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Low-Lying Excitations in a Bose Gas of Hard Spheres

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The pseudopotential method is used to calculate that low-lying excitation energies of a Bose gas of hard spheres at $T=0$ to an order beyond that previously calculated by this technique. The results are in agreement with those obtained by Beliaev using a different approach. The phonon velocity is found to be equal to the velocity of compressional waves to the order of approximation considered.

INTRODUCTION

RECENTLY, a number of methods have been developed for systematically calculating the properties of a many-body system at low temperatures. Although most of these methods have so far only been applied to the calculation of ground-state properties, several, e.g., the pseudopotential method[1,2] and the binary collision method of quantum statistics,[3] have also been used for calculating thermodynamical properties at finite temperatures.

Considerable attention has been given to the application of these methods to the Bose gas of hard spheres. The reason is that the small number of parameters in the problem greatly simplifies the theoretical investigation. Moreover, there is the hope that the study of this idealized problem will eventually lead to a better understanding of the observed properties of liquid He^4. One considers the case of infinite matter in which both the number, N, of bosons and the volume, Ω, of the confining box approach infinity while the density, $\rho=N/\Omega$, is held constant. For this problem Lee, Huang, and Yang have shown that a low density expansion of the thermodynamical properties can be made in terms of the parameter $(\rho a^3)^{\frac{1}{2}}$ where "a" is the diameter of the hard spheres.[4]

At any temperature less than the critical temperature T_c, the low-lying excitation energies of a system of bosons are associated with unique wave numbers and, therefore, they exhibit particle-like properties. The energy-momentum relation for these quasi-particles can be measured for He^4 by neutron scattering experiments.[5] For the idealized hard sphere model the leading dependence in the low density approximation was first calculated by Lee and Yang for all $T<T_c$.[2] Independently, Beliaev has also derived the energy-momentum dependence of the low-lying excitations from a study of the one-particle propagator in the Bose many-body system.[6] Beliaev's results, which apply only to the ground state (i.e., $T=0$) of the system, include the leading correction to the excitation energy in terms of the parameter $(\rho a^3)^{\frac{1}{2}}$. This correction includes an imaginary part which is, of course, to be associated with the line width of the excitations.

In the present paper we apply the pseudopotential method to calculate the $O(\rho a^3)^{\frac{1}{2}}$ correction to the energies of the low-lying excitations above the ground state of a Bose gas of hard spheres. The interest of this investigation is twofold. On the one hand, the pseudopotential approach is one of the methods best understood in principle for the treatment of the Bose gas of hard spheres. Therefore, it is of considerable interest to extend the application of this method beyond the approximations used in previous calculations. On the other hand, there is the hope that the present technique can be extended to the case $T\neq 0$.

We begin in Sec. I by writing down the Hamiltonian for the many-body system in an approximate form which is useful for our calculation. Our final results, given in Secs. IV and V, are in agreement with those of Beliaev.

1. HAMILTONIAN OF THE SYSTEM

In the low density approximation the interaction energy, V, in a box of Bose hard spheres (spin$=0$) can be approximated by the sum over all two-body pseudo-

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[1] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).

[2] T. D. Lee and C. N. Yang, Phys. Rev. **112**, 1419 (1958).

[3] T. D. Lee and C. N. Yang, Phys. Rev. **113**, 1165 (1959) (and papers to be published).

[4] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[5] D. G. Henshaw, Phys. Rev. Letters **1**, 127 (1958); H. Palevsky, K. Otnes, E. Larsson, R. Pauli and R. Stedman, Phys. Rev. **108**, 1346 (1957); J. L. Yarnell, G. P. Arnold, P. J. Bendt and E. C. Kerr, Phys. Rev. Letters **1**, 9 (1958).

[6] S. T. Beliaev, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 417 (1958) [translation: Soviet Phys. JETP **7**, 289 (1958)] and J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 433 (1958) [translation: Soviet Phys. JETP and **7**, 299 (1958)].

potentials, i.e.,

$$V \cong \sum_{i<j} V(\mathbf{x}_{ij}),$$

where

$$V(\mathbf{x}_{ij}) = \frac{8\pi \tan ka}{k}\delta^{(3)}(\mathbf{x}_{ij})\frac{\partial}{\partial x_{ij}}x_{ij}$$

$$+[\text{terms corresponding to partial waves with } L \geqq 1], \quad \text{(I.1)}$$

and we use units for which $\hbar = 2m = 1$.[1] The interaction V_{ij} has meaning only when it operates on an eigenfunction of relative momentum $\mathbf{k} = \frac{1}{2}(\mathbf{k}_i - \mathbf{k}_j)$. In the many-body problem it is convenient to use the language of quantized fields. One therefore introduces the operators $\psi(\mathbf{x}) \equiv (1/\Omega^{\frac{1}{2}})\sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$ and

$$\psi^\dagger(\mathbf{x}) \equiv (1/\Omega^{\frac{1}{2}})\sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}},$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ are, respectively, the annihilation and creation operators of free-particle states with momentum $\mathbf{k}$ in a box of volume Ω with periodic boundary conditions. In this representation the approximate interaction $\sum_{i<j} V(\mathbf{x}_{ij})$ is written as:

$$V \cong \frac{1}{2}\int d^3x_1 d^3x_2\, \psi^\dagger(\mathbf{x}_2)\psi^\dagger(\mathbf{x}_1)V(\mathbf{x}_{12})\psi(\mathbf{x}_1)\psi(\mathbf{x}_2). \quad \text{(I.2)}$$

It is usually assumed that $ka \ll 1$, for then the pseudopotential $V(\mathbf{x}_{ij})$ is greatly simplified by the replacement $\tan ka/k \cong a$. This assumption is inherent to the low density approximation. The dominant part of the interaction energy, V', can then be written in the following form:

$$V' = \lim_{r\to 0}\frac{4\pi a}{\Omega}\frac{\partial}{\partial r}\left\{ r\sum_{\mathbf{k}_3,\mathbf{k}_4}\exp\left[\frac{i}{2}(\mathbf{k}_3-\mathbf{k}_4)\cdot\mathbf{r}\right] \times \sum_{\mathbf{k}_1,\mathbf{k}_2} a_1^\dagger a_2^\dagger a_3 a_4 \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\right\}. \quad \text{(I.3a)}$$

Wu has shown that the S-wave interaction V' can also be expressed as[7]

$$V' = \frac{4\pi a}{\Omega}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} a_1^\dagger a_2^\dagger a_3 a_4 \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4) \times \cos\frac{\epsilon}{2}|\mathbf{k}_3-\mathbf{k}_4|. \quad \text{(I.3b)}$$

This last form for V' is useful because it greatly simplifies calculations. It is to be understood that the limit $\epsilon \to 0+$ is to be taken *after* all momentum sums over matrix elements have been performed. In the limit of infinite volume, i.e., Ω, $N \to \infty$ and $\rho = N/\Omega = \text{constant}$, the limits may be performed in the order $\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}$ with the ϵ limit taken last.

The total Hamiltonian, H, of the gas of hard spheres is given by the sum of the interaction energy V and the kinetic energy, $T = \sum_{\mathbf{k}} k^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$. In Appendix I we show that the part of H which is necessary for our present calculation consists of four terms.

$$H \cong \sum_{\mathbf{k}} k^2 N_{\mathbf{k}} + V'$$
$$= H_0 + H_1 + H_2 + H_3 + \text{(higher order terms)} \quad \text{(I.4)}$$

where $N_{\mathbf{k}} \equiv a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ is the number operator for particles with momentum $\mathbf{k}$, and

$$H_0 \equiv 4\pi\rho a N + \sum_{\mathbf{k}}{}'(k^2+k_0^2)N_{\mathbf{k}} + \frac{k_0^2}{2}\sum_{\mathbf{k}}{}'(a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}}a_{-\mathbf{k}}\cos k\epsilon), \quad \text{(I.5a)}$$

$$H_1 \equiv -\frac{4\pi a}{\Omega}\Big(\sum_{\mathbf{k}}{}' N_{\mathbf{k}}\Big)^2 - \frac{4\pi a}{\Omega}\Big(\sum_{\mathbf{k}}{}' N_{\mathbf{k}}\Big)\sum_{\mathbf{l}}{}'(a_{\mathbf{l}}^\dagger a_{-\mathbf{l}}^\dagger + a_{\mathbf{l}}a_{-\mathbf{l}}\cos l\epsilon), \quad \text{(I.5b)}$$

$$H_2 \equiv \frac{8\pi a\sqrt{N}}{\Omega}\sum_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'}{}'\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger a_{\mathbf{k}+\mathbf{k}'}\cos\frac{\epsilon}{2}|\mathbf{k}+\mathbf{k}'| + a_{\mathbf{k}+\mathbf{k}'}^\dagger a_{\mathbf{k}}a_{\mathbf{k}'}\cos\frac{\epsilon}{2}|\mathbf{k}-\mathbf{k}'|\right), \quad \text{(I.5c)}$$

$$H_3 \equiv \frac{4\pi a}{\Omega}\sum_{\mathbf{k}\neq\pm\mathbf{k}'}{}' a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger a_{\mathbf{k}'}a_{-\mathbf{k}'}\cos k'\epsilon, \quad \text{(I.5d)}$$

and

$$k_0^2 \equiv 8\pi a\rho. \quad \text{(I.6)}$$

The summation symbol $\sum'$ means that the zero momentum term is to be omitted. The Hamiltonian H_0 includes the dominant (for low densities) diagonal terms in the plane wave representation plus those terms in the interaction which describe the annihilation or creation of a pair of particles of momentum $(\mathbf{k}, -\mathbf{k})$. When the interaction is switched off, only the states with $\mathbf{k}=0$ are occupied so that the zero momentum occupation number $\langle N_0\rangle$ equals N. When the interaction is present, some of the zero momentum states are excited so that $\langle N_0\rangle < N$. The depletion of the zero-momentum state due to the pair terms in H_0 leads to higher order corrections in H, and these are given by

[7] T. T. Wu, Phys. Rev. **115**, 1390 (1959). The limitation on the validity of the pseudopotential method has been discussed by Wu in his examination of higher order terms.

H_1. In a similar fashion H_2 includes the leading terms in H which change the occupation number of the zero-momentum state by one. Finally, H_3 consists of those terms in which a pair of particles of momentum $(\mathbf{k}', -\mathbf{k}')$ is converted into a pair of momentum $(\mathbf{k}, -\mathbf{k})$: Our Hamiltonian differs somewhat from that given in Eq. (4.8) of reference 7 because, in our case, the depletion factor $\xi = 1-(1/N)\sum_{\mathbf{k}}' N_{\mathbf{k}}$ cannot be replaced by its ground state expectation value, but must be explicitly exhibited in operator form.

II. LOW-LYING EXCITATIONS IN FIRST APPROXIMATION

The program of this calculation is to obtain the eigenvalues and eigenstates of H_0, which is the dominant term in the Hamiltonian, and to evaluate the contributions of H_1, H_2, and H_3 by perturbation theory.

The Hamiltonian H_0 is not Hermitean and, therefore, it cannot clearly be diagonalized by a canonical transformation. However, by means of the transformation[8]

$$\xi_{\mathbf{k}} = (1-\alpha_k^2)^{-\frac{1}{2}}(a_{\mathbf{k}}+\alpha_k a_{-\mathbf{k}}^\dagger), \qquad \text{(II.1a)}$$

$$\xi_{-\mathbf{k}} = (1-\alpha_k^2)^{-\frac{1}{2}}(a_{-\mathbf{k}}+\alpha_k a_{\mathbf{k}}^\dagger), \qquad \text{(II.1b)}$$

where

$$\alpha_k = (k_0^2 \cos k\epsilon)^{-1}\{k^2+k_0^2 - [k^4+2k_0^2k^2+k_0^4(1-\cos k\epsilon)]^{\frac{1}{2}}\}. \qquad \text{(II.2)}$$

Wu has shown that H_0 may be diagonalized aside from a term of the form $-\frac{1}{2}k_0^2\sum_{\mathbf{k}}(1-\cos\epsilon k)\xi_{\mathbf{k}}\xi_{-\mathbf{k}}$.[7] Introducing "right" vacuum states $|0_{\mathbf{k}}\rangle$ by means of

$$\xi_{\mathbf{k}}|0_{\mathbf{k}}\rangle = \xi_{-\mathbf{k}}|0_{\mathbf{k}}\rangle = 0, \qquad \text{(II.3a)}$$

one can see immediately that

$$|0\rangle = \prod_{\mathbf{k}>0} |0_{\mathbf{k}}\rangle \qquad \text{(II.3b)}$$

is in fact a "right" eigenstate of H_0, corresponding to the lowest energy. Similarly, the first "right" excited eigenstate is given by

$$|1_{\mathbf{k}}\rangle = K_k^{(1)}\xi_{\mathbf{k}}^\dagger|0\rangle, \qquad \text{(II.3c)}$$

where $K_k^{(1)}$ is a normalization factor. The vector $|1_{\mathbf{k}}\rangle$ defines a one quasi-particle state of momentum $\mathbf{k}$. The eigenvalues of the eigenstates $|0\rangle$ and $|1_{\mathbf{k}}\rangle$ are the well-known quantities[9]

$$E_0 = 4\pi a\rho N[1+(128/15\sqrt{\pi})(\rho a^3)^{\frac{1}{2}}], \qquad \text{(II.4a)}$$

$$E_0(k) = E_0 + k(k^2+2k_0^2)^{\frac{1}{2}}. \qquad \text{(II.4b)}$$

Because of the presence of the off-diagonal term proportional to $\xi_{\mathbf{k}}\xi_{-\mathbf{k}}$ the Hermitean adjoints of $|0\rangle$ and $|1_{\mathbf{k}}\rangle$ are not "left" eigenstates of H_0. In order to construct these left eigenstates, Wu made use of a canonical transformation of the form of Eq. (II.1) with α_k replaced by

$$\bar{\alpha}_k = \alpha_k \cos\epsilon k. \qquad \text{(II.5)}$$

The corresponding operators are called $\bar{\xi}_{\mathbf{k}}$ and $\bar{\xi}_{-\mathbf{k}}$. By means of this transformation, the expression (I.5a) for H_0 can be diagonalized aside from an off-diagonal term of the form $(k_0^2/2)\sum_{\mathbf{k}}(1-\cos\epsilon k)\bar{\xi}_{\mathbf{k}}^\dagger\bar{\xi}_{-\mathbf{k}}^\dagger$. Introducing the "left" vacuum states by means of

$$\langle 0_{\mathbf{k}}|\bar{\xi}_{\mathbf{k}}^\dagger = \langle 0_{\mathbf{k}}|\bar{\xi}_{-\mathbf{k}}^\dagger = 0, \qquad \text{(II.6a)}$$

one can easily check that

$$\langle 0| = \prod_{\mathbf{k}>0}\langle 0_{\mathbf{k}}|, \qquad \text{(II.6b)}$$

and

$$\langle 1_{\mathbf{k}}| = \bar{K}_k^{(1)}\langle 0|\bar{\xi}_{\mathbf{k}}, \qquad \text{(II.6c)}$$

are the "left" eigenstates of H_0 corresponding to the eigenvalues of Eqs. (II.4a) and (II.4b), respectively. The constant $\bar{K}_k^{(1)}K_k^{(1)}$ may be determined from the normalization condition $\langle 1_{\mathbf{k}}|1_{\mathbf{k}}\rangle = 1$ which yields

$$\bar{K}_k^{(1)}K_k^{(1)} = (1-\bar{\alpha}_k^2)^{-\frac{1}{2}}(1-\alpha_k^2)^{-\frac{1}{2}}(1-\alpha_k\bar{\alpha}_k). \qquad \text{(II.7)}$$

In the limit $\epsilon \to 0+$, $\bar{K}_k^{(1)}K_k^{(1)} = 1$. The eigenstates of H_0 corresponding to several quasi particles of momenta $\mathbf{k}, \mathbf{k}'\cdots$ can be obtained in a similar manner by judicious application of the $\xi_{\mathbf{k}}^\dagger$ and $\bar{\xi}_{\mathbf{k}}$ operators.

By making use of the right and left eigenstates of H_0 constructed above, it is easy to see that the nondiagonal part of H_0 can be disregarded in the calculation of the ground and one quasi-particle energy levels.

The "bare" vacuum states (i.e., the vacuum states in the absence of interaction) are defined by $a_{\mathbf{k}}|\rangle_{\mathbf{k}} = {}_{\mathbf{k}}\langle|a_{\mathbf{k}}^\dagger = 0$. It can then be verified that the relations between the "real" and "bare" vacuum states $|0_{\mathbf{k}}\rangle$ and $|\rangle_{\mathbf{k}}$ are given by

$$|0_{\mathbf{k}}\rangle = K_k \exp(-\alpha_k a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger)|\rangle_{\mathbf{k}}, \qquad \text{(II.8a)}$$

$$\langle 0_{\mathbf{k}}| = \bar{K}_k\, {}_{k}\langle|\exp(-\bar{\alpha}_k a_{\mathbf{k}}a_{-\mathbf{k}}), \qquad \text{(II.8b)}$$

where the normalization condition $\langle 0_{\mathbf{k}}|0_{\mathbf{k}}\rangle = 1$ yields the result $\bar{K}_kK_k = 1-\alpha_k\bar{\alpha}_k$. Thus, the dominant feature of the ground state of a dilute Bose gas of hard spheres is seen to be the presence of pair excitations of momenta $(\mathbf{k}, -\mathbf{k})$.

III. LOW-LYING EXCITATIONS IN SECOND APPROXIMATION

We have seen in Sec. II that the eigenfunctions of H_0 are a combination of all possible pair excitations of momenta $(\mathbf{k}, -\mathbf{k})$. Because of this fact, we find that H_1 and H_3 have diagonal elements in the $\xi_{\mathbf{k}}$-representation, and, therefore, they contribute in first-order perturbation theory. Similarly, it is clear that H_2 has only off-diagonal matrix elements and contributes in second-order perturbation theory. We now make order of magnitude estimates of the contributions of these perturbations to the low-lying excitation energies of the system.

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[8] N. N. Bogoliubov, J. Phys. (U.S.S.R.) **11**, 23 (1947).

[9] T. T. Wu has shown in reference (7) that the next term beyond Eq. (II.4a) for the ground-state energy is of relative order $\rho a^3 \ln(\rho a^3)$.

Using first-order perturbation theory the relative contribution of H_3 to the ground-state energy is estimated as follows:

$$\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}\frac{1}{E_0}\langle 0|H_3|0\rangle=\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}\frac{1}{E_0}\left\langle 0\left|\frac{4\pi a}{\Omega}\sum_{\mathbf{k}\neq\pm\mathbf{k}'}{}' a_{\mathbf{k}}{}^{\dagger}a_{-\mathbf{k}}{}^{\dagger}a_{\mathbf{k}'}a_{-\mathbf{k}'}\cos k'\epsilon\right|0\right\rangle$$

$$=\lim_{\epsilon\to 0+}\frac{1}{\rho^2}\int d^3kd^3k'\,f(\alpha_k,\alpha_{k'},\cos k\epsilon,\cos k'\epsilon)\sim k_0{}^6/\rho^2\sim\rho a^3.$$

This same terms makes a contribution to the excitation energy of a quasi-particle of momentum $\mathbf{k}$ which is estimated to be:

$$\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}\{\langle 1_{\mathbf{k}}|H_3|1_{\mathbf{k}}\rangle-\langle 0|H_3|0\rangle\}=\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}\left\{\left\langle 1_{\mathbf{k}}\left|\frac{4\pi a}{\Omega}\sum_{\mathbf{k}\neq\pm\mathbf{k}'}{}' a_{\mathbf{k}}{}^{\dagger}a_{-\mathbf{k}}{}^{\dagger}a_{\mathbf{k}'}a_{-\mathbf{k}'}\cos k'\epsilon\right|1_{\mathbf{k}}\right\rangle\right.$$

$$\left.-\left\langle 0\left|\frac{4\pi a}{\Omega}\sum_{\mathbf{k}\neq\pm\mathbf{k}'}{}' a_{\mathbf{k}}{}^{\dagger}a_{-\mathbf{k}}{}^{\dagger}a_{\mathbf{k}'}a_{-\mathbf{k}'}\cos k'\epsilon\right|0\right\rangle\right\}\sim ak_0{}^3f(k/k_0)\sim(\rho a^3)^{\frac{1}{2}}k(k^2+2k_0{}^2)^{\frac{1}{2}}f'(k/k_0),$$

where $f(k/k_0)$ and $f'(k/k_0)$ are certain functions of k/k_0. Thus, the contribution of this term to the excitation energy is $\sim(\rho a^3)^{\frac{1}{2}}f'(k/k_0)$ times the leading terms which is $k(k^2+2k_0{}^2)^{\frac{1}{2}}$. Moreover, one can show that H_1 contributes to the same order of magnitude as H_3. In like manner we find that the second-order perturbation treatment of H_2 gives a result which is also $\sim(\rho a^3)^{\frac{1}{2}}k_0{}^2g(k/k_0)$. Similar estimates indicate that the contributions of H_1, H_2, and H_3 in higher orders of perturbation theory are negligible in the present approximation.

The calculation of the energy of a low-lying excitation to a given order in $(\rho a^3)^{\frac{1}{2}}$ involves the subtraction of the perturbation energies of the Hamiltonian H in the ground state from that of the one quasi-particle state. As H_2 connects a one quasi-particle state with two and four quasi-particle states, it gives rise to two contributions which we denote by $\omega_{2,2}(k)$ and $\omega_{2,4}(k)$, respectively. For convenience, $\omega_{2,4}(k)$ includes the subtraction of the ground state perturbation energy due to H_2. Thus, the second approximation to the quasi-particle energies results from an evaluation of the following terms:

$$\omega(k)=\omega_0(k)+\omega_1(k)+\omega_{2,2}(k)+\omega_{2,4}(k)+\omega_3(k)+O(\rho a^3)k_0{}^2f(k/k_0),\tag{III.1}$$

where

$$\omega_0(k)\equiv k(k^2+2k_0{}^2)^{\frac{1}{2}},\tag{III.2a}$$

$$\omega_1(k)\equiv\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}(\langle 1_{\mathbf{k}}|H_1|1_{\mathbf{k}}\rangle-\langle 0|H_1|0\rangle),\tag{III.2b}$$

$$\omega_{2,2}(k)\equiv\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}\sum_{\mathbf{l}_1\leq\mathbf{l}_2}{}'\left(\frac{1}{\omega_0(k)-\omega_0(l_1)-\omega_0(l_2)+i\delta}\right)\langle 1_{\mathbf{k}}|H_2|1_{\mathbf{l}_1}1_{\mathbf{l}_2}\rangle\langle 1_{\mathbf{l}_1}1_{\mathbf{l}_2}|H_2|1_{\mathbf{k}}\rangle,\tag{III.2c}$$

$$\omega_{2,4}(k)\equiv-\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}\sum_{\mathbf{l}_1\leq\mathbf{l}_2\leq\mathbf{l}_3}{}'\left(\frac{1}{\omega_0(l_1)+\omega_0(l_2)+\omega_0(l_3)}\right)\{\langle 1_{\mathbf{k}}|H_2|1_{\mathbf{l}_1}1_{\mathbf{l}_2}1_{\mathbf{l}_3}1_{\mathbf{k}}\rangle\langle 1_{\mathbf{l}_1}1_{\mathbf{l}_2}1_{\mathbf{l}_3}1_{\mathbf{k}}|H_2|1_{\mathbf{k}}\rangle$$
$$-\langle 0|H_2|1_{\mathbf{l}_1}1_{\mathbf{l}_2}1_{\mathbf{l}_3}\rangle\langle 1_{\mathbf{l}_1}1_{\mathbf{l}_2}1_{\mathbf{l}_3}|H_2|0\rangle\},\tag{III.2d}$$

$$\omega_3(k)\equiv\lim_{\epsilon\to 0+}\lim_{\Omega\to\infty}(\langle 1_{\mathbf{k}}|H_3|1_{\mathbf{k}}\rangle-\langle 0|H_3|0\rangle).\tag{III.2e}$$

The momenta in the sums of $\omega_{2,2}$ and $\omega_{2,4}$ are assumed to be ordered so that each different intermediate state is counted only once. In the limit of infinite volume the contribution from terms with $\mathbf{l}_1=\mathbf{l}_2$, say, can be neglected, and therefore in the evaluation of the matrix elements we can assume that the $\mathbf{l}$-momenta are all different. The $(+i\delta)$ in the energy denominator of $\omega_{2,2}$ is included because it is possible for both $\omega_0(k)=\omega_0(l_1)+\omega_0(l_2)$ and $\mathbf{k}=\mathbf{l}_1+\mathbf{l}_2$ to hold simultaneously and therefore this denominator can vanish.

As shown in Appendix II, the matrix elements which appear in the above equations can be written in terms of the quantities α_k and $\cos k\epsilon$. After introducing the dimensionless parameter

$$y\equiv k/k_0,\tag{III.3}$$

and going to the limit of infinite volume, we obtain the following integrals

$$\omega_1=-\lim_{\epsilon\to 0+}\frac{8\pi a k_0^3}{(1-\alpha_y\bar{\alpha}_y)}\left\{[1+\alpha_y\bar{\alpha}_y-2\bar{\alpha}_y]\int\frac{d^3y_1}{(2\pi)^3}\left(\frac{\alpha_1\bar{\alpha}_1}{1-\alpha_1\bar{\alpha}_1}\right)-(1+\alpha_y\bar{\alpha}_y)\int\frac{d^3y_1}{(2\pi)^3}\left(\frac{\bar{\alpha}_1}{1-\alpha_1\bar{\alpha}_1}\right)\right\}$$

$$=-\frac{8\pi a k_0^3}{(1-\alpha_y^2)}\left[(1-\alpha_y)^2\int\frac{d^3y_1}{(2\pi)^3}\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)-(1+\alpha_y^2)\int\frac{d^3y_1}{(2\pi)^3}\left(\frac{\alpha_1}{1-\alpha_1^2}-\frac{1}{2y_1^2}\right)\right], \quad \text{(III.4a)}$$

$$\omega_{2,2}=16\pi k_0^3 a\lim_{\epsilon\to 0+}\left(\frac{1}{1-\alpha_y\bar{\alpha}_y}\right)\int\frac{d^3y_1}{(2\pi)^3}\left(\frac{1}{\omega_0(y)-\omega_0(y_1)-\omega_0(y_2)+i\delta}\right)\left(\frac{1}{1-\alpha_1\bar{\alpha}_1}\right)\left(\frac{1}{1-\alpha_2\bar{\alpha}_2}\right)$$
$$\times\{\bar{\alpha}_y(\alpha_1\cos\tfrac{1}{2}y_1\epsilon+\alpha_2\cos\tfrac{1}{2}y_2\epsilon-\alpha_1\alpha_2\cos\tfrac{1}{2}|\mathbf{y}_1-\mathbf{y}_2|\epsilon)+(\cos\tfrac{1}{2}y\epsilon-\alpha_1\cos\tfrac{1}{2}|\mathbf{y}+\mathbf{y}_1|\epsilon-\alpha_2\cos\tfrac{1}{2}|\mathbf{y}+\mathbf{y}_2|\epsilon)\}$$
$$\times\{\alpha_y(\bar{\alpha}_1\cos\tfrac{1}{2}|\mathbf{y}+\mathbf{y}_2|\epsilon+\bar{\alpha}_2\cos\tfrac{1}{2}|\mathbf{y}+\mathbf{y}_1|\epsilon-\bar{\alpha}_1\bar{\alpha}_2\cos\tfrac{1}{2}y\epsilon)+(\cos\tfrac{1}{2}|\mathbf{y}_1-\mathbf{y}_2|\epsilon-\bar{\alpha}_1\cos\tfrac{1}{2}y_2\epsilon-\bar{\alpha}_2\cos\tfrac{1}{2}y_1\epsilon)\}$$
$$=\frac{2k_0^3a}{\pi^2(1-\alpha_y^2)}\int d^3y_1\left(\frac{1}{\omega_0(y)-\omega_0(y_1)-\omega_0(y_2)+i\delta}\right)\left(\frac{1}{1-\alpha_1^2}\right)\left(\frac{1}{1-\alpha_2^2}\right)$$
$$\times[1-\alpha_1-\alpha_2+\alpha_1\alpha_y+\alpha_2\alpha_y-\alpha_1\alpha_2\alpha_y]^2+\frac{2'k_0^3a}{\pi^2(1-\alpha_y^2)}\int d^3y_1\frac{2}{(\mathbf{y}_1-\mathbf{y}_2)^2}, \quad \text{(III.4b)}$$

$$\omega_{2,4}=-16\pi k_0^3 a\lim_{\epsilon\to 0+}\left(\frac{1}{1-\alpha_y\bar{\alpha}_y}\right)\int\frac{d^3y_1}{(2\pi)^3}\left(\frac{1}{\omega_0(y)+\omega_0(y_1)+\omega_0(y_2)}\right)\left(\frac{1}{1-\alpha_1\bar{\alpha}_1}\right)\left(\frac{1}{1-\alpha_2\bar{\alpha}_2}\right)$$
$$\times\{\alpha_1\alpha_2\cos\tfrac{1}{2}|\mathbf{y}_1-\mathbf{y}_2|\epsilon+\alpha_y\alpha_1\cos\tfrac{1}{2}|\mathbf{y}+\mathbf{y}_1|\epsilon+\alpha_y\alpha_2\cos\tfrac{1}{2}|\mathbf{y}+\mathbf{y}_2|\epsilon-\alpha_1\cos\tfrac{1}{2}y_1\epsilon-\alpha_2\cos\tfrac{1}{2}y_2\epsilon-\alpha_y\cos\tfrac{1}{2}y\epsilon\}$$
$$\times\{\bar{\alpha}_1\bar{\alpha}_2\cos\tfrac{1}{2}y\epsilon+\bar{\alpha}_y\bar{\alpha}_1\cos\tfrac{1}{2}y_2\epsilon+\bar{\alpha}_y\bar{\alpha}_2\cos\tfrac{1}{2}y_1\epsilon-\bar{\alpha}_1\cos\tfrac{1}{2}|\mathbf{y}+\mathbf{y}_2|\epsilon-\bar{\alpha}_2\cos\tfrac{1}{2}|\mathbf{y}+\mathbf{y}_1|\epsilon-\bar{\alpha}_y\cos\tfrac{1}{2}|\mathbf{y}_1-\mathbf{y}_2|\epsilon\}$$
$$=-\frac{2k_0^3a}{\pi^2(1-\alpha_y^2)}\int d^3y_1\left[\frac{1}{\omega_0(y)+\omega_0(y_1)+\omega_0(y_2)}\right]\left(\frac{1}{1-\alpha_1^2}\right)\left(\frac{1}{1-\alpha_2^2}\right)$$
$$\times(\alpha_1+\alpha_2+\alpha_y-\alpha_1\alpha_2-\alpha_y\alpha_1-\alpha_y\alpha_2)^2+\frac{2k_0^3a}{\pi^2}\left(\frac{\alpha_y^2}{1-\alpha_y^2}\right)\int d^3y_1\frac{2}{(\mathbf{y}_1-\mathbf{y}_2)^2}, \quad \text{(III.4c)}$$

$$\omega_3=16\pi a k_0^3\lim_{\epsilon\to 0+}\left(\frac{\bar{\alpha}_y}{1-\alpha_y\bar{\alpha}_y}\right)\int\frac{d^2y_1}{(2\pi)^3}\left(\frac{\bar{\alpha}_1}{1-\alpha_1\bar{\alpha}_1}\right)$$
$$=16\pi a k_0^3\left(\frac{\alpha_y}{1-\alpha_y^2}\right)\int\frac{d^3y_1}{(2\pi)^3}\left(\frac{\alpha_1}{1-\alpha_1^2}-\frac{1}{2y_1^2}\right), \quad \text{(III.4d)}$$

where

$$\mathbf{y}_2=\mathbf{y}-\mathbf{y}_1,$$
$$\alpha_y=1+y^2-y(2+y^2)^{\frac{1}{2}};\quad \alpha_1\equiv\alpha_{y_1}, \quad \text{(III.5)}$$
$$\omega_0(y)=y(2+y^2)^{\frac{1}{2}}.$$

The procedure used to evaluate

$$F\equiv\lim_{\epsilon\to 0+}\int d^3y_1\,f(y_1,\epsilon)$$

is the following: (*A*) We write $F=\int d^3y_1\,f(y_1,0)$ when the latter integral exists. (*B*) When this is not the case, we first subtract

$$\lim_{\epsilon\to 0+}\int d^3y_1\,f_{as}(y_1,\epsilon)$$

where $f_{as}(y_1,\epsilon)$ is the asymptotic value of $f(y_1,\epsilon)$ for $y_1\to\infty$ (in the cases of interest the latter integral is zero). We then observe that in the integrals in question

$$\int d^3y_1\,[f(y_1,\epsilon)-f_{as}(y_1,\epsilon)]$$

is defined for $\epsilon>0$ and for $\epsilon=0$ so that we can interchange the operations of limit and integration to finally obtain[10]

$$F=\int d^3y_1\,[f(y_1,0)-f_{as}(y_1,0)].$$

[10] See reference (4) below Eq. (25).

In the next two sections the results for the above quantities in low and high momentum limits will be stated and then discussed.

IV. LOW MOMENTUM EXCITATIONS, $k \ll k_0$

The various integrals in the previous section can be performed in the low momentum region, i.e., for $y = k/k_0 \ll 1$. In this region we expect the excitation energy to be of the form $\omega(k) = (\text{velocity})k$, as was the case for $\omega_0(k) = k(k^2+2k_0^2)^{\frac{1}{2}}$. We obtain the following expressions for the corrections to $\omega_0(k)$ from Eqs. (III.4).[11]

$$\omega_1 = -\frac{16}{3\sqrt{\pi}} \frac{k_0^2(\rho a^3)^{\frac{1}{2}}}{(1-\alpha_y^2)} [2(1+\alpha_y^2) - \alpha_y]$$

$$= -\frac{8}{3\sqrt{\pi}} k_0^2 (\rho a^3)^{\frac{1}{2}} \left(\frac{3+4y^2}{y(2+y^2)^{\frac{1}{2}}} \right), \qquad \text{(IV.1a)}$$

$$\omega_{2,2} = (\omega_{2,2})_R - i(\omega_{2,2})_I, \qquad \text{(IV.1b)}$$

$$(\omega_{2,2})_R = \frac{8}{\sqrt{\pi}} k_0^2 (\rho a^3)^{\frac{1}{2}} \frac{1}{y(2+y^2)^{\frac{1}{2}}} + \frac{k_0^2(\rho a^3)^{\frac{1}{2}}}{\sqrt{\pi}} \left[5\pi + \frac{37}{3\sqrt{2}} y + O(y^2) \right],$$

$$(\omega_{2,2})_I = (3\sqrt{\pi}/10\sqrt{2}) k_0^2 (\rho a^3)^{\frac{1}{2}} y^5 [1+O(y)]$$

$$\cong 3k^5/320\pi\rho, \qquad \text{(IV.1c)}$$

$$\omega_{2,4} = \frac{8}{\sqrt{\pi}} k_0^2 (\rho a^3)^{\frac{1}{2}} \frac{1}{y(2+y^2)^{\frac{1}{2}}} + \frac{k_0^2(\rho a^3)^{\frac{1}{2}}}{\sqrt{\pi}} \left[-5\pi + \frac{43}{3\sqrt{2}} y + O(y^2) \right],$$

$$\omega_3 = -\frac{16}{\sqrt{\pi}} k_0^2 (\rho a^3)^{\frac{1}{2}} \left(\frac{\alpha_y}{1-\alpha_y^2} \right)$$

$$= -\frac{8}{\sqrt{\pi}} k_0^2 (\rho a^3)^{\frac{1}{2}} \frac{1}{y(2+y^2)^{\frac{1}{2}}}. \qquad \text{(IV.1d)}$$

Upon adding these expressions together, we obtain for the energy of a low momentum quasi-particle [see Eq. (III.1)][12]:

$$\omega(k) \xrightarrow[k \ll k_0]{} \sqrt{2} k_0 k \left[1 + \frac{8}{\sqrt{\pi}} (\rho a^3)^{\frac{1}{2}} - i \frac{3\sqrt{\pi}}{20} (\rho a^3)^{\frac{1}{2}} \left(\frac{k}{k_0} \right)^4 + O\left(\frac{k}{k_0} \right)^2 \right]. \qquad \text{(IV.2)}$$

The velocity, v_{ph}, of the low momentum quasi-particles, which are also called *phonons*, is seen to be:

$$v_{ph} = 4(\pi\rho a)^{\frac{1}{2}} [1 + (8/\sqrt{\pi})(\rho a^3)^{\frac{1}{2}} + O(\rho a^3)]. \qquad \text{(IV.3)}$$

For the ground state, the phonon velocity must be directly related to the velocity of compressional waves, v_c, in the gas, and in fact we can verify that these two velocities are equal to the order of our calculation. The velocity v_c is related to the compressibility by:

$$v_c = (2dP/d\rho)^{\frac{1}{2}}, \qquad \text{(IV.4)}$$

where in our units $2m=1$. The pressure, P, at zero temperature is given by:

$$P = -\frac{d(E_0/N)}{d(1/\rho)} = \rho^2 \frac{d(E_0/N)}{d\rho}$$

$$\cong 4\pi a \rho^2 [1 + (64/5\sqrt{\pi})(\rho a^3)^{\frac{1}{2}}], \qquad \text{(IV.5)}$$

where we have used Eq. (II.4a). Thus, it can readily be proved up to terms of order ρa^2 that the phonon velocity equals the velocity of compressional waves in the gas, i.e., $v_{ph} = v_c$.

The imaginary part of the quasi-particle energy, corresponds in this calculation to the decay of one quasi-particle into two other quasi-particles with a mean life, $\tau = 1/[2(\omega_{2,2})_I]$. In the low momentum limit we obtain for τ:

$$\tau \xrightarrow[k \ll k_0]{} \frac{160\pi\rho}{3k^5} = \frac{5\sqrt{2}}{3\sqrt{\pi}} \frac{1}{k^2(\rho a^3)^{\frac{1}{2}}} (k_0/k)^3. \qquad \text{(IV.6)}$$

Thus, in the low momentum region the mean life of the phonons becomes very long and we conclude that these states are essentially eigenstates of the system.[13]

V. HIGH MOMENTUM EXCITATIONS, $k \gg k_0$

In the high momentum regions characterized by $k \gg k_0$, we have $\omega_0(k) = k^2 + k_0^2 + O(k_0^4/k^2)$. We will keep only those terms in $\omega(k)$ which are $\gtrsim k_0^2(\rho a^3)^{\frac{1}{2}}$. It then becomes unnecessary to calculate $(\omega_{2,2})_R$ and $(\omega_{2,4})$ for

---

[11] The function α_y has several very useful properties, e.g.:

$$\alpha_y = 1 + y^2 - y(2+y^2)^{\frac{1}{2}}; \quad \alpha_y^{-1} = 1 + y^2 + y(2+y^2)^{\frac{1}{2}}$$

$$(1-\alpha_y)^2 = 2y^2\alpha_y; \quad (1-\alpha_y^2) = 2y(2+y^2)^{\frac{1}{2}}\alpha_y; \quad (1+\alpha_y^2) = 2(1+y^2)\alpha_y.$$

[12] Note that our results are given in terms of the density of all particles, N/Ω, and not the density of zero-momentum particles, $\langle N_0 \rangle / \Omega$. For the ground state, the relation between these two quantities is:

$$\frac{\langle N_0 \rangle}{\Omega} = \frac{N}{\Omega} - \frac{1}{\Omega} \sum_{\mathbf{k}}' \langle N_{\mathbf{k}} \rangle = \rho \left[1 - \frac{8}{3\sqrt{\pi}} (\rho a^3)^{\frac{1}{2}} + O(\rho a^3) \right].$$

[13] Equations (IV.6) and (V.3) for the quasi-particle lifetimes were obtained previously by T. D. Lee and C. N. Yang [reference (2)].

we can show that $(\omega_{2,2})_R \sim k_0^2(\rho a^3)^{\frac{1}{2}}(k_0/k)$ and $\omega_{2,4} \sim k_0^2(\rho a^3)^{\frac{1}{2}}(k_0/k)^2$ when $k \gg k_0$. The quantity ω_3 is also negligible in this approximation, and therefore we only need to write down the terms ω_1 and $(\omega_{2,2})_I$.

$$\omega_1 = -(32/3\sqrt{\pi})k_0^2(\rho a^3)^{\frac{1}{2}}[1+O(k_0/k)^2], \tag{V.1a}$$

$$(\omega_{2,2})_I = 2(2\pi)^{\frac{1}{2}}k_0 k(\rho a^3)^{\frac{1}{2}}[1+O(k_0/k)^2]. \tag{V.1b}$$

Thus, the expression for the energy of the quasi-particles in the high momentum region is:

$$\omega(k) = k^2 + k_0^2[1-(32/3\sqrt{\pi})(\rho a^3)^{\frac{1}{2}}] - i2(2\pi)^{\frac{1}{2}}k_0 k(\rho a^3)^{\frac{1}{2}} + k_0^2 O[(\rho a^3)^{\frac{1}{2}}k_0/k]. \tag{V.2}$$

We observe that when we speak of high momentum excitations, $k \gg k_0$, we are always limited by our original approximation to momenta for which $ka \ll 1$. But these two inequalities can be compatible, for we may have $1 \gg ka \gg k_0 a \sim (\rho a^3)^{\frac{1}{2}}$.

The imaginary part of the quasi-particle energy corresponds to a mean life which for high momenta is:

$$\tau \xrightarrow[k\gg k_0]{} \frac{1}{16\pi\rho a^2 k}. \tag{V.3}$$

In this region, as for the phonon region, the line width due to quasi-particle decay is small compared to the excitation energy.

CONCLUSION

We have shown that the pseudopotential method can be used to calculate the energies of low-lying excitations in a gas of Bose hard spheres at $T=0$ to an order beyond that previously calculated by the method. The phonon velocity, v_{ph}, as calculated in this paper is equal to the velocity of compressional waves, v_c, as determined from a lower order calculation of the ground-state energy. One feels intuitively that these velocities must be equal, since at $T=0$ the phonon excitations are the only degrees of freedom of the gas which can coherently add up to give a compressional wave. The detailed results of Secs. IV and V are in agreement with those obtained by Beliaev by means of his investigation of the Green's functions of the Bose many body system.[6]

It would be of interest to obtain the generalization of the present results to an arbitrary temperature $T<T_c$. Our calculation suggests that it might indeed be possible to perform such an investigation using the method of pseudopotentials.

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APPENDIX I. DERIVATION OF THE APPROXIMATE HAMILTONIAN

In this appendix we outline the derivation of the approximate Hamiltonian given in Sec. I starting from the low-density, many-body interaction Hamiltonian of Eq. (I.3b).

$$V' = \frac{4\pi a}{\Omega}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} a_1^\dagger a_2^\dagger a_3 a_4 \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\cos\frac{\epsilon}{2}|\mathbf{k}_3-\mathbf{k}_4|.$$

The diagonal terms of V' can be written down in a straightforward manner as follows:

$$\begin{aligned} V_{\text{diag.}}' &= \frac{4\pi a}{\Omega}\left\{\sum_{\mathbf{k}}(N_{\mathbf{k}}^2-N_{\mathbf{k}}) + 2\sum_{\mathbf{k}_1\neq\mathbf{k}_2} N_1 N_2 \cos\frac{\epsilon}{2}|\mathbf{k}_1-\mathbf{k}_2|\right\} \\ &= 4\pi a\rho(N-1) + 8\pi a\rho\sum_{\mathbf{k}}{}' N_{\mathbf{k}}\left(2\cos\frac{k\epsilon}{2}-1\right) - \frac{4\pi a}{\Omega}\left(\sum_{\mathbf{k}}{}' N_{\mathbf{k}}\right)\left[\sum_{\mathbf{k}}{}' N_{\mathbf{k}}\left(2\cos\frac{k\epsilon}{2}-1\right)\right] \\ &\quad + \frac{8\pi a}{\Omega}\sum_{\mathbf{k}_1\neq\mathbf{k}_2}{}' N_1 N_2\left(\cos\frac{\epsilon}{2}|\mathbf{k}_1-\mathbf{k}_2| - \cos\frac{k_1\epsilon}{2}\right) - \frac{4\pi a}{\Omega}\sum_{\mathbf{k}}{}' N_{\mathbf{k}}^2\left(2\cos\frac{k\epsilon}{2}-1\right) \\ &= 4\pi a\rho N + 8\pi a\rho\sum_{\mathbf{k}}{}' N_{\mathbf{k}} - \frac{4\pi a}{\Omega}\Big[\sum_{\mathbf{k}}{}' N_{\mathbf{k}}\Big]^2 + \rho a O(1), \end{aligned} \tag{A.1}$$

where we can consistently replace $(2\cos k\epsilon/2-1)$ by 1 without affecting calculations.[14]

Because of the special role played by the zero-momentum state, we classify the off-diagonal terms in V' according to the number of operators a_0 and $a_0^\dagger$. Thus:

[14] This replacement is justified in reference (7) below Eq. (3.32).

$$V'-V_{\text{diag.}}'=\frac{4\pi a}{\Omega}\sum_{\mathbf{k}}{}'\left[a_{\mathbf{k}}^{\dagger}a_{-\mathbf{k}}^{\dagger}a_0a_0+a_0^{\dagger}a_0^{\dagger}a_{\mathbf{k}}a_{-\mathbf{k}}\cos k\epsilon\right]$$
$$+\frac{8\pi a}{\Omega}\sum_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'}{}'\left(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'}^{\dagger}a_{\mathbf{k}+\mathbf{k}'}a_0\cos\frac{\epsilon}{2}|\mathbf{k}+\mathbf{k}'|+a_0^{\dagger}a_{\mathbf{k}+\mathbf{k}'}^{\dagger}a_{\mathbf{k}}a_{\mathbf{k}'}\cos\frac{\epsilon}{2}|\mathbf{k}-\mathbf{k}'|\right)$$
$$+\frac{4\pi a}{\Omega}\sum_{\mathbf{k}_1,\mathbf{k}_2\neq\mathbf{k}_3,\mathbf{k}_4}{}'a_1^{\dagger}a_2^{\dagger}a_3a_4\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\cos\frac{\epsilon}{2}|\mathbf{k}_3-\mathbf{k}_4|.\quad\text{(A.2)}$$

Now, in the representation in which $N_{\mathbf{k}}$ is diagonal and $\langle N_{-\mathbf{k}}\rangle=\langle N_{\mathbf{k}}\rangle=n_{\mathbf{k}}$, the matrix elements of the first term in Eq. (A.2) are

$$\frac{4\pi a}{\Omega}\langle n_0-2,\,n_{\mathbf{k}}+1|a_0a_0\sum_{\mathbf{l}}{}'a_{\mathbf{l}}^{\dagger}a_{-\mathbf{l}}^{\dagger}|n_0n_{\mathbf{k}}\rangle$$
$$=(8\pi a/\Omega)n_{\mathbf{k}}n_0(1-1/n_0)^{\frac{1}{2}}\cong(8\pi a/\Omega)n_{\mathbf{k}}n_0.$$

Although the total number of particles is conserved, we may suppress the quantum number $n_0=\langle N_0\rangle$ in matrix elements if instead we always write

$$\langle N_0\rangle=N-\sum_{\mathbf{k}}{}'\langle N_{\mathbf{k}}\rangle.$$

With this notation, the above matrix element can be written

$$\frac{4\pi a}{\Omega}\langle n_{\mathbf{k}}+1|a_0a_0\sum_{\mathbf{l}}{}'a_{\mathbf{l}}^{\dagger}a_{-\mathbf{l}}^{\dagger}|n_{\mathbf{k}}\rangle=\frac{8\pi a}{\Omega}n_{\mathbf{k}}[N-\sum_{\mathbf{k}}{}'n_{\mathbf{k}}],$$

where terms of $O(1/N)$ have been neglected. Thus, we can calculate matrix elements correctly by making the following substitutions into Eq. (A.2)

$$a_0\rightarrow[N-\sum_{\mathbf{k}}{}'N_{\mathbf{k}}]^{\frac{1}{2}},$$
$$a_0^{\dagger}\rightarrow[N-\sum_{\mathbf{k}}{}'N_{\mathbf{k}}]^{\frac{1}{2}}.\qquad\text{(A.3)}$$

The quantity $\sum_{\mathbf{k}}{}'N_{\mathbf{k}}$ is, of course, an operator and cannot be replaced by its ground state expectation value, because

$$\langle 1_{\mathbf{k}}|\sum_{\mathbf{l}}{}'N_{\mathbf{l}}|1_{\mathbf{k}}\rangle\neq\langle 0|\sum_{\mathbf{l}}{}'N_{\mathbf{l}}|0\rangle.$$

The approximate Hamiltonian given by Eqs. (I.4) and (I.5) now follows from Eqs. (A.1) and (A.2) with the substitutions (A.3). The justification for the terms retained is made according to the order of magnitude estimates of Sec. III. For example, in the term with no zero-momentum operators, only

$$H_3=\frac{4\pi a}{\Omega}\sum_{\mathbf{k}\neq\pm\mathbf{k}'}{}'a_{\mathbf{k}}^{\dagger}a_{-\mathbf{k}}^{\dagger}a_{\mathbf{k}'}a_{-\mathbf{k}'}\cos k'\epsilon$$

contributes to the order which we calculate.

APPENDIX II. EVALUATION OF A TYPICAL MATRIX ELEMENT

Consider the matrix element $\langle 1_{\mathbf{k}_1}1_{\mathbf{k}_2}|H_2|1_{\mathbf{k}}\rangle$ with $\mathbf{k}_1\neq\mathbf{k}_2$, which occurs in the expression for $\omega_{2,2}$, Eq. (III.2c). Using Eqs. (I.5c), (II.1), (II.3c), and (II.6c) we have:

$$\langle 1_{\mathbf{k}_1}1_{\mathbf{k}_2}|H_2|1_{\mathbf{k}}\rangle=\frac{8\pi a\sqrt{N}}{\Omega}\bar{K}_1^{(1)}\bar{K}_2^{(1)}K_k^{(1)}\sum_{\substack{\mathbf{l}_1,\mathbf{l}_2\\(\mathbf{l}=\mathbf{l}_1+\mathbf{l}_2)}}{}'\langle 0|\,\xi_1\xi_2\{a_{\mathbf{l}_1}^{\dagger}a_{\mathbf{l}_2}^{\dagger}a_{\mathbf{l}}\cos\tfrac{1}{2}\epsilon|\mathbf{l}_1+\mathbf{l}_2|+a_{\mathbf{l}}^{\dagger}a_{\mathbf{l}_1}a_{\mathbf{l}_2}\cos\tfrac{1}{2}\epsilon|\mathbf{l}_1-\mathbf{l}_2|\}\xi_{\mathbf{k}}^{\dagger}|0\rangle$$
$$=\frac{8\pi a\sqrt{N}}{\Omega}\bar{K}_1^{(1)}\bar{K}_2^{(1)}K_k^{(1)}(1-\bar{\alpha}_1^2)^{-\frac{1}{2}}(1-\bar{\alpha}_2^2)^{-\frac{1}{2}}(1-\alpha_k^2)^{-\frac{1}{2}}\sum_{\substack{\mathbf{l}_1,\mathbf{l}_2\\(\mathbf{l}=\mathbf{l}_1+\mathbf{l}_2)}}{}'\langle 0|\,(a_{\mathbf{k}_1}+\bar{\alpha}_1a_{-\mathbf{k}_1}^{\dagger})(a_{\mathbf{k}_2}+\bar{\alpha}_2a_{-\mathbf{k}_2}^{\dagger})$$
$$\times\{a_{\mathbf{l}_1}^{\dagger}a_{\mathbf{l}_2}^{\dagger}a_{\mathbf{l}}\cos\tfrac{1}{2}l\epsilon+a_{\mathbf{l}}^{\dagger}a_{\mathbf{l}_1}a_{\mathbf{l}_2}\cos\tfrac{1}{2}\epsilon|\mathbf{l}_1-\mathbf{l}_2|\}(a_{\mathbf{k}}^{\dagger}+\alpha_ka_{-\mathbf{k}})|0\rangle$$
$$=\frac{8\pi a\sqrt{N}}{\Omega}\bar{K}_1^{(1)}\bar{K}_2^{(1)}K_k^{(1)}\frac{(1-\bar{\alpha}_1^2)^{\frac{1}{2}}}{\bar{\alpha}_1}\frac{(1-\bar{\alpha}_2^2)^{\frac{1}{2}}}{\bar{\alpha}_2}(1-\alpha_k^2)^{\frac{1}{2}}\sum_{\substack{\mathbf{l}_1,\mathbf{l}_2\\(\mathbf{l}=\mathbf{l}_1+\mathbf{l}_2)}}{}'\langle 0|a_{-\mathbf{k}_1}^{\dagger}a_{-\mathbf{k}_2}^{\dagger}$$
$$\times\{a_{\mathbf{l}_1}^{\dagger}a_{\mathbf{l}_2}^{\dagger}a_{\mathbf{l}}\cos\tfrac{1}{2}l\epsilon+a_{\mathbf{l}}^{\dagger}a_{\mathbf{l}_1}a_{\mathbf{l}_2}\cos\tfrac{1}{2}\epsilon|\mathbf{l}_1-\mathbf{l}_2|\}a_{\mathbf{k}}^{\dagger}|0\rangle,\quad\text{(A.4)}$$

where we have also used Eqs. (II.3a) and (II.6a) in order to obtain the last line. We now apply the identities

$$\langle 0_{\mathbf{k}}|a_{\mathbf{k}}a_{-\mathbf{k}}|0_{\mathbf{k}}\rangle=-\left(\frac{\alpha_k}{1-\alpha_k\bar{\alpha}_k}\right),\quad\langle 0_{\mathbf{k}}|a_{\mathbf{k}}^{\dagger}a_{-\mathbf{k}}^{\dagger}|0_{\mathbf{k}}\rangle=-\left(\frac{\bar{\alpha}_k}{1-\alpha_k\bar{\alpha}_k}\right),\qquad\text{(A.5)}$$

which can readily be verified with the aid of Eqs. (II.1), (II.3a), and (II.6a), to derive the final expression for this

matrix element

$$\langle 1_{\mathbf{k}_1}1_{\mathbf{k}_2}|H_2|1_{\mathbf{k}}\rangle=\frac{16\pi a\sqrt{N}}{\Omega}\bar{K}_1{}^{(1)}\bar{K}_2{}^{(1)}K_k{}^{(1)}\frac{(1-\bar{\alpha}_1{}^2)^{\frac{1}{2}}}{(1-\alpha_1\bar{\alpha}_1)}\frac{(1-\bar{\alpha}_2{}^2)^{\frac{1}{2}}}{(1-\alpha_2\bar{\alpha}_2)}\frac{(1-\alpha_k{}^2)^{\frac{1}{2}}}{(1-\alpha_k\bar{\alpha}_k)}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})$$
$$\times\{\bar{\alpha}_k[\alpha_1\cos\tfrac{1}{2}k_1\epsilon+\alpha_2\cos\tfrac{1}{2}k_2\epsilon-\alpha_1\alpha_2\cos\tfrac{1}{2}\epsilon|\mathbf{k}_1-\mathbf{k}_2|]+[\cos\tfrac{1}{2}k\epsilon-\alpha_1\cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}_1|-\alpha_2\cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}_2|]\}. \quad \text{(A.6a)}$$

In a similar fashion we obtain for $\langle 1_{\mathbf{k}}|H_2|1_{\mathbf{k}_1}1_{\mathbf{k}_2}\rangle\neq\langle 1_{\mathbf{k}_1}1_{\mathbf{k}_2}|H_2|1_{\mathbf{k}}\rangle^*$

$$\langle 1_{\mathbf{k}}|H_2|1_{\mathbf{k}_1}1_{\mathbf{k}_2}\rangle=\frac{16\pi a\sqrt{N}}{\Omega}\bar{K}_k{}^{(1)}K_1{}^{(1)}K_2{}^{(1)}\frac{(1-\alpha_1{}^2)^{\frac{1}{2}}}{(1-\alpha_1\bar{\alpha}_1)}\frac{(1-\alpha_2{}^2)^{\frac{1}{2}}}{(1-\alpha_2\bar{\alpha}_2)}\frac{(1-\bar{\alpha}_k{}^2)^{\frac{1}{2}}}{(1-\alpha_k\bar{\alpha}_k)}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})$$
$$\times\{\alpha_k[\bar{\alpha}_1\cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}_2|+\bar{\alpha}_2\cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}_1|-\bar{\alpha}_1\bar{\alpha}_2\cos k\epsilon/2]+[\cos\tfrac{1}{2}\epsilon|\mathbf{k}_1-\mathbf{k}_2|-\bar{\alpha}_1\cos\tfrac{1}{2}k_2\epsilon-\bar{\alpha}_2\cos k_1\epsilon/2]\}. \quad \text{(A.6b)}$$

After using Eq. (II.7), the product of these two matrix elements can be substituted into Eq. (III.2c) to yield Eq. (III.4b).

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Transport Phenomena in Slightly Ionized Gases: High Electric Fields*

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Starting with the electron velocity distribution obtained by Chapman and Cowling for a Lorentzian gas, in the presence of an electric field, the author has investigated the variation with electric field of a number of transport properties, arising from a magnetic field, perpendicular to the electric field and temperature gradient in the gas. The applicability of the results to semiconductors has also been pointed out. A constant mean free path has been assumed, which is validated by experiments for helium.

INTRODUCTION

NUMEROUS investigations of the velocity distribution of electrons in a Lorentzian gas, consisting of a large number of neutral molecules and a small number of electrons, when an electric field is present have been carried out. However no detailed discussion of transport phenomena has been made except in the case of vanishingly small electric fields, when the Maxwellian distribution of velocities is valid. In a recent communication[1] (called Part I hereafter) Sodha[1] has discussed the transport phenomena in the case of low electric fields (when terms involving fourth and higher power of electric field are negligible) assuming the time of relaxation $\tau\propto x^n$, $x=(m/2kT)^{\frac{1}{2}}v$ being the dimensionless velocity.

In this paper the author has investigated the variation of transport properties with electric field when the time of relaxation τ is given by

$$\tau=l/x. \tag{1}$$

This corresponds to a constant mean free path and is valid in many cases of interest, e.g., helium[2] (for slow electrons with energies below 1.5 ev or 10^4°K). It may be added that l has the dimensions of time.

VELOCITY DISTRIBUTION FUNCTION

In Part I, the electron velocity distribution function in the presence of an electric field was expressed as

$$f_0(x)=A\exp\left(-\int\frac{2xdx}{(1+z\tau^2)}\right), \tag{2}$$

where the symbols have the usual[1] meanings.

Substituting for τ from Eq. (1) in Eq. (2) we obtain

$$f_0(x)=A\exp(-x^2)(x^2+a)^a, \tag{3}$$

where

$$a=zl^2=(m_1/3m_2{}^2)(q^2E^2/kT)l^2. \tag{4}$$

Equations (1) and (3) are also valid[3] for nondegenerate semiconductors, if acoustic modes of lattice vibrations are considered to be the sole source of scattering and

$$a=(1/m_2{}^2)(q^2E^2/c_1{}^2)l^2, \tag{4A}$$

where m_2 is the effective electronic mass and c_1 the velocity of sound in the crystal.

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