^3\Bigg], \tag{16}$$

in which N is the concentration of normal electrons, v_F is the Fermi velocity associated with the normal electrons, and $\bar{l}$ is related to the mean free path $l=v_F\tau$ by

$$\bar{l}=v_F\bar{\tau}=v_F\tau(1+i\omega\tau)^{-1}. \tag{17}$$

The functions K_{ij}, L_{ij}, and M_{ij} are given explicitly in Appendix A; the eight nonvanishing functions are K_{ii}, L_{xz}, L_{zx}, M_{ii}, $i=x, y, z$. Thus, the component of the rf normal electron current along the static field (the y direction) couples only to the rf electric field $E_{q,y}$. Physically, the vanishing of the functions L_{yi} and L_{iy} ($i=x, z$) implies the absence of an rf Hall effect for the longitudinal static field orientation. However, Eq. (16) does not eliminate the possibility of an rf Hall effect in the transverse field experiment.

A complete solution for the rf fields and currents can be obtained by combining the Maxwell equations with Eq. (16). In terms of the Fourier transform for the ith component of the rf electric field, the Maxwell equations (4a) and (4b) can be written as

$$(-q^2c^2-\omega_s^2+\omega^2)E_{q,i}=4\pi i\omega j_{nq,i}+2c^2E_i'(0)$$
$$-\delta_{i,z}[c^2q^2E_{q,z}+4\pi\omega^2P_{q,z}+2c^2E_z'(0)], \tag{18}$$

in which $\delta_{i,z}$ is the Kronecker delta function. The second term on the right-hand side of Eq. (18) is the boundary current term [Eq. (5b)] in which $E_i'(0)=(\partial E_i/\partial z)_{z=0}$. The third and fourth terms arise from the polarization of the medium associated with the fluctuation of charge density in space and time. The supercurrent term $j_{sq,i}$ of Eq. (4b) has been replaced by the term in $(\omega_s^2E_{q,i})$ through the London Eq. (2b).

Although the equilibrium charge density ρ_0 vanishes, the presence of rf fields and currents and of electronic collisions gives rise to charge density fluctuations $\delta\rho=\delta\rho_n+\delta\rho_s$ which must satisfy the Maxwell equation

$$\mathrm{div}\mathbf{D}=4\pi\delta\rho. \tag{19}$$

The normal component $\delta\rho_n$ is related to the distribution function by

$$\delta\rho_n=2em^{*3}h^{-3}\int\delta f v^2 dv\sin\theta d\theta d\phi \tag{20}$$

and to the normal current density by the continuity equation

$$\mathrm{div}\mathbf{j}_n+(\partial\rho_n/\partial t)_{\mathrm{drift}}=0. \tag{21}$$

[13] The reduced field, h, at a temperature T is defined as $h(T)=H_0/H_c(T)$, where $H_c(T)$ is the critical field, and H_0 is the applied static field.

In writing either the Boltzmann equation or the continuity equation for the normal electron assembly, it is assumed that the number of normal electrons is a constant of the motion. If these density fluctuations do not give rise to a conversion of normal to superconducting electrons (and *vice-versa*), then this model implies the existence of local temperature fluctuations or waves. The solution of Eq. (21) yields

$$\delta\rho_{nq}=iq\bar{\tau}j_{nq,z}. \tag{22}$$

Equations (18) and (19) result in the relation

$$\delta\rho_{nq}+\delta\rho_{sq}=q\omega^{-1}(j_{nq,z}+j_{sq,z}), \tag{23}$$

in which $\delta\rho_{sq}$ is related to the supercurrent density by the equation of continuity

$$\delta\rho_{sq}=q\omega^{-1}j_{sq,z}, \tag{24}$$

since the superconducting electrons suffer no collisions. The equations (22)–(24) imply $qj_{nq,z}(1+i\omega\tau)^{-1}=0$, which has only the trivial solution

$$j_{nq,z}\equiv 0. \tag{25}$$

An rf Hall current can be carried by the normal electrons only in the xy plane; any Hall current in the z direction must be transported by superconducting electrons.

The field variation of the surface impedance Z is formally contained in the simultaneous solution of Eqs. (16) and (18), since

$$Z=R+iX=4\pi c^{-1}(E/H)_{z=0}. \tag{26}$$

Explicit expressions for the surface resistance $R(H_0)$ and surface reactance $X(H_0)$ are derived only for the longitudinal and transverse field orientations.

(a) Longitudinal Field

For this geometry, there is only one nonvanishing component of the rf normal current given by

$$j_{nq,y}=\frac{3\omega_n^2\bar{\tau}}{16\pi}\left[K_{yy}(i\bar{l}q)E_{q,y}-\left(\frac{\omega_c\bar{\tau}}{\pi}\right)^2\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{yy}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')E_{q-q'-q'',y}+\Theta(\omega_c\bar{\tau})^4\right], \tag{27}$$

in which the plasma frequency ω_n is defined by

$$\omega_n^2=4\pi Ne^2/m^*. \tag{28}$$

Substitution of Eq. (27) into Maxwell's Eq. (18) yields the integral equation

$$\left[-q^2c^2+\omega^2-\omega_s^2-\tfrac{3}{4}i\omega\omega_n^2\bar{\tau}K_{yy}(i\bar{l}q)\right]E_{q,y}$$
$$=2c^2E_y'(0)-\frac{3i\omega\omega_n^2\omega_c^2\bar{\tau}^3}{4\pi^2}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{yy}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')E_{q-q'-q'',y}+\Theta(\omega_c\bar{\tau})^4. \tag{29}$$

Assuming that the effect of the static field on the rf field distribution is small (a good approximation for the longitudinal field orientation), Eq. (29) can be solved by iteration. The result can be written as

$$E_{q,y}=2E_y'(0)c^2\Gamma_y(q)\left[1-\frac{3i\omega\omega_n^2\omega_c^2\bar{\tau}^3}{4\pi^2}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{yy}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')\Gamma_y(q-q'-q'')+\Theta(\omega_c\bar{\tau})^4\right], \tag{30}$$

in which

$$\Gamma_y(q)=\left[-q^2c^2+\omega^2-\omega_s^2-\tfrac{3}{4}i\omega\omega_n^2\bar{\tau}K_{yy}(i\bar{l}q)\right]^{-1}. \tag{31}$$

Since no power is coupled from the rf field component E_y to E_x or to E_z, it is sufficient to take $E_x=E_z=0$. Within the approximation of a scalar m^*, the longitudinal static field orientation gives rise to no rf Hall field.

The longitudinal field variation of the surface impedance is determined by Eq. (30), with

$$Z_L(H_0)=-\frac{4\pi i\omega}{c^2}\frac{E_y(0)}{E_y'(0)}=-\frac{2i\omega}{c^2E_y'(0)}\int_{-\infty}^{\infty}E_{q,y}dq. \tag{32}$$

Since the surface impedance in the absence of a static field is

$$Z(0)=R(0)+iX(0)=-4i\omega\int_{-\infty}^{\infty}\Gamma_y(q)dq, \tag{33}$$

the longitudinal field variation of the surface impedance is

$$\Delta Z_L=Z_L(H_0)-Z_L(0)=\Delta R_L+i\Delta X_L$$
$$=-\frac{3\omega^2\omega_n^2\omega_c^2\bar{\tau}^3}{\pi^2}\int_{-\infty}^{\infty}\Gamma_y(q)dq\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{yy}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')\Gamma_y(q-q'-q'')+\Theta(H_0^4). \tag{34}$$

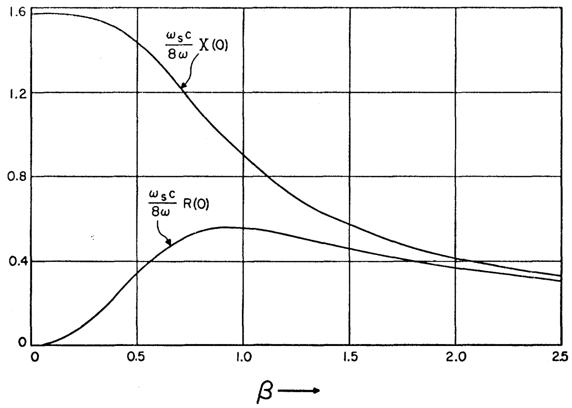

FIG. 2. Plot of the zero field surface resistance and surface reactance for a superconducting metal in the limit of the classical skin effect. The dimensionless quantities $(\omega_s c/8\omega)R(0)$ and $(\omega_s c/8\omega)X(0)$ are plotted vs the dimensionless parameter $\beta=\lambda/\delta_{Cl}$. The points $\beta=0$ corresponds to $T=0°K$ and $\beta=\infty$ to $T=T_c$. See text for definition of symbols.

The integrals of Eqs. (33) and (34) are carried out in Appendix B for the two limiting cases, the normal electrons described by either the classical skin effect ($|q\bar{l}|\ll 1$) or by the extreme anomalous skin effect ($|q\bar{l}|\gg 1$).

The zero field result for the classical case is

$$Z_{Cl}(0)=-4\pi\omega k_c^{-1}c^{-2}, \tag{35}$$

with

$$k_c=(-\lambda^{-2}+\omega^2c^{-2}-i\omega\omega_n^2\bar{\tau}c^{-2})^{\frac{1}{2}}, \tag{36}$$

$\mathrm{Re}(k_c)<0$ and $\mathrm{Im}(k_c)>0$. The corresponding result for the extreme limit of the anomalous skin effect is

$$Z(0)=-8\omega c^{-1}\omega_s^{-1}\sum_{i=1}^{3}\epsilon_i\mathcal{C}_i\ln(-\epsilon_i) \tag{37}$$

in which ϵ_i denotes the three roots of the dimensionless equation,

$$\epsilon^3-\epsilon-\alpha^3=0. \tag{38}$$

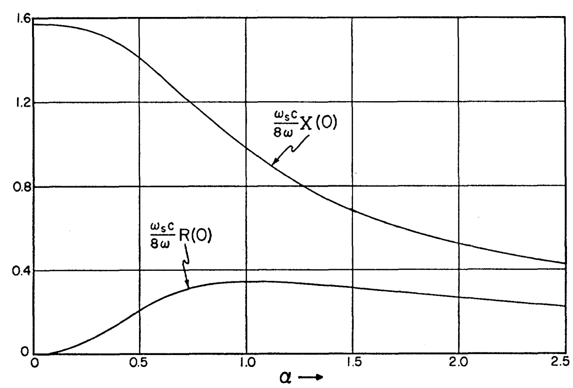

FIG. 3. Plot of the zero field surface resistance and surface reactance for a superconducting metal in the limit of the extreme anomalous skin effect. The dimensionless quantities $(\omega_s c/8\omega)R(0)$ and $(\omega_s c/8\omega)X(0)$ are plotted vs the dimensionless parameter $\alpha=\lambda/\delta_n$. The points $\alpha=0$ corresponds to $T=0°K$ and $\alpha=\infty$ to $T=T_c$. See text for definition of symbols.

The dimensionless quantities ϵ and α are defined by

$$\epsilon=iq\lambda \tag{39}$$

and

$$\alpha^3=\tfrac{3}{4}\pi\omega\omega_n^2\omega_s^{-3}cv_F^{-1}=\lambda^3/\delta_n^3. \tag{40}$$

The rf skin depth in the extreme anomalous limit for the normal electrons δ_n is given by[14]

$$\delta_n=(\tfrac{3}{4}\pi\omega\omega_n^2c^{-2}v_F^{-1})^{-\frac{1}{3}}, \tag{41}$$

and the quantities $\mathcal{C}_i$ are defined by

$$\mathcal{C}_i=(\epsilon_i-\epsilon_j)^{-1}(\epsilon_i-\epsilon_k)^{-1}, \tag{42}$$

with $i=x, y, x$ and $i\neq j\neq k$. The derivation of Eq. (37) is valid for $\nu<10$ kMc/sec under the conditions of the extreme limit of the anomalous skin effect, except in the limit $T\rightarrow 0°K$, at which one of the roots ϵ_i of Eq. (38)

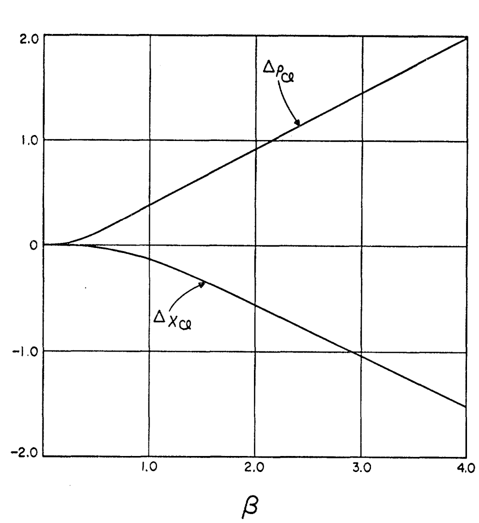

FIG. 4. Plot of the longitudinal field variation of the surface impedance for a superconducting metal in the limit of the classical skin effect. The dimensionless quantities $\Delta\rho_{Cl}$ and $\Delta\chi_{Cl}$ are plotted vs β and are related to $\Delta R_{L,Cl}$ and $\Delta X_{L,Cl}$ by $(\Delta\rho_{Cl}+i\Delta\chi_{Cl})=[5c^4v_F^3/2\pi\omega^2\omega_n^2\omega_c^2\lambda\bar{l}^5](\Delta R_{L,Cl}+i\Delta X_{L,Cl})$. Except for normalization factors, these curves give the quadratic field terms of $R(H_0)$ and of $X(H_0)$ as a function of β in the classical limit.

vanishes. However, in tin it is a good approximation even for this pathological root in the experimentally accessible range $1.2°K\leqslant T<T_c$, since $|iq\bar{l}|\sim 1$ for reduced temperature $t\sim 0.2$ (or $T\sim 1°K$).[15] The results for the zero field surface impedance as a function of (λ/δ) are presented for the classical limit in Fig. 2 and for the extreme anomalous limit in Fig. 3. Comments on these plots are reserved for Sec. III.

The longitudinal field variation of the surface impedance in the classical limit is

$$\Delta Z_{L,Cl}=\frac{2\pi i\omega^2\omega_n^2\omega_c^2\bar{l}^5}{5c^4v_F^3(k_c+i\lambda^{-1})}+\mathcal{O}(H_0^4). \tag{43}$$

[14] J. Reuter and E. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).

[15] Serber (reference 9) has obtained the same result as Eq. (37) in the limit $T\rightarrow 0°K$ by another derivation valid at low T.

This variation is small and quadratic in H_0. The nonzero value of $\Delta Z_{L,Cl}$ arises entirely from the nonlocal terms in the current-field relation of Eq. (27)—i.e., in this limit $M_{yy}(iq\bar{l},iq'\bar{l},iq''\bar{l})\approx\mathcal{O}(q^2\bar{l}^2)$. On the other hand, the dominant term in the zero field surface impedance is a local term (i.e., Ohm's law). In Fig. 4 a plot is presented of the longitudinal field variation of the surface impedance in the classical limit as a function of the dimensionless parameter $\beta=\lambda/\delta_{Cl}$, in which δ_{Cl} is the classical skin depth, $\delta_{Cl}=c(2\pi\omega_n{}^2\bar{\tau})^{-\frac{1}{2}}$.

The result corresponding to Eq. (43) in the extreme limit of the anomalous skin effect is

$$\Delta Z_L=-\frac{4i\alpha^6}{3\pi^2 v_F}\frac{\omega_c{}^2}{\omega_n{}^2}\Bigg[\frac{1}{2}\bigg(\sum_{i=1}^{3}\frac{\mathcal{C}_i}{\epsilon_i-1}\ln(\epsilon_i)\bigg)^2-\frac{1}{2}\bigg(\sum_{i=1}^{3}\frac{\mathcal{C}_i}{\epsilon_i+1}\ln(-\epsilon_i)\bigg)^2+\sum_{i,j=1}^{3}\mathcal{C}_i\mathcal{C}_j\frac{\ln(\epsilon_i)\ln(-\epsilon_j)}{\epsilon_j-\epsilon_i+2}\bigg(\frac{1}{\epsilon_i-1}+\frac{1}{\epsilon_j+1}\bigg)$$
$$-\frac{2\pi^2}{9\alpha^9}-\frac{1}{12}\sum_{i=1}^{3}\frac{\mathcal{C}_i(\epsilon_i+2)(\epsilon_i+3)}{(\epsilon_i+1)^3}\bigg(\frac{\ln^2(-\epsilon_i)+\ln^2(2+\epsilon_i)}{3(\epsilon_i+1)+\epsilon_i(\epsilon_i-1)}\bigg)+\frac{1}{12}\sum_{i=1}^{3}\frac{\mathcal{C}_i(\epsilon_i-2)(\epsilon_i-3)}{(\epsilon_i-1)^3}\bigg(\frac{\ln^2(\epsilon_i)+\ln^2(2-\epsilon_i)}{3(\epsilon_i-1)-\epsilon_i(\epsilon_i+1)}\bigg)$$
$$+\frac{1}{3}\sum_{i=1}^{3}\frac{\mathcal{C}_iG(-1-2/\epsilon_i)}{(\epsilon_i+1)^3}+\frac{1}{3}\sum_{i=1}^{3}\frac{\mathcal{C}_iG(-1+2/\epsilon_i)}{(\epsilon_i-1)^3}\Bigg].\quad(44)$$

The functions $G(x)$ appearing in Eq. (44) are defined by the series for $|x|<1$,

$$G(x)=\sum_{n=1}^{\infty}(-1)^{n+1}n^{-2}x^n,\quad(45)$$

with $G(x)$ for $|x|>1$ determined by the recurrence relation

$$G(x)+G(x^{-1})=(\pi^2/6)+\tfrac{1}{2}\ln^2 x.\quad(46)$$

From Eqs. (37) and (44) it is seen that both $Z(0)$ and ΔZ_L are independent of the relaxation time τ, a result characteristic of the extreme anomalous limit of the normal metal. The plot of ΔZ_L vs α in the extreme anomalous limit is illustrated in Fig. 5 and is discussed in Sec. III. For convenience the results are presented logarithmically in terms of the dimensionless quantities $\Delta\rho+i\Delta\chi$.

(b) Transverse Field

The rf normal electron current perpendicular to the static magnetic field is directed along the x direction and is given by

$$j_{nq,x}=\frac{3\omega_n{}^2\bar{\tau}}{16\pi}\Bigg[K_{xx}(iq\bar{l})E_{q,x}-\frac{i\omega_c\bar{\tau}}{\pi}\int_{\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}L_{xz}(iq\bar{l},iq'\bar{l})E_{q-q',z}$$
$$-\bigg(\frac{\omega_c\bar{\tau}}{\pi}\bigg)^2\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{xx}(iq\bar{l},iq'\bar{l},iq''\bar{l})E_{q-q'-q'',x}+\mathcal{O}(\omega_c\bar{\tau})^3\Bigg].\quad(47)$$

The Hall current $j_{nq,z}$ vanishes, but gives rise to the relation

$$j_{nq,z}=0=\frac{3\omega_n{}^2\bar{\tau}}{16\pi}\Bigg[K_{zz}(iq\bar{l})E_{q,z}-\frac{i\omega_c\bar{\tau}}{\pi}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}L_{zx}(iq\bar{l},iq'\bar{l})E_{q-q',x}$$
$$-\bigg(\frac{\omega_c\bar{\tau}}{\pi}\bigg)^2\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{zz}(iq\bar{l},iq'\bar{l},iq''\bar{l})E_{q-q'-q'',z}+\mathcal{O}(\omega_c\bar{\tau})^3\Bigg].\quad(48)$$

For this geometry, the Maxwell Eq. (18) yields the relations for $\mathbf{E}_q$,

$$[-q^2c^2+\omega^2-\omega_s{}^2-\tfrac{3}{4}i\omega\omega_n{}^2\bar{\tau}K_{xx}(i\bar{l}q)]E_{q,x}=2c^2E_x'(0)+\tfrac{3}{4}i\omega\omega_n{}^2\bar{\tau}\Bigg[-\frac{i\omega_c\bar{\tau}}{\pi}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}L_{xz}(i\bar{l}q,i\bar{l}q')E_{q-q',z}$$
$$-\bigg(\frac{\omega_c\bar{\tau}}{\pi}\bigg)^2\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{xx}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')E_{q-q'-q'',x}+\mathcal{O}(\omega_c\bar{\tau})^3\Bigg],\quad(49)$$

and

$$E_{q,z}=-\frac{i}{\pi}\frac{\omega_c\bar{\tau}}{K_{zz}(i\bar{l}q)}\left(\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}L_{zx}(i\bar{l}q,i\bar{l}q')E_{q-q',x}\right.$$
$$\left.-\frac{i}{\pi}\omega_c\bar{\tau}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}M_{zz}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')E_{q-q'-q'',z}+\mathcal{O}(\omega_c\bar{\tau})^2\right). \quad (50)$$

Solution of Eqs. (49) and (50) is accomplished by iteration, with the result

$$E_{q,x}=2c^2E_x'(0)\Gamma_x(q)\left(1+\frac{3i\omega\omega_n^2\omega_c^2\bar{\tau}^3}{4\pi^2}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}\frac{\Gamma_x(q-q'-q'')}{K_{zz}(i\bar{l}(q-q'))}\right.$$
$$\left.\times[L_{xz}(i\bar{l}q,i\bar{l}q')L_{zx}(i\bar{l}(q-q'),\,i\bar{l}q'')-M_{xx}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')K_{zz}(i\bar{l}(q-q'))]+\mathcal{O}(\omega_c\bar{\tau})^4\right), \quad (51)$$

and

$$E_{q,z}=\frac{2ic^2E_x'(0)\omega_c\bar{\tau}}{\pi K_{zz}(i\bar{l}q)}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}L_{zx}(i\bar{l}q,i\bar{l}q')\Gamma_x(q-q')+\mathcal{O}(\omega_c\bar{\tau})^3, \quad (52)$$

in which

$$\Gamma_x(q)=\Gamma_y(q)=[-q^2c^2+\omega^2-\omega_s^2-\tfrac{3}{4}i\omega\omega_n^2\bar{\tau}K_{xx}(i\bar{l}q)]^{-1}. \quad (53)$$

The surface impedance for the transverse field orientation,

$$Z_T(H_0)=-4\pi i\omega c^{-2}[E_x(z)/E_x'(z)]_{z=0} \quad (54)$$

gives rise to a field variation $\Delta Z_T=[Z_T(H_0)-Z(0)]$ given by

$$\Delta Z_T=-\frac{3}{\pi^2}\omega^2\omega_n^2\omega_c^2\bar{\tau}^3\int_{-\infty}^{\infty}\Gamma_x(q)dq\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}\frac{\Gamma_x(q-q'-q'')}{K_{zz}(i\bar{l}(q-q'))}$$
$$\times[M_{xx}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')K_{zz}(i\bar{l}(q-q'))-L_{xz}(i\bar{l}q,i\bar{l}q')L_{zx}(i\bar{l}(q-q'),\,i\bar{l}q'')]+\mathcal{O}(H_0^4). \quad (55)$$

The integrals of Eq. (55) are carried out in Appendix C for the limiting cases of the classical skin effect and the extreme anomalous skin effect. The transverse field variation of the surface impedance in the classical limit is

$$\Delta Z_{T,Cl}=4\Delta Z_{L,Cl}+\mathcal{O}(H_0^4). \quad (56)$$

Although in the classical limit $M_{xx}\approx-\frac{4}{3}$, nevertheless, the contribution to $\Delta Z_{T,Cl}$ arises from purely nonlocal terms in the current-field relation. The presence of the Hall field $E_z(z)$ exactly cancels the contribution from the local term.

The transverse field variation in the extreme anomalous limit is found to be

$$\Delta Z_T=3\Delta Z_L+\mathcal{O}(H_0^4). \quad (57)$$

A relation of this sort was first predicted by Landau and Ginsburg[16] and discussed by Pippard.[17] In this case, the leading term in ΔZ_T is independent of the relaxation time and of the mean free path. The contribution to ΔZ_T from the Hall field term is of order $(\lambda/\bar{l})$. Therefore, this term is not included in the calculation of ΔZ_T, since it vanishes in the limit of infinite mean free path.

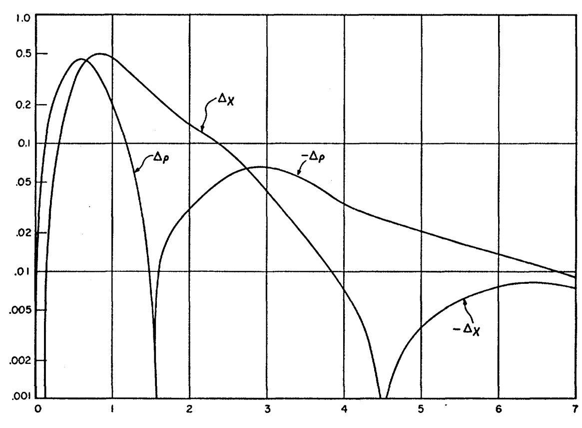

FIG. 5. Semilogarithmic plot of the longitudinal field variation of the surface impedance for a superconducting metal in the limit of the extreme anomalous skin effect. The dimensionless quantities $\Delta\rho$ and $\Delta\chi$ are plotted as a function of α and are related to ΔR_L and ΔX_L by $(\Delta\rho+i\Delta\chi)=(3\pi^2v_F\omega_n^2/4\alpha^3\omega_c^2)(\Delta R_L+i\Delta X_L)$. $\Delta\rho<0$ for $\alpha>1.55$ and $\Delta\chi<0$ for $\alpha>4.5$. For much larger $\alpha\,(\alpha\gg10)$, $\Delta\rho$ again becomes positive. Except for normalization factors, these curves give the quadratic field terms of $R(H_0)$ and of $X(H_0)$ as a function of α in the extreme anomalous limit.

[16] V. L. Ginsburg and L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **20**, 1064 (1950).

[17] A. B. Pippard, *Advances in Electronics and Electron Physics*, edited by L. Marton (Academic Press, New York, 1954), Vol. 6, pp. 38–40.

III. DISCUSSION

The plots of $Z(0)$ and $\Delta Z(H_0)$ shown in Sec. II are in terms of the dimensionless parameters $\beta=\lambda/\delta_{Cl}$ for the classical limit and $\alpha=\lambda/\delta_n$ for the extreme anomalous limit. Assuming the validity of the present two-fluid model, these curves are universal for all superconductors. The specific characteristics of a given superconductor are contained in the parameters α and β —e.g., v_F and m^*. It is a major problem to obtain the temperature dependences $\alpha(T)$ and $\beta(T)$ which are needed for a comparison between the present theory and experimental results.[1,2] This analysis is left for a subsequent paper.

In the limit $\alpha, \beta \rightarrow \infty$, the surface impedance results quoted here should reduce to those for a normal metal. For the case of the classical skin effect (Figs. 2 and 4), the ratio $[X(0)/R(0)]\rightarrow 1$ as $\beta\rightarrow\infty$, in agreement with the results for the normal state. It should be noted that in Fig. 2, the quantities $R(0)$, $X(0)$, ω_s and β are all temperature dependent. At the transition temperature T_c there is a dramatic decrease in $R(0)$ and in $X(0)$ as the magnetic flux is expelled in the normal-superconducting transition. This effect is masked in Fig. 2 by the strong temperature dependence of ω_s. Similar remarks are pertinent to the corresponding plot for the extreme anomalous skin effect, Fig. 3.

Both in the normal and superconducting states, $[\Delta X(H_0)/\Delta R(H_0)]\rightarrow 0$ for all temperatures in the strict classical limit. However, as the metal becomes slightly anomalous, a nonzero value is obtained for this ratio. Since the effect of the magnetic field is to curl up the electron trajectories, thereby shortening them, the presence of H_0 causes the metal to be less anomalous. A comparison of Figs. 2 and 3 indicates that for large β, as the sample becomes less anomalous, R decreases while X increases, a situation also found in the normal metal. Thus, Fig. 4 shows the slightly anomalous superconducting metal to have a negative value for $[\Delta X(H_0)/\Delta R(H_0)]$ at all temperatures. This situation is encountered in the superconducting state for H_0 both longitudinal and transverse, the effect being larger by a factor 4 in the transverse case.

A similar comparison between the normal and superconducting metal in the extreme anomalous limit is more difficult. The zero field result given by Eq. (37) and Fig. 3 reduces to $[X(0)/R(0)]\rightarrow\sqrt{3}$ in the limit $\alpha\rightarrow\infty$, in agreement with the behavior in the normal metal.[14] The solution of the field dependence problem for normal metals in the extreme anomalous limit[10,18] allows R and X to either increase or decrease with increasing H_0, for H_0 in either the longitudinal or transverse orientations. In fact for the limit of low fields and long relaxation times, the oscillatory character of $R(H_0)$ and $X(H_0)$ preclude the possibility of determining the sign of $\Delta R(H_0)$ and $\Delta X(H_0)$ individually. This behavior complicates a comparison with the superconducting case, where both $\Delta R(H_0)$ and $\Delta X(H_0)$ become infinite as $\beta\rightarrow\infty$, indicating the failure of the power series expansion in H_0.

In the superconducting state there is no oscillatory variation of $R(H_0)$ and $X(H_0)$ as is observed in the cyclotron resonance studies on normal metals. According to Fig. 5, $\Delta R(H_0)$ and $\Delta X(H_0)$ can each be either positive or negative. In agreement with the results in the normal state, $[\Delta X(H_0)/\Delta R(H_0)]$ can assume either sign, as is also observed experimentally. At low temperatures, experiments yield $\Delta R>0$ and $\Delta X>0$,[1,2] a result also found in the present theory. This result is reasonable since both R and X increase in the superconducting to normal transition. One might expect that the application of a static field would have a monotonic effect on the electromagnetic behavior of the superconductor as H_0 increases from zero to the critical field H_c. The opposite result, $\Delta R<0$ and $\Delta X<0$ is found experimentally at 1 kMc/sec in the limit of high T,[2] as is also predicted by the present theory. Actually, for T very close to T_c, a temperature range inaccessible experimentally, the theory predicts that ΔR once again becomes positive. Not only is there merely agreement in the signs of ΔR and ΔX, but where agreement exists the predicted order of magnitude is also correct. This subject is to be treated in detail elsewhere.

The present theory predicts that ΔR and ΔX both $\rightarrow 0$ in the limit $T\rightarrow 0°$K. Since the experimental results in this limit are as yet inconclusive, no test of this aspect of the theory can be made.

There are two important qualitative features of the experimental results at 1 kMc/sec which are not contained in the theory.[2] Experimentally ΔR changes sign from negative to positive at a higher temperature than does ΔX, in contrast with the results shown in Fig. 5. However, studies at 9.4 kMc/sec yield $\Delta X \geqslant 0$ and a sign change in ΔR.[1] These results at the higher frequency do not actually violate the theory, since the sign change in ΔX could be so close to T_c as to be experimentally inaccessible. Although this theory predicts a larger field dependence of Z for the transverse field orientation, as is also observed experimentally, it implies that in the two limiting cases considered, those of the classical and extreme anomalous skin effects, the results in longitudinal and transverse H_0 should be proportional. However, the experimental data at 1 kMc/sec show that ΔR and ΔX can, in fact, each change sign as H_0 changes from longitudinal to transverse.

Nevertheless, these failures of the theory may be less serious than they appear at first sight. It is not clear that the samples strictly satisfy the conditions of the extreme anomalous skin effect. Furthermore, the geometrical arrangement employed in the longitudinal and transverse experiments involve electrons associated with different parts of the Fermi surface; thus, at a given temperature, the appropriate α is different for

[18] M. Ia. Azbel, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 969 (1958) [translation: Soviet Phys.-JETP 34(7), 669 (1958)].

the two field orientations and no simple comparison can be made. Clarification is needed of the experimental situation, as can be obtained from careful measurements of ΔR and ΔX in longitudinal and transverse fields as a function of temperature, rf frequency, and crystalline orientation.

The above treatment has assumed a local theory for the superconducting electrons. This formalism can easily be extended to take into account a nonlocal current-field relation by replacing Eqs. (2a) and (2b) by the appropriate nonlocal equations. In general, the explicit solutions would be more difficult to obtain. However, for small departures from a local theory, as would for example be given by the kernel[19]

$$\mathcal{K}(q)=\lambda^{-2}[1-9\hbar^2 qk_F(8m\epsilon_{gap})^{-1}+\cdots], \tag{58}$$

no further calculation, but merely a reidentification of terms need be made in order to take into account the lowest order nonlocal term. The first term in Eq. (58) is the London term, and the kernel $\mathcal{K}(q)$ relates the Fourier transform of the supercurrent $\mathbf{j}_{sq}$ to that of the vector potential $\mathbf{A}_q$ by $\mathbf{j}_{sq}=-(c/4\pi)\mathcal{K}(q)\mathbf{A}_q$. At present the experimental evidence in the frequency range $\nu<10$ kMc/sec does not motivate the introduction of a nonlocal current-field relation for the supercurrent.

Measurement of the field variation of the surface impedance can yield information on the band structure for the normal electrons in the superconducting state. A comparison between theory and experiment provides a measure of m^*v_F for the normal electrons, in which m^*v_F represents a mean value averaged over the electron trajectory. It is hoped that various models of the electronic band structure in a superconductor can be tried to fit the data. The proposed program is much the same as has been carried out by Pippard for Cu in the normal state[20]; in Pippard's work the same parameter is determined.

The authors would like to thank Mr. Jack Mochel for help with the computations used in Fig. 5.

APPENDIX A. PROPERTIES OF THE FUNCTIONS K, L, M, AND P

The integrals which occur in the Fourier transform of the rf electric current are examined here. The angular integration of the functions K_{ij} associated with zero static field can be carried out by elementary methods. The results for the nonvanishing functions are

$$\begin{aligned}K_{xx}(t)&=\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin^3\theta\cos^2\phi\,\Omega(t)\\&=t^{-3}\left[2t-(1-t^2)\ln\left(\frac{1+t}{1-t}\right)\right],\\K_{yy}(t)&=\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\cos^2\theta\,\Omega(t)=K_{xx}(t),\\K_{zz}(t)&=\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin^3\theta\sin^2\phi\,\Omega(t)\\&=-2t^{-3}\left[2t-\ln\left(\frac{1+t}{1-t}\right)\right],\end{aligned} \tag{A-1}$$

in which $\Omega(t)$ is defined by Eq. (13). The vanishing of the functions K_{ij} $(i\neq j)$ implies that for $H_0=0$, the normal current and the rf electric field are parallel. The nonvanishing $K_{ii}(t)$ functions are even functions, with

$$K_{ii}(t)=K_{ii}(-t). \tag{A-2}$$

The singularities of $K_{ii}(t)$ include branch points at $t=\pm1$, but no poles. In the limit of small argument, $t\ll1$, the expansions for the $K_{ii}(t)$ functions are:

$$\begin{aligned}K_{xx}(t)&=K_{yy}(t)=\tfrac{4}{3}(1+\tfrac{1}{5}t^2+\cdots),\\K_{zz}(t)&=\tfrac{4}{3}(1+\tfrac{3}{5}t^2+\cdots).\end{aligned} \tag{A-3}$$

In the limit of large argument, $t\gg1$, the expansions for the $K_{ii}(t)$ functions are:

$$\begin{aligned}K_{xx}(t)&=K_{yy}(t)=\pm\frac{\pi i}{t}\left(1-\frac{1}{t^2}\right)\\&\quad+\frac{4}{t^2}\left(1-\frac{1}{3t^2}-\frac{1}{15t^4}-\cdots\right),\\K_{zz}(t)&=-\frac{4}{t^2}\left[1-\frac{1}{t^2}-\frac{1}{3t^4}-\cdots\right]\pm\frac{2\pi i}{t^3}.\end{aligned} \tag{A-4}$$

Note that the symmetry relation (A-2) implies that a sign change of t requires a phase change of 2π in i for the $K_{ii}(t)$ functions.

Of the nine integrals $L_{ij}(t,t')$, associated with terms linear in H_0, only two are nonvanishing,

$$L_{xz}(t,t')=\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin^3\theta\cos\phi\,\Omega(t)\frac{\partial}{\partial\phi}[\Omega(t-t')\sin\phi]=\frac{t^2}{t'^2}K_{xx}(t)-\frac{t(t-t')}{t'^2}K_{xx}(t-t')-\frac{(t-t')}{t'}K_{zz}(t-t'),$$

and

$$L_{zx}(t,t')=\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin^3\theta\sin\phi\,\Omega(t)\frac{\partial}{\partial\phi}[\Omega(t-t')\cos\phi]=-\frac{(t-t')^2}{t'^2}K_{xx}(t-t')+\frac{t(t-t')}{t'^2}K_{xx}(t)-\frac{t}{t'}K_{zz}(t). \tag{A-5}$$

[19] J. Bardeen, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. XV, p. 305, Sec. 20.
[20] A. B. Pippard, Trans. Roy. Soc. (London) **A250**, 325 (1957).

For the nonvanishing components, the following relations hold:

$$L_{ij}(t,t')=L_{ij}(-t,\,-t'),$$

and

$$L_{ij}(t,t')=-L_{ji}(t-t',\,-t'). \tag{A-6}$$

The singularities of $L_{ij}(t,t')$ include branch points at $t=\pm1$, and $(t-t')=\pm1$, but no poles. In the limit of small argument, $t,\ t'\ll1$, the expansion of the $L_{ij}(t,t')$ gives

$$\begin{aligned}L_{xz}(t,t')&=\tfrac{4}{3}[1+\tfrac{1}{5}(3t'^2-8tt'+6t^2)+\cdots],\\ L_{zx}(t,t')&=-\tfrac{4}{3}[1+\tfrac{1}{5}(t'^2-4tt'+6t^2)+\cdots].\end{aligned} \tag{A-7}$$

Expansion of the $L_{ij}(t,t')$ in the limit of large argument, $t,\ t'\gg1$, yields

$$L_{xz}(t,t')=\pm\frac{\pi i}{t(t-t')^2}+\frac{4}{3}\frac{(t'-3t)}{t^2(t-t')^3}+\cdots,$$

and

$$L_{zx}(t,t')=\mp\frac{\pi i}{t^2(t-t')}-\frac{4}{3}\frac{(2t'-3t)}{t^3(t-t')^2}+\cdots. \tag{A-8}$$

The nonvanishing functions $M_{ij}(t,t',t'')$, associated with terms quadratic in H_0, of Eq. (16) are M_{xx}, M_{yy}, and M_{zz}. Of these, only M_{xx} and M_{yy} enter in the calculation of the field dependence of the surface impedance to order H_0^2, where M_{xx} and M_{yy} are given by

$$\begin{aligned}M_{xx}(\tau+t',\,t',\,t'')=&\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin^3\theta\,\cos\phi\,\Omega(\tau+t')\frac{\partial}{\partial\phi}\left\{\Omega(t)\frac{\partial}{\partial\phi}[\Omega(\tau-t'')\cos\phi]\right\}=\frac{4\tau^2}{t't''(1-\tau^2)}\\ &+\frac{3(\tau+t')(\tau-t'')}{t't''(t'+t'')^2}[(\tau+t')t''-(\tau-t'')t']+K_{xx}(\tau)\left(-\frac{\tau^3(t'-t'')}{t'^2t''^2}-\frac{2\tau^4}{t't''(1-\tau^2)}+\frac{3}{4}\frac{\tau^2(1-\tau^2)(\tau+t')(\tau-t'')}{t'^2t''^2}\right)\\ &+K_{xx}(\tau+t')\left(-\frac{(\tau+t')^3}{t'^2(t'+t'')}-\frac{3(\tau+t')^2(\tau-t'')}{t'(t'+t'')^2}-\frac{3}{4}\frac{(\tau+t')^2(\tau-t'')}{(t'+t'')^3t'^2}[1-(\tau+t')^2][\tau(t'+t'')+2t'(\tau-t'')]\right)\\ &+K_{xx}(\tau-t'')\left(\frac{(\tau-t'')^3}{t''^2(t'+t'')}+\frac{3(\tau+t')(\tau-t'')^2}{t''(t'+t'')^2}-\frac{3}{4}\frac{(\tau+t')(\tau-t'')^2}{(t'+t'')^3t''^2}[1-(\tau-t'')^2][\tau(t'+t'')+2t''(\tau+t')]\right),\end{aligned}$$

and

$$\begin{aligned}M_{yy}(\tau+t',\,t',\,t'')=&\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\,\cos^2\theta\,\Omega(\tau+t')\frac{\partial}{\partial\phi}\left\{\Omega(\tau)\frac{\partial}{\partial\phi}[\Omega(\tau-t'')]\right\}=(\tau+t')(\tau-t'')\left(\frac{2t't''-\tau(t'-t'')}{t't''(t'+t'')^2}\right.\\ &+\frac{K_{xx}(\tau)}{4}\frac{\tau^2(1-\tau^2)}{t'^2t''^2}+\frac{K_{xx}(\tau+t')}{4}\frac{(\tau+t')}{t'^2(t'+t'')^3}[(\tau+t')^2(3\tau t'+\tau t''-2t't'')-(3\tau t'+\tau t''+2t't''+4t'^2)]\\ &\left.+\frac{K_{xx}(\tau-t'')}{4}\frac{(\tau-t'')}{t''^2(t'+t'')^3}[(\tau-t'')^2(3\tau t''+\tau t'+2t't'')-(3\tau t''+\tau t'-2t't''-4t''^2)]\right).\end{aligned} \tag{A-9}$$

The symmetry relations obeyed by the three nonvanishing $M_{ij}(t,t',t'')$ functions are

$$M_{ii}(t,t',t'')=M_{ii}(-t,\,-t',\,-t''), \tag{A-10}$$

and

$$M_{ii}(t,t',t'')=M_{ii}(t-t'-t'',\,-t'',\,-t'), \tag{A-11}$$

for $i=x,\ y,\ z$. The singularities of $M_{ii}(t,t',t'')$ include branch points at $t=\pm1$, $(t-t')=\pm1$, and $(t-t'-t'')=\pm1$, but no poles. In the limit of small argument, $t,\ t',\ t''\ll1$, the expansions for the pertinent $M_{ii}(t,t',t'')$ functions are:

$$\begin{aligned}M_{xx}(t,t',t'')=&-\tfrac{4}{3}\{1+\tfrac{1}{5}[13(t-t')^2+8(t-t')(t'-t'')\\ &+(t'^2-4t't''+t''^2)]+\cdots\},\end{aligned} \tag{A-12}$$

$$M_{yy}(t,t',t'')=-(4/15)t(t-t'-t'')+\cdots.$$

In the limit of large argument, $t,\ t',\ t''\gg1$, the asymptotic expansions are:

$$\begin{aligned}M_{xx}(t,t',t'')&=\pm\frac{3\pi i}{4t(t-t')(t-t'-t'')}+\mathcal{O}(t^{-4}),\\ M_{yy}(t,t',t'')&=\pm\frac{\pi i}{4t(t-t')(t-t'-t'')}+\mathcal{O}(t^{-4}).\end{aligned} \tag{A-13}$$

Because of the symmetry relations (A-6) for the $L_{ij}(t,t')$ and (A-10) for the $M_{ii}(t,t',t'')$, a sign change of all the variables in the asymptotic expansions (A-7) and (A-13) requires a phase change of 2π in evaluating $\ln(-1)$.

The combination of the $M_{yy}(t,t',t'')$ functions which occurs in the evaluation of the rf electric field for the longitudinal magnetic field orientation is

$$P(t,t',t'')=M_{yy}(t,t',t'')+M_{yy}(t,t'',t'). \tag{A-14}$$

In the limit of large argument, $\tau,\ t',\ t''\gg1$, the asymp-

totic expansion of $P(\tau+t', t', t'')$ yields

$$P(\tau+t', t', t'') = \mp\frac{\pi i}{4}\frac{2\tau+t'-t''}{(\tau+t')(\tau-t'')(\tau+t'-t'')}+\mathcal{O}(t^{-4}). \quad \text{(A-15)}$$

The sign in Eq. (A-15) is determined to be consistent with the symmetry relations

$$P(t,t',t'')=P(-t+t'+t'', t', t''),$$

and

$$P(t,t',t'')=P(-t, -t', -t''). \quad \text{(A-16)}$$

APPENDIX B. INTEGRALS FOR LONGITUDINAL FIELD ORIENTATION

The calculation of the variation of the surface impedance in a longitudinal static magnetic field involves only one component of the rf electric field, $E_{q,y}$, given by Eq. (30). The functions $\Gamma_y(q)$ and $M_{yy}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')$ of Eq. (30) are defined by Eqs. (31) and (A-9), respectively. The symmetry relation (A-10) insures that $E_q=E_{-q}$. The integration of Eq. (30) is simplified with the substitution $p=q'+q''$ to yield $E_{q,y}$ in the form

$$E_{q,y}=2E_y'(0)c^2\Gamma_y(q)\Bigg(1-\frac{3i\omega\omega_n^2\omega_c^2\bar{\tau}^3}{4\pi^2}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}} \times\int_{-\infty}^{\infty}\frac{(p-q')dp}{(p-q')^2+\lambda^{-2}}M_{yy}(i\bar{l}q, -i\bar{l}q', i\bar{l}(q'-p))\Gamma_y(q+p)+\mathcal{O}(\omega_c\bar{\tau})^4\Bigg). \quad \text{(B-1)}$$

The q' integration can be done by contour integration in which q and p are treated as real quantities. The singularities of $[q'(p-q')M_{yy}(iq\bar{l}, -iq'\bar{l}, i(q'-p)\bar{l})]$ include branch points at $q'=-q\pm i\bar{l}^{-1}$, but no poles. By proper choice of the contour, no contribution arises from the branch cut because of the invariance of the integrand under the transformation $i\bar{l}\rightarrow -i\bar{l}$. The result is

$$E_{q,y}=2E_y'(0)c^2\Gamma_y(q)\Bigg\{1+\frac{3}{8\pi}\omega\omega_n^2\omega_c^2\bar{\tau}^3\int_{-\infty}^{\infty}dp\Bigg[\frac{\Gamma_y(q+p)}{p}\Bigg(\frac{p-i\lambda^{-1}}{p-2i\lambda^{-1}}M_{yy}(i\bar{l}q, \bar{l}\lambda^{-1}, i\bar{l}(i\lambda^{-1}-p)) - \frac{p+i\lambda^{-1}}{p+2i\lambda^{-1}}M_{yy}(i\bar{l}q, -i\bar{l}(i\lambda^{-1}+p), -\bar{l}\lambda^{-1})\Bigg)+\frac{\Gamma_y(-q+p)}{p}\Bigg(\frac{p-i\lambda^{-1}}{p-2i\lambda^{-1}}M_{yy}(-i\bar{l}q, \bar{l}\lambda^{-1}, i\bar{l}(i\lambda^{-1}-p)) - \frac{p+i\lambda^{-1}}{p+2i\lambda^{-1}}M_{yy}(-i\bar{l}q, -i\bar{l}(p+i\lambda^{-1}), -\bar{l}\lambda^{-1})\Bigg)\Bigg]+\mathcal{O}(\omega_c\bar{\tau})^4\Bigg\}. \quad \text{(B-2)}$$

The explicit calculation of $E_{q,y}$ and the field variation of the surface impedance ΔZ_L is tractable for the two limiting cases: $\bar{l}\lambda^{-1}$, $\bar{l}\delta_{Cl}^{-1}\ll 1$ (classical skin effect), and $\bar{l}\lambda^{-1}$, $\bar{l}\delta_n^{-1}\gg 1$ (extreme anomalous skin effect).

(i) The Classical Skin Effect

In this limit, the functions $M_{yy}(iq\bar{l},iq'\bar{l}, iq''\bar{l})$ and $\Gamma_y(q)$ become

$$M_{yy}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')\cong(4/15)\bar{l}^2q(q-q'-q''), \quad \text{(B-3)}$$

and

$$\Gamma_y(q)\cong-[c^2(q+k_c)(q-k_c)]^{-1}, \quad \text{(B-4)}$$

in which k_c is given by Eq. (36). With this substitution the integral of Eq. (B-2) can be carried out exactly (e.g., by contour integration in the upper half plane) and yields

$$E_{q,y}=-\frac{2E_y'(0)}{q^2-k_c^2}\Bigg(1-\frac{i\omega\omega_n^2\omega_c^2\bar{\tau}^3\bar{l}^2}{5c^2}\frac{q^2}{q^2-(k_c+2i\lambda^{-1})^2}+\mathcal{O}(\omega_c\bar{\tau})^4\Bigg). \quad \text{(B-5)}$$

From Eqs. (B-5) and (6b), the spatial dependence of the rf electric field is found for $z\geqslant 0$,

$$E_y(z)=-\frac{iE_y'(0)}{k_c}e^{ik_cz}\Bigg(1-\frac{\omega\omega_n^2\omega_c^2\bar{\tau}^3\bar{l}^2\lambda k_c}{20c^2(k_c+i\lambda^{-1})}[2i\lambda^{-1}e^{-2z/\lambda}-k_c(1-e^{-2z/\lambda})]+\mathcal{O}(\omega_c\bar{\tau})^4\Bigg). \quad \text{(B-6)}$$

The surface impedance is thus given by

$$Z_L(H_0)=-\frac{4\pi\omega}{k_c c^2}\left(1-\frac{i\omega\omega_n{}^2\omega_c{}^2\bar{\tau}^3\bar{l}^2k_c}{10c^2(k_c+i\lambda^{-1})}+\mathcal{O}(\omega_c\bar{\tau})^4\right), \tag{B-7}$$

with

$$\Delta Z_L(H_0)=\frac{2\pi i\omega^2\omega_n{}^2\omega_c{}^2\bar{\tau}^3\bar{l}^2}{5c^4(k_c+i\lambda^{-1})}. \tag{B-8}$$

(ii) The Extreme Anomalous Skin Effect

In this limit the functions $M_{yy}(iq\bar{l},iq'\bar{l},iq''\bar{l})$ and $\Gamma_y(q)$ are given by

$$M_{yy}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')\simeq\mp\pi[4\bar{l}^3q(q-q')(q-q'-q'')]^{-1}, \tag{B-9}$$

and

$$\Gamma_y(q)\simeq(-q^2c^2+\omega^2-\omega_s{}^2\mp3\pi i\omega\omega_n{}^2\bar{\tau}/4q\bar{l})^{-1}. \tag{B-10}$$

The symmetry relations (A-2) and (A-10) imply a consistent choice of sign in applying Eqs. (B-9) and (B-10) to a physical problem. The same physical results are obtained for either choice of sign provided that the choice is made consistently throughout the calculation. The integration of Eq. (B-2) is conveniently carried out in terms of the variable $\mu>0$ and of the function $P(\tau+t',t',t'')$ defined in Eq. (A-14). The resulting form of (B-2) is

$$E_{q,y}=2E_y'(0)c^2\Gamma_y(q)[1+(3/16\pi)\omega\omega_n{}^2\omega_c{}^2\bar{\tau}^3\mathcal{G}_q+\mathcal{O}(\omega_c\bar{\tau})^4], \tag{B-11}$$

with

$$\begin{aligned}\mathcal{G}_q=\int_0^\infty d\mu\Gamma_y(\mu)\{&(\mu-q)^{-1}[P(i\bar{l}q,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}+q-\mu))-P(i\bar{l}\mu,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}-q+\mu))]\\&-(\mu+q)^{-1}[P(i\bar{l}q,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}+q+\mu))-P(-i\bar{l}\mu,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}-q-\mu))]\\&+(\mu-q-2i\lambda^{-1})^{-1}P(i\bar{l}q,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}+q-\mu))-(\mu+q+2i\lambda^{-1})^{-1}P(i\bar{l}q,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}+q+\mu))\\&-(\mu-q+2i\lambda^{-1})^{-1}P(i\bar{l}\mu,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}-q+\mu))+(\mu+q-2i\lambda^{-1})^{-1}P(-i\bar{l}\mu,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}-q-\mu))\}.\end{aligned} \tag{B-12}$$

The asymptotic expansions of $P(t,t',t'')$ appropriate to (B-12) are from Eqs. (A-15) and (A-16)

$$P(i\bar{l}q,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}+q-\mu))\simeq-\frac{\pi}{4\bar{l}^3}\frac{q+\mu}{\mu q(q+i\lambda^{-1})(\mu-i\lambda^{-1})},$$

$$P(i\bar{l}\mu,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}-q+\mu))\simeq-\frac{\pi}{4\bar{l}^3}\frac{q+\mu}{\mu q(\mu+i\lambda^{-1})(q-i\lambda^{-1})},$$

$$P(-i\bar{l}\mu,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}-q-\mu))\simeq\frac{\pi}{4\bar{l}^3}\frac{q-\mu}{\mu q(\mu-i\lambda^{-1})(q-i\lambda^{-1})},$$

and

$$P(i\bar{l}q,\bar{l}\lambda^{-1},i\bar{l}(i\lambda^{-1}+q+\mu))\simeq\frac{\pi}{4\bar{l}^3}\frac{q-\mu}{\mu q(q+i\lambda^{-1})(\mu+i\lambda^{-1})}. \tag{B-13}$$

Consistent with Eq. (B-13) is the form

$$\Gamma_y(\mu)\simeq-\mu c^{-2}\sum_{i=1}^{3}C_i(\mu-\mu_i)^{-1}, \tag{B-14}$$

in which μ_i is one of the three roots of the equation

$$\mu^3+(\omega_s{}^2-\omega^2)c^{-2}\mu-i\alpha^3\lambda^{-3}=0, \tag{B-15}$$

and $C_i^{-1}=(\mu_i-\mu_j)(\mu_i-\mu_k)$, $i\neq j\neq k\neq i$. The dimensionless quantity α is defined by Eq. (40), and the choice of phase in $\Gamma_y(\mu)$ is made as

$$\lim_{|\mu\bar{l}|\to\infty}\ln\left(\frac{1+i\bar{l}\mu}{1-i\bar{l}\mu}\right)=-\pi i. \tag{B-16}$$

The asymptotic expansions (B-13) and (B-14) yield

$$\mathcal{I}_q=-\frac{2\pi i}{\lambda\bar{l}^3c^2}\sum_{i=1}^{3}C_i\int_0^\infty\frac{(q^2+4\lambda^{-2}-5\mu^2)d\mu}{(\mu-\mu_i)(\mu^2+\lambda^{-2})[(\mu^2-q^2-4\lambda^{-2})^2+(4q\lambda^{-1})^2]},\tag{B-17}$$

which can be integrated to obtain

$$\begin{aligned}\mathcal{I}_q=\frac{\pi}{\bar{l}^3c^2}\sum_{i=1}^{3}C_i\biggl(&\frac{\ln(-i\lambda^{-1}\mu_i^{-1})}{(\mu_i+i\lambda^{-1})(q^2+\lambda^{-2})}-\frac{\ln(i\lambda^{-1}\mu_i^{-1})}{(\mu_i-i\lambda^{-1})(q^2+\lambda^{-2})}-\frac{\ln(-q\mu_i^{-1}-2i\mu_i^{-1}\lambda^{-1})}{2q(q+i\lambda^{-1})(\mu_i+q+2i\lambda^{-1})}\\&+\frac{\ln(q\mu_i^{-1}+2i\mu_i^{-1}\lambda^{-1})}{2q(q+i\lambda^{-1})(\mu_i-q-2i\lambda^{-1})}+\frac{\ln(-q\mu_i^{-1}+2i\mu_i^{-1}\lambda^{-1})}{2q(q-i\lambda^{-1})(\mu_i+q-2i\lambda^{-1})}-\frac{\ln(q\mu_i^{-1}-2i\mu_i^{-1}\lambda^{-1})}{2q(q-i\lambda^{-1})(\mu_i-q+2i\lambda^{-1})}\biggr).\end{aligned}\tag{B-18}$$

The Fourier transform $E_{q,y}$, given explicitly by Eqs. (B-11) and (B-18), permits calculation of the rf electric field everywhere in the metal. In particular, the value $E_y(0)$ at the surface is determined, thus specifying the surface impedance. The expression for $E_y(0)$ can be written as

$$E_y(0)=\frac{c^2E_y'(0)}{\pi}\left[\int_{-\infty}^{\infty}\Gamma_y(q)dq+\frac{3}{16\pi}\omega\omega_n^2\omega_c^2\bar{\tau}^3\mathcal{J}+\mathcal{O}(\omega_c\bar{\tau})^4\right],\tag{B-19}$$

in which

$$\mathcal{J}=\int_{-\infty}^{\infty}\Gamma_y(q)\mathcal{I}_q dq.\tag{B-20}$$

In the limit of the extreme anomalous skin effect, the integration of Eq. (B-20) can be carried out. A convenient form for $\mathcal{J}$ is written in terms of the dimensionless variable ϵ defined in Eq. (39), as

$$\begin{aligned}\mathcal{J}\simeq\frac{2\pi\lambda^6}{\bar{l}^3}\sum_{i,j=1}^{3}\mathfrak{C}_i\mathfrak{C}_j\int_0^{i\infty}d\epsilon\biggl\{&\left[\frac{\ln(-\epsilon_i)}{\epsilon_i+1}-\frac{\ln(\epsilon_i)}{\epsilon_i-1}\right]\frac{\epsilon}{(\epsilon^2-1)(\epsilon-\epsilon_j)}-\frac{\ln(2-\epsilon)-\ln(\epsilon_i)}{2(\epsilon-1)(\epsilon-\epsilon_j)(\epsilon+\epsilon_i-2)}\\&-\frac{\ln(2-\epsilon)-\ln(-\epsilon_i)}{2(\epsilon-1)(\epsilon-\epsilon_j)(\epsilon-\epsilon_i-2)}+\frac{\ln(2+\epsilon)-\ln(-\epsilon_i)}{2(\epsilon+1)(\epsilon-\epsilon_j)(\epsilon+\epsilon_i+2)}+\frac{\ln(2+\epsilon)-\ln(\epsilon_i)}{2(\epsilon+1)(\epsilon-\epsilon_j)(\epsilon-\epsilon_i+2)}\biggr\},\end{aligned}\tag{B-21}$$

in which

$$\mathfrak{C}_i=(\epsilon_i-\epsilon_k)^{-1}(\epsilon_i-\epsilon_j)^{-1}=-\lambda^{-2}C_i.\tag{B-22}$$

Integration of Eq. (B-21) gives the explicit expressions for $Z(0)$ and for $\Delta Z_L(H_0)$ presented in Eqs. (37) and (44).

APPENDIX C. INTEGRALS FOR TRANSVERSE FIELD ORIENTATION

The calculation of the variation of the surface impedance in a transverse field involves both the rf electric field components, $E_{q,x}$, and the rf Hall field, $E_{q,z}$, given by Eqs. (51) and (52), respectively. The functions $\Gamma_x(q)$, $K_{zz}(i\bar{l}q)$, $L_{xz}(i\bar{l}q,i\bar{l}q')$, $L_{zx}(i\bar{l}q,i\bar{l}q'')$, and $M_{xx}(i\bar{l}q,i\bar{l}q',i\bar{l}q'')$ appearing in Eqs. (51) and (52) are defined in Eqs. (53), (A-1), (A-5), and (A-9), respectively. Explicit solutions for $E_{q,x}$ and $E_{q,z}$ can be derived in the limit of the classical skin effect ($|\bar{l}q|\ll1$) and in the extreme limit of the anomalous skin effect ($|\bar{l}q|\gg1$).

(i) Classical Skin Effect

In the limit $|\bar{l}q|\ll1$, the functions K_{ij}, L_{ij}, and M_{ij} of Eqs. (51) and (52) assume the expansions given by Eqs. (A-3), (A-7), and (A-12), respectively. The function $\Gamma_x(q)$ is given by Eq. (B-4). With this substitution, Eqs. (51) and (52) become

$$E_{q,x}=-\frac{2E_x'(0)}{q^2-k_c^2}\left(1+\frac{4i\omega\omega_n^2\omega_c^2\bar{\tau}^3\bar{l}^2}{5\pi^2c^2}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}\frac{q(q-q'-q'')}{[(q-q'-q'')^2-k_c^2]}+\mathcal{O}(\omega_c\bar{\tau})^4\right),\tag{C-1}$$

and

$$E_{q,z}=-\frac{2iE_x'(0)\omega_c\bar{\tau}}{\pi}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\frac{1}{[(q-q')^2-k_c^2]}+\mathcal{O}(\omega_c\bar{\tau})^3.\tag{C-2}$$

Comparison of the expression for $E_{q,x}$ for the transverse field orientation with $E_{q,y}$ for the longitudinal field case

yields for $z \geqslant 0$,

$$E_x(z) = -\frac{iE_x'(0)}{k_c} e^{ik_c z}\left(1 - \frac{\omega\omega_n^2\omega_c^2\bar{\tau}^3\bar{l}^2\lambda k_c}{5c^2(k_c+i\lambda^{-1})}[2i\lambda^{-1}e^{-2z/\lambda} - k_c(1-e^{-2z/\lambda})] + \mathcal{O}(\omega_c\bar{\tau})^4\right), \tag{C-3}$$

so that

$$\Delta Z_T(H_0) = 4\Delta Z_L(H_0) + \mathcal{O}(H_0^4). \tag{C-4}$$

(ii) Extreme Anomalous Skin Effect

For $|\bar{l}q| \gg 1$, the appropriate expansions of the K_{ij}, L_{ij}, and M_{ij} functions of Eqs. (51) and (52) are:

$$\begin{aligned} K_{xx}(i\bar{l}q) &\simeq \mp\pi(\bar{l}q)^{-1}, \\ K_{zz}(i\bar{l}q) &\simeq 4(\bar{l}q)^{-2}, \\ L_{xz}(i\bar{l}q, i\bar{l}q') &\simeq \mp\pi[\bar{l}^3 q(q-q')^2]^{-1}, \\ L_{zx}(i\bar{l}q, i\bar{l}q') &\simeq \pm\pi[\bar{l}^3 q^2(q-q')]^{-1}, \\ M_{xx}(i\bar{l}q, i\bar{l}q', i\bar{l}q'') &\simeq \mp\tfrac{3}{4}\pi[\bar{l}^3 q(q-q')(q-q'-q'')]^{-1}. \end{aligned} \tag{C-5}$$

Substitution of Eq. (C-5) into Eq. (51) yields

$$E_{q,x} \simeq 2c^2E_x'(0)\Gamma_x(q)\left(1 - \frac{3i\omega\omega_n^2\omega_c^2\bar{\tau}^3}{4\pi^2}\int_{-\infty}^{\infty}\frac{q'dq'}{q'^2+\lambda^{-2}}\int_{-\infty}^{\infty}\frac{q''dq''}{q''^2+\lambda^{-2}}\Gamma_x(q-q'-q'')M_{xx}(i\bar{l}q, i\bar{l}q', i\bar{l}q'') + \mathcal{O}(\omega_c\bar{\tau})^4\right). \tag{C-6}$$

To obtain Eq. (C-6) from Eq. (51) use has been made of the orders of magnitude of the quantities listed in Eq. (C-5) (i.e., $L_{xz}L_{zx} \approx \mathcal{O}(\bar{l}q)^{-6}$ and $M_{xx}K_{zz} \approx \mathcal{O}(\bar{l}q)^{-5}$). Thus, the Hall field $E_{q,z}$ does not affect the field variation of the surface impedance in this limit, in contrast with the result obtained for the classical skin effect. Since $\Gamma_x(q) = \Gamma_y(q)$, and since in the extreme limit of the anomalous skin effect $M_{xx} = 3M_{yy}$, comparison of Eq. (C-6) with Eq. (30) shows that

$$\Delta Z_T = 3\Delta Z_L + \mathcal{O}(H_0^4). \tag{C-7}$$