matrix element

$$\langle 1_{\mathbf{k}_1} 1_{\mathbf{k}_2} | H_2 | 1_{\mathbf{k}} \rangle = \frac{16\pi a\sqrt{N}}{\Omega} \bar{K}_1{}^{(1)} \bar{K}_2{}^{(1)} K_k{}^{(1)} \frac{(1-\bar{\alpha}_1{}^2)^{\frac{1}{2}}}{(1-\alpha_1\bar{\alpha}_1)} \frac{(1-\bar{\alpha}_2{}^2)^{\frac{1}{2}}}{(1-\alpha_2\bar{\alpha}_2)} \frac{(1-\alpha_k{}^2)^{\frac{1}{2}}}{(1-\alpha_k\bar{\alpha}_k)} \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})$$

$$\times\{\bar{\alpha}_k[\alpha_1 \cos\tfrac{1}{2}k_1\epsilon + \alpha_2 \cos\tfrac{1}{2}k_2\epsilon - \alpha_1\alpha_2 \cos\tfrac{1}{2}\epsilon|\mathbf{k}_1-\mathbf{k}_2|] + [\cos\tfrac{1}{2}k\epsilon - \alpha_1 \cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}_1| - \alpha_2 \cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}_2|]\}. \quad \text{(A.6a)}$$

In a similar fashion we obtain for $\langle 1_{\mathbf{k}} | H_2 | 1_{\mathbf{k}_1} 1_{\mathbf{k}_2} \rangle \neq \langle 1_{\mathbf{k}_1} 1_{\mathbf{k}_2} | H_2 | 1_{\mathbf{k}} \rangle^*$

$$\langle 1_{\mathbf{k}} | H_2 | 1_{\mathbf{k}_1} 1_{\mathbf{k}_2} \rangle = \frac{16\pi a\sqrt{N}}{\Omega} \bar{K}_k{}^{(1)} K_1{}^{(1)} K_2{}^{(1)} \frac{(1-\alpha_1{}^2)^{\frac{1}{2}}}{(1-\alpha_1\bar{\alpha}_1)} \frac{(1-\alpha_2{}^2)^{\frac{1}{2}}}{(1-\alpha_2\bar{\alpha}_2)} \frac{(1-\bar{\alpha}_k{}^2)^{\frac{1}{2}}}{(1-\alpha_k\bar{\alpha}_k)} \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})$$

$$\times\{\alpha_k[\bar{\alpha}_1 \cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}_2| + \bar{\alpha}_2 \cos\tfrac{1}{2}\epsilon|\mathbf{k}+\mathbf{k}_1| - \bar{\alpha}_1\bar{\alpha}_2 \cos k\epsilon/2] + [\cos\tfrac{1}{2}\epsilon|\mathbf{k}_1-\mathbf{k}_2| - \bar{\alpha}_1 \cos\tfrac{1}{2}k_2\epsilon - \bar{\alpha}_2 \cos k_1\epsilon/2]\}. \quad \text{(A.6b)}$$

After using Eq. (II.7), the product of these two matrix elements can be substituted into Eq. (III.2c) to yield Eq. (III.4b).

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Transport Phenomena in Slightly Ionized Gases: High Electric Fields*

MAHENDRA SINGH SODHA
Physics Division, Armour Research Foundation, Chicago, Illinois
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Starting with the electron velocity distribution obtained by Chapman and Cowling for a Lorentzian gas, in the presence of an electric field, the author has investigated the variation with electric field of a number of transport properties, arising from a magnetic field, perpendicular to the electric field and temperature gradient in the gas. The applicability of the results to semiconductors has also been pointed out. A constant mean free path has been assumed, which is validated by experiments for helium.

INTRODUCTION

NUMEROUS investigations of the velocity distribution of electrons in a Lorentzian gas, consisting of a large number of neutral molecules and a small number of electrons, when an electric field is present have been carried out. However no detailed discussion of transport phenomena has been made except in the case of vanishingly small electric fields, when the Maxwellian distribution of velocities is valid. In a recent communication[1] (called Part I hereafter) Sodha[1] has discussed the transport phenomena in the case of low electric fields (when terms involving fourth and higher power of electric field are negligible) assuming the time of relaxation $\tau \propto x^n$, $x=(m/2kT)^{\frac{1}{2}}v$ being the dimensionless velocity.

In this paper the author has investigated the variation of transport properties with electric field when the time of relaxation τ is given by

$$\tau = l/x. \tag{1}$$

This corresponds to a constant mean free path and is valid in many cases of interest, e.g., helium[2] (for slow electrons with energies below 1.5 ev or 10^4°K). It may be added that l has the dimensions of time.

VELOCITY DISTRIBUTION FUNCTION

In Part I, the electron velocity distribution function in the presence of an electric field was expressed as

$$f_0(x) = A \exp\left(-\int \frac{2xdx}{(1+z\tau^2)}\right), \tag{2}$$

where the symbols have the usual[1] meanings.

Substituting for τ from Eq. (1) in Eq. (2) we obtain

$$f_0(x) = A \exp(-x^2)(x^2+a)^a, \tag{3}$$

where

$$a = zl^2 = (m_1/3m_2{}^2)(q^2E^2/kT)l^2. \tag{4}$$

Equations (1) and (3) are also valid[3] for nondegenerate semiconductors, if acoustic modes of lattice vibrations are considered to be the sole source of scattering and

$$a = (1/m_2{}^2)(q^2E^2/c_1{}^2)l^2, \tag{4A}$$

where m_2 is the effective electronic mass and c_1 the velocity of sound in the crystal.

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[1] M. S. Sodha, Phys. Rev. 116, 486 (1959).

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[3] J. Yamashita and N. Watanabe, Progr. Theoret. Phys. Kyoto 12, 443 (1954).

TABLE I. Transport properties under the influence of a magnetic field. [Subscripts 1, 2, and s denote the first ($\omega^2\tau^2 \ll 1$), second ($\omega^4\tau^4 \ll 1$), and saturation ($\omega^2\tau^2 \gg 1$) approximations.]

Auxillary condition	Property	Definition	Expression
I	Drift mobility	$\mu = I_x/nqE_x$	$\mu = (q/3m)\Phi(1,3)$ $\mu_1 = (q/3m)\Phi_1(1,3)$ $\mu_2 = (q/3m)[\Phi_1(1,3) - \omega^2\Phi_1(3,3)]$ $\mu_s = (q/3m\omega^2)\Phi_1(-1,3)$
I	Hall mobility	$\mu' = c_0 I_y/HI_x$	$\mu' = (q/m)\Phi(2,3)/\Phi(1,3)$ $\mu_1' = (q/m)\Phi_1(2,3)/\Phi_1(1,3)$ $\mu_s' = (q/m)\Phi_1(0,3)/\Phi_1(-1,3)$
I	Magnetoresistance coefficient	$B = (\rho - \rho_0)/\rho_0 H^2$ $= -(\mu - \mu_1)/\mu_1 H^2$	$B = -[\Phi(1,3) - \Phi_1(1,3)]/H^2\Phi_1(1,3)$ $B_2 = (q/mc_0)^2\Phi_1(3,3)/\Phi_1(1,3)$
I	Corbino coefficient	$A_C = I_y/HI_x$	$A_C = \mu'/c_0$
II	Thermal conductivity	$K = -C_x \Big/ \left(\dfrac{\partial T}{\partial x}\right)$	$K = \dfrac{nm}{12T}[\Phi(1,7) - \Phi^2(1,5)/\Phi(1,3)]$ $K_1 = \dfrac{nm}{12T}[\Phi_1(1,7) - \Phi_1^2(1,5)/\Phi_1(1,3)]$ $K_2 = K_1 - \dfrac{nm\omega^2}{12T}\left[\Phi_1(3,7) + \dfrac{\Phi_1^2(1,5)\Phi_1(3,3)}{\Phi_1^2(1,3)} - 2\dfrac{\Phi_1(3,5)\Phi_1(1,5)}{\Phi_1(1,3)}\right]$
II	Coefficient of thermal magnetoresistance	$B' = -(K - K_1)/K_1 H^2$	$B_2' = \left(\dfrac{q}{mc_0}\right)^2 [\Phi_1(1,7) - \Phi_1^2(1,5)/\Phi_1(1,3)]^{-1}$ $\times\left(\Phi_1(3,7) + \dfrac{\Phi_1^2(1,5)\Phi_1(3,3)}{\Phi_1^2(1,3)} - 2\dfrac{\Phi_1(3,5)\Phi_1(1,5)}{\Phi_1(1,3)}\right)$
II	Magnetothermo-electric effect	$S = E_x \Big/ \left(\dfrac{\partial T}{\partial x}\right)$ $\Sigma = (S - S_1)/S_1 H^2$	$S = -(m/2qT)\Phi_1(1,5)/\Phi_1(1,3)$ $S_1 = -m/2qT\Phi(1,5)/\Phi(1,3)$ $S_2 = S_1\left[1 - \omega^2\left(\dfrac{\Phi_1(3,5)}{\Phi_1(1,5)} - \dfrac{\Phi_1(3,3)}{\Phi_1(1,3)}\right)\right]$ $\Sigma_2 = (q/mc_0)^2\left(\dfrac{\Phi_1(3,3)}{\Phi_1(1,3)} - \dfrac{\Phi_1(3,5)}{\Phi_1(1,5)}\right)$
I	Coefficient of diffusion	$\partial n/\partial t = \nabla^2(\mathbf{D}n)$	$(m/2kT)D_x = \frac{1}{3}\left\langle \dfrac{\tau x^2}{1+\omega^2\tau^2}\right\rangle$ $(m/2kT)D_y = \frac{1}{3}\left\langle \dfrac{\tau^2 x^2}{1+\omega^2\tau^2}\right\rangle$ (Standard results) $(m/2kT)D_{x,1} = \frac{1}{3}\langle \tau x^2\rangle$ $(m/2kT)D_{y,1} = \frac{1}{3}\omega\langle \tau^2 x^2\rangle$

Yamashita and Watanabe[3] and Sodha and Eastman[4] have used Eq. (3) to investigate the variation of drift mobility and Hall mobility (at low magnetic fields) of electrons with the electric field. The present investigation is much wider in scope.

Eq. (3) is valid when $a \gg \omega^2 l^2$ or $(E/H) \gg (3kT/m_1c^2)^{\frac{1}{2}}$ which is true up to reasonably high magnetic fields.

TRANSPORT PROPERTIES

The electric current density $\mathbf{I}$ and the thermal current density $\mathbf{C}$ due to electrons when the magnetic

[4] M. S. Sodha and P. C. Eastman, Phys. Rev. **110**, 1314 (1958).

field **H** is in the z direction and perpendicular to the electric field **E** are given by[5]

$$I_x=\frac{nq}{3}\left(\frac{\Phi(1,5)}{2T}\frac{\partial T}{\partial x}+\Phi(1,3)\frac{qE_x}{m}-\frac{\omega}{2T}\Phi(2,5)\frac{\partial T}{\partial y}-\omega\Phi(2,3)\frac{qE_y}{m}\right), \tag{5A}$$

$$I_y=\frac{nq}{3}\left(\frac{\Phi(1,5)}{2T}\frac{\partial T}{\partial y}+\Phi(1,3)\frac{qE_y}{m}+\frac{\omega}{2T}\Phi(2,5)\frac{\partial T}{\partial x}+\omega\Phi(2,3)\frac{qE_x}{m}\right), \tag{5B}$$

$$C_x=-\frac{nm}{6}\left(\frac{\Phi(1,7)}{2T}\frac{\partial T}{\partial x}+\Phi(1,5)\frac{qE_x}{m}-\frac{\omega}{2T}\Phi(2,7)\frac{\partial T}{\partial y}-\omega\Phi(2,5)\frac{qE_y}{m}\right), \tag{5C}$$

and

$$C_y=-\frac{nm}{6}\left(\frac{\Phi(1,7)}{2T}\frac{\partial T}{\partial y}+\Phi(1,5)\frac{qE_y}{m}+\frac{\omega}{2T}\Phi(2,7)\frac{\partial T}{\partial x}+\omega\Phi(2,5)\frac{qE_x}{m}\right), \tag{5D}$$

where

$$\Phi(r,s)=-\frac{4\pi}{n}\int_0^\infty\frac{\tau^r v^s}{1+\omega^2\tau^2}\frac{\partial f_0}{\partial v}dv,$$

$$\omega=qH/mc_0$$

and the symbols have the usual meanings.

When $\tau^r v^s/(1+\omega^2\tau^2)$ is not independent of v we may write

$$\Phi(r,s)=\left\langle\frac{1}{v^2}\frac{d}{dv}\left(\frac{\tau^r v^s}{1+\omega^2\tau^2}\right)\right\rangle.$$

Equations (5) represent four relations between eight quantities I_x, I_y, C_x, C_y, E_x, E_y, $\partial T/\partial x$ and $\partial T/\partial y$. Hence a relation between two quantities is unique only when three auxillary conditions are also specified. For the sake of convenience we use the following two sets of auxillary conditions

I. $E_y=\partial T/\partial x=\partial T/\partial y=0$,

II. $E_y=I_x=\partial T/\partial y=0$.

In the second auxillary condition $\partial T/\partial x$ is assumed small so that Eq. (3) is valid.

A few transport properties and expressions obtained from Eqs. (5) and the auxillary conditions are listed in Table I.

Using Eq. (1), Table I, and Eq. (3), and remembering that $4\pi f_0x^2dx$ is the number of electrons having velocities between x and $x+dx$, we obtain

$$(m/2kT)D_{x,1}=l\alpha_1/3, \tag{6}$$

$$(m/2kT)D_{y,1}=\omega l^2/3, \tag{7}$$

$$\mu_1=2ql\alpha_2/3m, \tag{8}$$

$$\mu_s=4q\alpha_1/3m\omega^2 l, \tag{9}$$

$$\mu_1'=ql\alpha_3/2m, \tag{10}$$

$$\mu_s'=3ql\alpha_4/4m, \tag{11}$$

$$B_2=(1/2)(ql/mc_0)^2\alpha_5, \tag{12}$$

$$K_1=2(k^2nlT/3m)\alpha_6, \tag{13}$$

$$B_2'=2(ql/mc_0)^2\alpha_7, \tag{14}$$

$$S_1=-2k\alpha_8/q, \tag{15}$$

and

$$\textstyle\sum_2=(ql/mc_0)^2(\alpha_9/2), \tag{16}$$

where

$$\alpha_1=I(3)/I(2),$$

$$\alpha_2=I(1)/I(2),$$

$$\alpha_3=I(0)/I(1),$$

$$\alpha_4=\alpha_1^{-1},$$

$$\alpha_5=a^a/I(1),$$

$$\alpha_6=\frac{3I(5)}{I(2)}-\frac{4I^2(3)}{I(2)I(1)},$$

$$\alpha_7=\left[\frac{I^2(3)a^a}{I^2(1)I(2)}-\frac{I(3)}{I(2)}\right],$$

$$\alpha_8=I(3)/I(1),$$

$$\alpha_9=[a^a/I(1)]-[I(1)/I(3)],$$

$$I(r)=\int_0^\infty x^r(x^2+a)^a\exp(-x^2)dx,$$

and

$$I^2(r)=[I(r)]^2,$$

since

$$\Phi_1(3,3)=-\frac{4\pi}{n}\int_0^\infty\tau^3v^3\frac{\partial f_0}{\partial v}dv=\frac{l^3a^a}{I(2)}.$$

VERY HIGH ELECTRIC FIELDS

For very high electric fields $z\tau^2\gg1$. Hence using Eqs. (1) and (2) we obtain

$$f_0(x)=A\exp(-x^4/2a). \tag{17}$$

[5] Sommerfeld and Frank, Revs. Modern Phys. 3, 1 (1931).

TABLE II. Variation of α's with a.

a \ α	α_1	α_2	α_3	α_4	α_5	α_6	$-\alpha_7$	α_8	$-\alpha_9$
0.0	1.1253	1.1510	1.8602	0.8886	0.0652	2.3490	0.4485	0.9777	0.9576
0.1	1.1633	1.0959	1.7256	0.8596	0.0533	2.3616	0.4647	1.0614	0.8888
0.2	1.1938	1.0588	1.6487	0.8376	0.0480	2.4022	0.4701	1.1275	0.8389
0.3	1.2203	1.0299	1.5921	0.8194	0.0445	2.4496	0.4719	1.1848	0.7994
0.4	1.2439	1.0059	1.5469	0.8038	0.0418	2.4993	0.4719	1.2365	0.7668
0.5	1.2654	0.9855	1.5093	0.7902	0.0397	2.5497	0.4710	1.2841	0.7391
0.6	1.2852	0.9676	1.4769	0.7781	0.0379	2.6002	0.4694	1.3283	0.7149
0.7	1.3036	0.9516	1.4486	0.7671	0.0364	2.6502	0.4673	1.3699	0.6935
0.8	1.3208	0.9372	1.4233	0.7571	0.0351	2.6997	0.4650	1.4093	0.6744
0.9	1.337	0.9241	1.4005	0.7478	0.0340	2.7487	0.4625	1.4469	0.6571
1.0	1.3525	0.9121	1.3799	0.7393	0.0329	2.7969	0.4600	1.4828	0.6414
2.0	1.4759	0.8276	1.2382	0.6775	0.0264	3.2454	0.4333	1.7835	0.5343
3.0	1.5676	0.7752	1.1531	0.6379	0.0229	3.6457	0.4101	2.0222	0.4716
4.0	1.6422	0.7375	1.0930	0.6089	0.0206	4.0128	0.3905	2.2266	0.4285
5.0	1.7059	0.7083	1.0469	0.5862	0.0189	4.3555	0.3739	2.4084	0.3963
6.0	1.7619	0.6846	1.0096	0.5676	0.0176	4.6794	0.3595	2.5737	0.3710
7.0	1.8120	0.6646	0.9786	0.5518	0.0165	4.9880	0.3469	2.7263	0.3503
8.0	1.8577	0.6476	0.9521	0.5383	0.0156	5.2841	0.3358	2.8688	0.3329
9.0	1.8997	0.6326	0.9290	0.5264	0.0149	5.5696	0.3258	3.0030	0.3181
10.0	1.9387	0.6194	0.9086	0.5158	0.0142	5.8459	0.3169	3.1302	0.3052

Hence

$$\Phi_1(r,s) = -\frac{4\pi}{n}\int_0^\infty \tau^r v^s \frac{\partial f_0}{\partial v} dv$$
$$= 4l^r (2kT/m)^{(s-3)/2}(2a)^{(s-r-3)/4}\frac{\Pi((s-r-3)/4)}{\Pi(1/2)},$$

where

$$\Pi(t) = \int_0^\infty z^t e^{-z} dz = \Gamma(t+1)$$

is a tabulated function.[6]

[6] E. Jahnke and F. Emde, *Tables oς Functions* (Dover Publications, New York, 1945), p. 9.

NUMERICAL RESULTS

Table II illustrates the variation of α's with a, obtained numerically on the Univac 1105.

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Plasma Oscillations of a Large Number of Electron Beams*

JOHN M. DAWSON
Project Matterhorn, Princeton University, Princeton, New Jersey
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Longitudinal oscillations of a large number of electron beams are investigated. The normal modes for the beams are found. An orthogonality relation between the modes is obtained and is used to solve the initial value problem and the problem of forced oscillations. It is demonstrated that no signal propagates faster than the fastest beam. The problem of passing to the limit of a continuous velocity distribution is considered in detail. It is shown that in the limit the results of Landau, Van Kampen, and others are recovered. The problem of Landau damping is discussed from the point of view of the beams.

I. INTRODUCTION

IN this paper a theory for the longitudinal oscillation of a large number of electron beams is presented. The term beam is used here to denote a stream of electrons which is infinite in extent and which has a definite velocity (no random motion within a beam). This work is, of course, closely related to the many papers which have appeared on the subject of plasma oscillations.[1–6] A large portion of this paper is devoted

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