change in the energy spectrum. The exponent of the integral energy spectrum of S nuclei, obtained in this experiment, over the range of energies from 0.23 to 9 Bev/nucleon is 1.78 ± 0.24; this is, within limits of experimental error, consistent with the values 1.54 ±0.16[9] and 1.60 ± 0.15[10] obtained in other experiments for energy intervals above 1.5 Bev/nucleon. Similarly, in the experiments of McDonald,[5,11] for α particles of energy between 0.28 and 0.9 Bev/nucleon, the exponent in the integral energy spectrum is 1.5 (as calculated by us from his data) for the March 13, 1956 flight and 1.4 ± 0.2 for the July 7, 1955 flight.

We conclude, therefore, that there is evidence for a large Forbush type of decrease in the intensity of the heavy nuclei ($Z\geqslant6$) of the primary cosmic radiation; unfortunately, one can say very little about the energy dependence of the decrease.

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Relativistic Pion-Hyperon Dispersion Relations*†

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Relativistic, fixed momentum-transfer dispersion relations are derived (but not proved) for pion scattering from Σ and Λ particles and the processes $\pi+\Lambda \leftrightarrows \pi+\Sigma$. Separate equations for the S- and P-wave amplitudes are obtained under the assumptions that high-energy processes and baryon recoil may be neglected. The P-wave equations are identical to those derived from Chew-Low theory for these processes. A brief discussion is given of the behavior of the P-wave amplitudes under the assumption of global symmetry. It is pointed out that the production of $K-N$ pairs may play an important role in both the S- and P-wave equations.

I. INTRODUCTION

In recent years relativistic dispersion relations have become a useful tool in the theoretical investigation of the pion-nucleon interaction. Dispersion relations have also been applied to K meson-nucleon scattering,[1] and there is every reason to believe that the dispersion approach to the strong interactions involving strange particles will become more and more useful as the experimental data concerning these interactions becomes more and more abundant.

We consider here the possible usefulness of the dispersion approach to the following three types of interactions involving systems of strangeness minus one:

$$\pi+Y\rightarrow\pi+Y, \qquad (1a)$$

$$\bar{K}+N\rightarrow\pi+Y, \qquad (1b)$$

$$\bar{K}+N\rightarrow\bar{K}+N, \qquad (1c)$$

where the symbol Y denotes either a Σ or a Λ hyperon. There is much evidence that the $K-N-Y$ interaction is somewhat weaker than the pion-nucleon interaction. On the other hand, the binding of Λ particles in nuclei is most easily explained by the hypothesis that the $\pi-\Sigma-\Lambda$ interaction is comparable to the pion-nucleon interaction. Hence it is probable that the pion-hyperon interactions are somewhat stronger than the $K-Y-N$ interactions. Thus the relationships among the three processes, (1a) through (1c), are analogous to those among the following three processes: (a) pion-nucleon scattering, (b) photopion production from nucleons, and (c) photon-nucleon scattering. However, there are two important points of difference between the (K,π) processes [Eq. (1)] and the corresponding (γ,π) processes mentioned above (besides the obvious differences in mass, charge, and spin). First, the K interactions are not really weak as are the electromagnetic interactions. This nonweakness complicates the relations between the amplitudes for the three processes. For example, the $\pi-Y$ scattering amplitudes may be affected appreciably by the $K-Y-N$ interactions.[2] The second point of difference is that in the reactions of Eq. (1) the total rest mass of the particles

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† Part of this work was done while one of the authors (M.N.) was at the Lawrence Radiation Laboratory, Berkeley, California.
[1] See, for example, P. T. Matthews and Abdus Salam, Phys. Rev. **110**, 565 and 569 (1958).

[2] R. H. Dalitz and S. F. Tuan, Phys. Rev. Letters **2**, 425 (1959).

in the strong channel ($\pi-Y$ channel) is less than the total rest mass of the particles in the other channel. Because of this fact, the dispersion relations for processes (1b) and (1c) necessarily involve an important and bothersome unphysical region.

Despite these differences between the processes of Eq. (1) and the corresponding (γ,π) processes, we believe that much can be learned from the analogy. It is likely that the strong $\pi-Y$ interaction leads to one or more resonances in the $\pi-Y$ scattering amplitude similar to the low-energy pion-nucleon resonance, and if such resonances exist, they will certainly be felt in the inelastic processes

$$\bar{K}+N\rightarrow\pi+Y.$$

Because of the interconnections among the amplitudes for processes (1a) through (1c), it is worthwhile to derive the dispersion relations for pion-hyperon scattering, even though this process is not directly observable. In this paper we write down (without proof) these pion-hyperon dispersion relations for fixed, arbitrary values of the momentum transfer. In Sec. III we derive approximate equations for the S- and P-wave amplitudes by neglecting various recoil terms and high-energy processes in the dispersion relations. The P-wave equations are analogous to the Chew-Low equations for pion-nucleon scattering; these have previously been discussed by one of the authors,[3] and by Amati, Stanghellini, and Vitale.[4] A brief discussion is given of the effects of the $\bar{K}-N$ channel on the location of possible resonances in the $\pi-Y$ system.

II. THE DISPERSION RELATIONS

We assume that the strong interactions are charge independent, and make the usual isotopic spin assignments. It is further assumed that the Σ and Λ are spin $\frac{1}{2}$ particles of the same parity. For each different angular momentum and parity state there are five different pion-hyperon scattering amplitudes, corresponding to $\pi-\Sigma$ scattering in states of total isotopic spin 2, 1, and 0, $\pi-\Lambda$ scattering in the state $I=1$, and the reaction $\pi+\Lambda\rightarrow\pi+\Sigma$ in the state $I=1$. These amplitudes are denoted by the symbols M_2, $M_{1\Sigma}$, M_0, $M_{1\Lambda}$, and $M_{1\Sigma\Lambda}$, respectively. In order that the crossing relations be expressible in simple form, it is convenient to work with the following five combinations of the isotopic spin amplitudes,

$$\begin{aligned}
M^{(1)}&=(7/12)M_2+\tfrac{1}{4}M_{1\Sigma}+\tfrac{1}{6}M_0,\\
M^{(2)}&=(5/12)M_2-\tfrac{1}{4}M_{1\Sigma}-\tfrac{1}{6}M_0,\\
M^{(3)}&=\tfrac{1}{2}M_2+\tfrac{1}{2}M_{1\Sigma},\\
M^{(4)}&=M_{1\Lambda},\\
M^{(5)}&=M_{1\Sigma\Lambda}/\sqrt{2}.
\end{aligned}\tag{2}$$

The amplitudes $M^{(\lambda)}$ are simply related to amplitudes for scattering in states of particular pion and hyperon charge, i.e.,

$$\begin{aligned}
M^{(1)}&=\tfrac{1}{2}[M(\pi^+\Sigma^+;\pi^+\Sigma^+)+M(\pi^-\Sigma^+;\pi^-\Sigma^+)]\\
&=\tfrac{1}{2}[M(\pi_1\Sigma^+;\pi_1\Sigma^+)+M(\pi_2\Sigma^+;\pi_2\Sigma^+)],
\end{aligned}\tag{3a}$$

$$\begin{aligned}
M^{(2)}&=\tfrac{1}{2}[M(\pi^+\Sigma^+;\pi^+\Sigma^+)-M(\pi^-\Sigma^+;\pi^-\Sigma^+)]\\
&=\tfrac{1}{2}i[M(\pi_1\Sigma^+;\pi_2\Sigma^+)-M(\pi_2\Sigma^+;\pi_1\Sigma^+)],
\end{aligned}\tag{3b}$$

$$M^{(3)}=M(\pi^0\Sigma^+;\pi^0\Sigma^+),\tag{3c}$$

$$\begin{aligned}
M^{(4)}&=\tfrac{1}{2}[M(\pi^+\Lambda;\pi^+\Lambda)+M(\pi^-\Lambda;\pi^-\Lambda)]\\
&=\tfrac{1}{2}[M(\pi_1\Lambda;\pi_1\Lambda)+M(\pi_2\Lambda;\pi_2\Lambda)],
\end{aligned}\tag{3d}$$

$$\begin{aligned}
M^{(5)}&=\tfrac{1}{2}[M(\pi^+\Sigma^0;\pi^+\Lambda^0)-M(\pi^-\Sigma^0;\pi^-\Lambda^0)]\\
&=\tfrac{1}{2}i[M(\pi_1\Sigma^0;\pi_2\Lambda^0)-M(\pi_2\Sigma^0;\pi_1\Lambda^0)],
\end{aligned}\tag{3e}$$

where $M(\beta;\alpha)$ is the amplitude for the production of the state β from the state α. The states π_1, π_2, and π_3 are defined by the equations, $\pi^\pm=2^{-\frac{1}{2}}(\pi_1\pm i\pi_2)$ and $\pi^0=\pi_3$.

The method of derivation of the dispersion relations to be used here is essentially the same as that used for other meson reactions by Capps and Takeda[5] and by Jin[6]; hence only a brief sketch of the derivation is given. As in references 5 and 6 no attempt is made to prove the relations. The derivation makes use of Heisenberg picture matrix elements of the pion current operators, J_1, J_2, and J_3; these operators are defined by the equations

$$(\Box^2-\mu^2)\phi_\alpha(x)=-J_\alpha(x),\tag{4}$$

where ϕ_α is the pion field operator for pions of charge state α in the Heisenberg picture. The form of $J_\alpha(x)$ depends on the nature of the interaction; we assume the pion-hyperon interaction Hamiltonian density H to be of the form,

$$\begin{aligned}
H(x)=G_\Lambda[\bar{\boldsymbol{\Sigma}}(x)i\gamma_5\Lambda(x)]\cdot\boldsymbol{\phi}(x)+\text{H.c.}\\
-iG_\Sigma[\bar{\boldsymbol{\Sigma}}(x)\times i\gamma_5\boldsymbol{\Sigma}(x)]\cdot\boldsymbol{\phi}(x),
\end{aligned}\tag{5}$$

where γ_5 is anti-Hermitian, $\boldsymbol{\Sigma}$ and $\Lambda(x)$ are the field operators for the Σ and Λ particles, and the dot and cross refer to scalar and vector products in charge space of the isotopic spin vectors $\boldsymbol{\Sigma}$, $\bar{\boldsymbol{\Sigma}}(x)$, and $\boldsymbol{\phi}$. The constants $\hbar$ and c are taken equal to one. (The global symmetry model corresponds to the choice[7] $G_\Sigma^2=G_\Lambda^2\cong 14$ in this equation.) The current $J_\alpha(x)$ may be computed by the method of Low,[8] and is given by

$$\begin{aligned}
J_1(x)=-G_\Lambda(\bar{\Sigma}_1i\gamma_5\Lambda)+\text{H.c.}+iG_\Sigma(\bar{\Sigma}_2i\gamma_5\Sigma_3\\
-\bar{\Sigma}_3i\gamma_5\Sigma_2)+\delta\mu^2\phi_1-\lambda\phi^2\phi_1,
\end{aligned}\tag{6}$$

with similar equations for $J_2(x)$ and $J_3(x)$. The last two terms result from renormalizations associated with the pion mass and the pion-pion interaction, respectively.

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The four momenta of the initial pion, initial hyperon, final pion, and final hyperon are denoted by the symbols, $k_i=(\mathbf{k}_i,\omega_i)$, $p_i=(\mathbf{p}_i,E_i)$, $k_f=(\mathbf{k}_f,\omega_f)$ and $p_f=(\mathbf{p}_f,E_f)$, respectively. We write the dispersion relations in the Breit-Lorentz system, defined as the system in which the three momenta of the initial and final hyperon are equal and opposite. We denote $\frac{1}{2}$ the three momentum-transfer by $\mathbf{q}$, and the average meson energy and three-momentum by ω and $\mathbf{k}$, i.e.,

$$\mathbf{q}=\mathbf{p}_f=-\mathbf{p}_i=\tfrac{1}{2}(\mathbf{k}_i-\mathbf{k}_f),$$
$$\omega=\tfrac{1}{2}(\omega_i+\omega_f),\quad \mathbf{k}=\tfrac{1}{2}(\mathbf{k}_i+\mathbf{k}_f).$$

We define the quantity $\mathcal{E}$ by the equation

$$\mathcal{E}=\tfrac{1}{2}(E_f-E_i)=\tfrac{1}{2}[(m_f^2+q^2)^{\frac{1}{2}}-(m_i^2+q^2)^{\frac{1}{2}}],$$

where m_i and m_f are the masses of the initial and final hyperons. This energy difference vanishes, of course, for the elastic processes. The average pion momentum $\mathbf{k}$ is related to ω and $\mathbf{q}$ by the equations[9]

$$\mathbf{k}=\mathbf{q}\omega\mathcal{E}q^{-2}+\boldsymbol{\nu}k_\perp, \tag{7a}$$

$$k_\perp=(\omega^2-\mu^2-q^2+\mathcal{E}^2-\omega^2\mathcal{E}^2q^{-2})^{\frac{1}{2}}, \tag{7b}$$

where $\boldsymbol{\nu}$ is a unit vector perpendicular to $\mathbf{q}$. The perpendicular momentum $k_\perp$ is defined for nonphysical energies by analytic continuation in the upper half ω plane. For real values of ω in the unphysical region $\omega^2<(\mu^2+q^2-\mathcal{E}^2)/(1-\mathcal{E}^2q^{-2})$, $k_\perp$ is imaginary, but $k_\perp$ satisfies the relation $k_\perp^*(-\omega)=-k_\perp(\omega)$ for all real ω.

It may be shown by well-known methods that the amplitude for the reaction $\pi_\alpha+Y_a\rightarrow\pi_\beta+Y_b$, as a function of $\mathbf{q}$, ω and $\boldsymbol{\nu}$, the initial spin state ζ_a, and the final spin state ζ_b, may be expressed in terms of the commutators of the meson current operators, i.e.,

$$\begin{aligned}M(Y_b\pi_\beta;\,&Y_a\pi_\alpha:\omega,\mathbf{q},\boldsymbol{\nu},\zeta_b,\zeta_a)\\ &=i2\pi^2(E_iE_f/m_im_f)^{\frac{1}{2}}\int d^4x\,e^{-ikx}\\ &\quad\times\langle\psi_b(\mathbf{q},\zeta_b)|\eta(t)[J_\beta^\dagger(\tfrac{1}{2}x),J_\alpha(-\tfrac{1}{2}x)]\\ &\qquad-\delta(t)[J_\beta^\dagger(\tfrac{1}{2}x),\phi_\alpha(-\tfrac{1}{2}x)]|\psi_a(-\mathbf{q},\zeta_a)\rangle,\end{aligned} \tag{8}$$

where kx is the four-dimensional scalar product $\mathbf{k}\cdot\mathbf{x}-\omega t$, $\eta(t)$ is unity for $t>0$ and zero for $t<0$, and ψ_a and ψ_b represent the Heisenberg state vectors of the initial and final baryon. The normalization is that of reference 5. The term involving ϕ is energy independent and will not enter into the dispersion relations if the proper subtractions are made; hence we will consistently disregard this term.[10] Equation (8) defines the "causal" amplitude for real values of the momentum-transfer and arbitrary values of the energy ω in the upper half complex plane. The dispersion relations result from the assumption that this amplitude, divided by a suitable polynomial in the energy, is analytic and bounded in the upper half energy plane.

In the usual manner we define the dispersive and absorptive parts of M (D and A) by writing the step function $\eta(t)$ as a sum of an odd and an even function, i.e.,

$$\begin{aligned}D=i\pi^2(E_iE_f/m_im_f)^{\frac{1}{2}}&\int d^4x\,e^{-ikx}[2\eta(t)-1]\\ &\times\langle\psi_b(\mathbf{q},\zeta_b)|[J_\beta^\dagger(\tfrac{1}{2}x),J_\alpha(-\tfrac{1}{2}x)]|\psi_a(-q,\zeta_a)\rangle,\end{aligned} \tag{9a}$$

$$\begin{aligned}iA=i\pi^2(E_iE_f/m_im_f)^{\frac{1}{2}}&\int d^4x\,e^{-ikx}\\ &\times\langle\psi_b(\mathbf{q},\zeta_b)|[J_\beta^\dagger(\tfrac{1}{2}x),J_\alpha(-\tfrac{1}{2}x)]|\psi_a(-\mathbf{q},\zeta_a)\rangle.\end{aligned} \tag{9b}$$

The casual amplitude may be written as a sum of spin independent and spin dependent amplitudes in the Breit system, i.e.,

$$M=M_N(\omega,q^2)\mathbf{1}+2iM_S(\omega,q^2)\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{k}_\perp, \tag{10}$$

where it is implied that the matrix elements in spin space of the unit operator $\mathbf{1}$ and Pauli spin operator $\boldsymbol{\sigma}$ are to be taken. The form of Eq. (10) follows from the invariance of the amplitude to three-dimensional rotations and reflections [note from Eqs. (7) that the scalars k^2 and $\mathbf{k}\cdot\mathbf{q}$ are functions of the energy and of q^2].

A. Dispersion Relations for the Elastic Scattering Amplitudes

In this section we treat the elastic scattering process involved in the amplitudes $M^{(1)}$ through $M^{(4)}$, [Eqs. (3a) through (3d)]; the more complicated $\pi+\Lambda\rightarrow\pi+\Sigma$ process $M^{(5)}$ is treated in Sec. IIB. The subscripts α and β in Eq. (8) are chosen to refer to pions in the states 1, 2, or 3; hence the current operators J_α and J_β are Hermitian. We derive the crossing relation by making use of the Hermitian property of the operator $i[J_\alpha(\frac{1}{2}x),J_\beta(-\frac{1}{2}x)]$. For any amplitude in which the initial and final hyperons are the same, it may be shown from Eq. (8) that the crossing relation is

$$\begin{aligned}M^*(\pi_\beta;\pi_\alpha:\omega,\mathbf{q},\boldsymbol{\nu},\zeta_a,\zeta_b)\\ =M(\pi_\beta;\pi_\alpha:-\omega,-\mathbf{q},\boldsymbol{\nu},\zeta_b,\zeta_a).\end{aligned} \tag{11}$$

This equation implies that the amplitudes $M^{(\lambda)}$ defined in Eqs. (3a) through (3d) satisfy the simple crossing relations,

$$M^{(\lambda)*}(\omega,\mathbf{q},\boldsymbol{\nu},\zeta_a,\zeta_b)=\epsilon_\lambda M^{(\lambda)}(-\omega,-\mathbf{q},\boldsymbol{\nu},\zeta_b,\zeta_a), \tag{12}$$

where

$$\epsilon_\lambda=1 \text{ for } \lambda=1,\,3,\text{ or } 4,$$
$$\epsilon_\lambda=-1 \text{ for } \lambda=2.$$

From Eq. (12) and the relation $k(\omega)=-k^*(-\omega)$ it is seen that the spin-independent and spin-dependent

[9] These equations are equivalent to Eq. (2.5) of reference 6.

[10] A discussion of this term is given for the analogous case of pion nuclear scattering by M. L. Goldberger, Phys. Rev. 99, 979 (1955).

amplitudes defined in Eq. (10) satisfy the relations

$$M_N^{(\lambda)*}(\omega,q^2)=\epsilon_\lambda M_N^{(\lambda)}(-\omega,q^2),$$
$$M_S^{(\lambda)*}(\omega,q^2)=-\epsilon_\lambda M_S^{(\lambda)}(-\omega,q^2). \qquad (13)$$

It may be shown that the dispersive parts (D) of the spin-independent and spin-dependent amplitudes are real, and the absorptive parts (iA) are imaginary; the proof of this is identical to that in reference 5 and will not be given here.

The form of the dispersion relations depends upon the behavior of the amplitude M_N or M_S as the energy approaches infinity along the real axis. If the integral $\int_\alpha^\infty |M|^2 d\omega$ exists, where α is any real energy, one may derive relations of the "unsubtracted" type. For each amplitude that satisfies the symmetry condition $M^*(\omega)=M(-\omega)$ one integrates the function $\omega' M(\omega')/(\omega'^2-\omega^2)$ around a contour including the real ω' axis and a semicircle of infinite radius in the upper-half ω' plane; for each amplitude that satisfies the condition $M^*(\omega)=-M(-\omega)$, one integrates the function $\omega M(\omega')/(\omega'^2-\omega^2)$ around the same contour. The results of such a procedure are

$$D_N^{(\lambda)}(\omega,q^2)=\frac{1}{\pi}P\int_0^\infty\left(\frac{1}{\omega'-\omega}+\frac{\epsilon_\lambda}{\omega'+\omega}\right)\times A_N^{(\lambda)}(\omega',q^2)d\omega', \qquad (14a)$$

$$D_S^{(\lambda)}(\omega,q^2)=\frac{1}{\pi}P\int_0^\infty\left(\frac{1}{\omega'-\omega}-\frac{\epsilon_\lambda}{\omega'+\omega}\right)\times A_S^{(\lambda)}(\omega',q^2)d\omega', \qquad (14b)$$

where the symbol P denotes the principal part of the integral. We assume that the spin-dependent amplitudes satisfy these unsubtracted relations. However, it is unlikely that all the spin-independent amplitudes are convergent enough to satisfy equations of this type, so that some subtraction procedure must be performed in order to derive correct dispersion relations. One possible subtraction procedure is that used in reference 5, and results in a dispersion relation for $D_N(\omega_1,q^2)-D_N(\omega_2,q^2)$. An alternate procedure is to derive equations for $D_N(\omega,q_1^2)-D_N(\omega,q_2^2)$; if this procedure is used one may simply subtract two equations of the type, Eq. (14a), to obtain the result. For simplicity we will write only the unsubtracted dispersion relations in this section; in the discussion of applications (Sec. III), the modifications resulting from appropriate subtraction procedures are described.

The contributions to the dispersion relations from the unphysical region $\omega<(\mu^2+q^2)^{\frac{1}{2}}$ are more complicated than in the pion-nucleon scattering case. The unphysical region may be investigated by expanding the products of current operators in Eq. (9b) in terms of intermediate states. There are two types of terms from the unphysical region. First, there is the contribution to the dispersion integral from intermediate continuum states; the lower limit ω_a of this contribution is determined by the intermediate states of type $\Lambda+\pi$, and is given by the formula

$$\omega_a=(m_\Lambda^2-m_i^2+2\mu m_\Lambda-2q^2)/(2E_i), \qquad (15)$$

where m_i is the mass of the initial (or final) hyperon. The value of the absorptive amplitude in this unphysical continuum region must be determined by analytic continuation in the upper half energy plane from the physical region. The unphysical continuum vanishes only for $\pi-\Lambda$ elastic scattering when $q^2=0$.

The second type contribution associated with the unphysical region are the poles corresponding to the single particle intermediate states Σ and Λ. The energy ω_b of a pole in the $\pi-Y_i$ scattering amplitude corresponding to the intermediate hyperon Y_n is given by the formula

$$\omega_b=|\Omega_{\rm in}|,$$
$$\Omega_{\rm in}=(m_n^2-m_i^2-\mu^2-2q^2)/(2E_i). \qquad (16)$$

The residues of the pole terms may be determined by the method used in references 1, 5, and 6. These residues depend on the Heisenberg picture matrix elements of the pion current operator given in Eq. (6). The symmetry properties of the meson current and of the real baryon states under Lorentz transformations and rotations in charge space may be used to show that the residues may be calculated by using Born approximation and replacing the unrenormalized coupling constants by renormalized ones.

If the Born approximation terms (real hyperon terms) are evaluated in Eq. (14), and the absorption integrals are cut off at the lower limit of the continuum, the dispersion relations for the eight elastic amplitudes may be written in the form

$$D_N^{(\lambda)}(\omega,q^2)=B_N^{(\lambda)}(\omega,q^2)+\frac{1}{\pi}P\int_{\omega_a}^\infty\left(\frac{1}{\omega'-\omega}+\frac{\epsilon_\lambda}{\omega'+\omega}\right)\times A_N^{(\lambda)}(\omega',q^2)d\omega', \qquad (17a)$$

$$D_S^{(\lambda)}(\omega,q^2)=B_S^{(\lambda)}(\omega,q^2)+\frac{1}{\pi}P\int_{\omega_a}^\infty\left(\frac{1}{\omega'-\omega}-\frac{\epsilon_\lambda}{\omega'+\omega}\right)\times A_S^{(\lambda)}(\omega',q^2)d\omega'. \qquad (17b)$$

The Born approximation terms $B_N^{(\lambda)}$ and $B_S^{(\lambda)}$ are given by the expressions,

$$B_N^{(1)}=\tfrac{1}{2}G_\Sigma^2\left(\frac{\Omega_{\Sigma\Sigma}}{\omega^2-\Omega_{\Sigma\Sigma}^2}\right)Q_{\Sigma\Sigma}+\tfrac{1}{2}G_\Lambda^2\left(\frac{\Omega_{\Sigma\Lambda}}{\omega^2-\Omega_{\Sigma\Lambda}^2}\right)Q_{\Sigma\Lambda},$$

$$B_S^{(1)}=\tfrac{1}{2}G_\Sigma^2\left(\frac{\omega}{\omega^2-\Omega_{\Sigma\Sigma}^2}\right)R_{\Sigma\Sigma}+\tfrac{1}{2}G_\Lambda^2\left(\frac{\omega}{\omega^2-\Omega_{\Sigma\Lambda}^2}\right)R_{\Sigma\Lambda},$$

$$B_N^{(2)}=-\tfrac{1}{2}G_\Sigma^2\left(\frac{\omega}{\omega^2-\Omega_{\Sigma\Sigma}^2}\right)Q_{\Sigma\Sigma}-\tfrac{1}{2}G_\Lambda^2\left(\frac{\omega}{\omega^2-\Omega_{\Sigma\Lambda}^2}\right)Q_{\Sigma\Lambda},$$

$$B_S^{(2)}=-\tfrac{1}{2}G_\Sigma^2\left(\frac{\Omega_{\Sigma\Sigma}}{\omega^2-\Omega_{\Sigma\Sigma}^2}\right)R_{\Sigma\Sigma}-\tfrac{1}{2}G_\Lambda^2\left(\frac{\Omega_{\Sigma\Lambda}}{\omega^2-\Omega_{\Sigma\Lambda}^2}\right)R_{\Sigma\Lambda}, \tag{18}$$

$$B_N^{(3)}=G_\Sigma^2\left(\frac{\Omega_{\Sigma\Sigma}}{\omega^2-\Omega_{\Sigma\Sigma}^2}\right)Q_{\Sigma\Sigma},$$

$$B_S^{(3)}=G_\Sigma^2\left(\frac{\omega}{\omega^2-\Omega_{\Sigma\Sigma}^2}\right)R_{\Sigma\Sigma},$$

$$B_\Lambda^{(4)}=G_\Lambda^2\left(\frac{\Omega_{\Lambda\Sigma}}{\omega^2-\Omega_{\Lambda\Sigma}^2}\right)Q_{\Lambda\Sigma},$$

$$B_S^{(4)}=G_\Lambda^2\left(\frac{\omega}{\omega^2-\Omega_{\Lambda\Sigma}^2}\right)R_{\Lambda\Sigma}.$$

where the Ω_{in} are given by Eq. (16). The quantities Q and R are the following functions of momentum transfer and the particle masses,

$$Q_{in}=\frac{E_i+m_i}{2m_i^2}\left[m_iE_i-\Gamma_{in}+\frac{q^2}{(E_i+m_i)^2}(m_iE_i+\Gamma_{in})\right],$$
$$R_{in}=1/2m_i^2, \tag{19}$$
$$\Gamma_{in}=\tfrac{1}{2}(m_i^2+m_n^2-\mu^2).$$

If only terms of lowest order in ω/m_i, q/m_i, and $(m_n-m_i)/m_i$ are considered, the expressions for Q and R are simplified to the following forms,

$$Q_{in}=\tfrac{1}{2}[-(m_i-m_n)^2+\mu^2+2q^2]/m_i^2,$$
$$R_{in}=1/(2m_i^2). \tag{20}$$

We shall make no attempt to prove these relations, or to determine for what range of momentum transfer the relations are valid. Only small values of q^2 are considered in deriving the S- and P-wave equations of Sec. III.

B. Dispersion Relations for the Inelastic Amplitude $M^{(5)}$

The derivation of the crossing relation is more complicated in the case of the inelastic amplitude $M^{(5)}$. If one follows the procedure used to derive the crossing relation, Eq. (11) of Sec. A, the result for the inelastic amplitude $M(\pi_1\Sigma^0;\pi_2\Lambda^0)$ is

$$M^*(\pi_1\Sigma^0;\pi_2\Lambda:\omega,\mathbf{q},\mathbf{v},\zeta_\Sigma,\zeta_\Lambda)=M(\pi_1\Lambda;\pi_2\Sigma^0:-\omega,-\mathbf{q},\mathbf{v},\zeta_\Lambda,\zeta_\Sigma). \tag{21}$$

Since the order of the hyperons in this equation is reversed, it is convenient to define the amplitude $M^{(5,r)}$ which is the reversed amplitude to that of Eq. (3e), i.e.,

$$M^{(5,r)}=\tfrac{1}{2}[M(\pi^+\Lambda;\pi^+\Sigma^0)-M(\pi^-\Lambda;\pi^-\Sigma^0)]$$
$$=\tfrac{1}{2}i[M(\pi_1\Lambda;\pi_2\Sigma^0)-M(\pi_2\Lambda;\pi_1\Sigma^0)].$$

The spin-independent and spin-dependent amplitudes for $M^{(5)}$ and $M^{(5,r)}$ are defined by Eq. (10). The crossing relation of Eq. (21), together with the relation $k_\perp^*(\omega)=-k_\perp(-\omega)$ implies that $M_N^{(5)}$ and $M_S^{(5)}$ satisfy the equations

$$M_N^{(5)*}(\omega,q^2)=-M_N^{(5,r)}(-\omega,q^2), \tag{22a}$$

$$M_S^{(5)*}(\omega,q^2)=M_S^{(5,r)}(-\omega,q^2). \tag{22b}$$

Because $M^{(5)}$ and $M^{(5,r)}$ refer to different processes, a further symmetry property of the amplitude is needed. We will derive the needed symmetry property by investigating the behavior of the causal amplitude under Wigner time reversal, working in a representation in which the pions are in states of definite charge ($+$ or $-$) rather than in the linear combinations of these states denoted by the indices 1 and 2. The behavior of the pion field operators under time reversal is given by the equation,

$$T\phi_+(t)T^{-1}=\eta\phi_+(-t);\quad T\phi_-(t)T^{-1}=\eta^*\phi_-(-t),$$

where η is a complex number of unit magnitude, and T is the Wigner time reversal operator that does *not* take a particle into its antiparticle or reflect the space axes.[11] Since the operators $\Box^2$ and μ^2 are invariant under time reversal, the time-reversal properties of the current operators $J_\pm(x)$ are the same as those of $\phi_\pm(x)$. These relations, together with the property $J_+(x)=[J_-(x)]^\dagger$, imply that the matrix elements of $J_\pm$ satisfy the relation

$$\langle a|J_+(t)|b\rangle=\eta^*\langle b^r|J_-(-t)|a^r\rangle,$$

where a^r and b^r are the time-reversed states to a and b. If this equation is applied to the current operators in Eq. (8), it may be shown that the causal amplitude satisfies the following time-reversal property,

$$M(\pi^\pm\Sigma^0;\pi^\pm\Lambda:\omega,\mathbf{q},\mathbf{v},\zeta_\Sigma,\zeta_\Lambda)=M(\pi^\pm\Lambda;\pi^\pm\Sigma^0:\omega,\mathbf{q},-\mathbf{v},\zeta_\Lambda{}^r,\zeta_\Sigma{}^r), \tag{23}$$

where the upper signs go together, and the lower signs go together. Equation (23) implies that $M_N^{(5)}$ and $M_S^{(5)}$ are identical to the corresponding reversed amplitudes, i.e.,

$$M_N^{(5)}=M_N^{(5,r)},\quad M_S^{(5)}=M_S^{(5,r)}. \tag{24}$$

Finally we combine Eqs. (22) and (24) to deduce the following symmetry of the inelastic amplitudes:

$$M_N^{(5)*}(\omega,q^2)=-M_N^{(5)}(-\omega,q^2),$$
$$M_S^{(5)*}(\omega,q^2)=M_S^{(5)}(-\omega,q^2). \tag{25}$$

The procedure of reference 5 may be used to show that $D_N^{(5)}$ and $D_S^{(5)}$ are real, and $iA_N^{(5)}$ and $iA_S^{(5)}$ are imaginary.

Since the amplitude $M_N^{(5)}$ satisfies the odd symmetry condition, the unsubtracted dispersion relations for this amplitude are those of Eqs. (14) with $\epsilon_\lambda=-1$. The lower limit of the unphysical continuum is again

[11] These equations are equivalent to Eq. (1) of T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. **106**, 340 (1957).

determined by intermediate states of the type $\Lambda+\pi$, and corresponds to the energy value

$$\omega_a'=(m_\Lambda\mu-E_\Lambda\mathcal{E}-q^2)/E. \qquad (26)$$

We use here the symbols E_Λ and E_Σ to represent the hyperon energies, rather than E_i and E_f, so that the formulas are valid for the amplitude $M^{(5,r)}$ as well as for $M^{(5)}$. The symbol E denotes the average of E_Λ and E_Σ.

Only the real Σ state contributes to the Born approximation term. The energy at which the Born approximation pole occurs is given by

$$\omega_b'=|\Omega'|,$$
$$\Omega'=(E_\Sigma\mathcal{E}-\tfrac{1}{2}\mu^2-q^2)/E. \qquad (27)$$

The calculation of the Born term is done in the same manner as that discussed in Sec. II A. The resulting dispersion relations for the $\pi\Lambda \leftrightarrows \pi\Sigma$ process are

$$D_N^{(5)}(\omega,q^2)=-G_\Lambda G_\Sigma[\omega/(\omega^2-\Omega'^2)]Q' +\frac{2\omega}{\pi}P\int_{\omega_a'}^{\infty}\frac{A_N^{(5)}(\omega',q^2)d\omega'}{(\omega'^2-\omega^2)}, \qquad (28a)$$

$$D_S^{(5)}(\omega,q^2)=-G_\Lambda G_\Sigma[\Omega'/(\omega^2-\Omega'^2)]R' +\frac{2}{\pi}P\int_{\omega_a'}^{\infty}\frac{\omega' A_S^{(5)}(\omega',q^2)d\omega'}{(\omega'^2-\omega^2)}. \qquad (28b)$$

If only terms of lowest order in ω/m, q/m, and $(m_\Sigma-m_\Lambda)/m$ are considered [where $m=\frac{1}{2}(m_\Sigma+m_\Lambda)$], the expressions for Q' and R' are quite simple, i.e.,

$$Q'=\tfrac{1}{2}[\mu^2-\tfrac{1}{2}(m_\Sigma-m_\Lambda)^2+2q^2]/m^2,$$
$$R'=1/(2m^2). \qquad (29)$$

Only this no-recoil limit is used in the applications. For completeness we write down the general expressions for Q' and R', however:

$$Q'=\frac{1}{2Em_\Sigma m_\Lambda}\left(\frac{-2Z[(l-q^2)(l-q^2+2E_\Sigma E)+q^2E^2]}{m_\Sigma^2+l+m_\Sigma E}+\frac{q^2(m_\Sigma^2+l+m_\Sigma E)}{2Z}\right), \qquad (30)$$

$$R'=\tfrac{1}{4}(m+E)/m_\Sigma m_\Lambda Z,$$

where l and Z are defined by the equations,

$$l=\mathcal{E}^2-\tfrac{1}{2}\mu^2,$$
$$2Z=(E_\Sigma+m_\Sigma)^{\frac{1}{2}}(E_\Lambda+m_\Lambda)^{\frac{1}{2}}.$$

III. APPROXIMATE EQUATIONS FOR S AND P WAVES

A. The Static Approximation

The relations between the Breit system and center-of-mass system values of the particle momenta may be obtained easily by evaluating the two scalars, $(p_i-p_f)^2$ and $-(k_i+k_f)(p_i+p_f)$ in the two systems. The resulting relations are

$$q^2-\mathcal{E}^2=\tfrac{1}{2}(-\mu^2+\omega_{i,c}\omega_{f,c}-\mathbf{k}_{i,c}\cdot\mathbf{k}_{f,c}), \qquad (31)$$

$$4\omega E=2W_c^2-m_i^2-m_f^2-2\mu^2-4(q^2-\mathcal{E}^2), \qquad (32)$$

where the index c denotes a center-of-mass system variable, and W_c is the total energy in the center-of-mass system. The separation of the center-of-mass amplitude into spin-independent and spin-dependent parts is of the form

$$M_c=M_{N,c}(\omega_c,\cos\theta_c)+i\boldsymbol{\sigma}\cdot\mathbf{k}_{i,c}\times\mathbf{k}_{f,c}M_{S,c}(\omega,\cos\theta_c). \qquad (33)$$

In general the relation between $M_{N,c}$ and $M_{S,c}$ and the corresponding Breit system amplitudes, [Eq. (10)], is quite complicated.[5] For similicity, however, we will make the no-recoil approximation by assuming that $m=\frac{1}{2}(m_\Lambda+m_\Sigma)$ is large compared with ω, q and $(m_\Sigma-m_\Lambda)$. The relationship between q^2 and θ_c, Eq. (31), is not changed in this approximation, but the other relations between the Breit system and center-of-mass system quantities are given by the following simple equations:

$$\omega=\omega_c,\quad \omega_{i,c}=\omega_c+\mathcal{E},\quad \omega_{f,c}=\omega_c-\mathcal{E}, \qquad (34a)$$

$$\mathbf{q}\times\mathbf{k}=\tfrac{1}{2}(\mathbf{k}_{i,c}\times\mathbf{k}_{f,c}), \qquad (34b)$$

$$M_N^{(\lambda)}=M_{N,c}^{(\lambda)}, \qquad (34c)$$

$$M_S^{(\lambda)}=M_{S,c}^{(\lambda)}. \qquad (34d)$$

Furthermore, the quantity $\mathcal{E}$ is now given simply by $\mathcal{E}=\frac{1}{2}(m_f-m_i)$. The center-of-mass differential cross section for unpolarized initial hyperons is related to the scattering amplitude by the equation

$$d\sigma=\frac{|\mathbf{k}_{f,c}|}{|\mathbf{k}_{i,c}|}(|M_{N,c}|^2+\sin^2\theta|M_{S,c}|^2).$$

In the no-recoil approximation, as is seen from Eqs. (20) and (29), the Born approximation term in the spin independent amplitude is linear in q^2 (or in $\cos\theta_c$), while the spin-dependent Born term is independent of q^2. Thus the Born term contributes only to the S and P waves. We make the further approximation of considering only S and P waves in all terms of the dispersion relations. This approximation, together with the no-recoil approximation, will be termed the static approximation. Inelastic processes, such as the process $\pi+Y\rightarrow\bar{K}+N$, are *not* neglected in this approximation.

B. Static P-Wave Equations

In this section all quantities will refer to the center-of-mass system, so the index c will be dropped. If only S and P waves are present, Eq. (33) may be written

$$M=T_0(\omega)+\mathbf{k}_i\cdot\mathbf{k}_fT_{1N}(\omega)+i\boldsymbol{\sigma}\cdot\mathbf{k}_i\times\mathbf{k}_fT_{1S}(\omega), \qquad (35)$$

where T_0, T_{1N}, and T_{1S} are the amplitudes for S wave,

spin-independent P wave, and spin-dependent P wave. If we denote by T_j the P-wave amplitudes corresponding to states of total angular momentum j, the T_j are given by

$$T_{\frac{3}{2}}=\tfrac{1}{3}(T_{1N}+T_{1S})\,;\; T_{\frac{1}{2}}=\tfrac{1}{3}(T_{1N}-2T_{1S}). \tag{36}$$

In order to exhibit the normalization of the amplitudes, we define the matrix $R=(S-1)/2i$, where S is the unitary scattering matrix [i.e., $R_{\alpha\alpha}=\exp(i\delta_\alpha)\sin\delta_\alpha$]. The relation between the amplitudes T_j and the corresponding matrix elements R_j of R are

$$T_j=R_j|\mathbf{k}_i|^{-\frac{3}{2}}|\mathbf{k}_f|^{-\frac{3}{2}}. \tag{37}$$

It is seen from comparing Eqs. (33), (34d), and (35) that

$$T_{1S}(\omega)=M_S(\omega),$$

so that in the no-recoil approximation, the spin-dependent equations are the equations for T_{1S}. Furthermore, as seen from Eqs. (31), (33), (34c) and (35), the spin-independent P amplitude is given by,

$$T_{1N}(\omega)=-\tfrac{1}{2}(\partial/\partial q^2)M_N(\omega,q^2).$$

Since a derivative with respect to q^2 is taken, one effectively uses a dispersion relation for the difference $M_N(\omega,q_1^2)-M_N(\omega,q_2^2)$. This subtraction improves the convergence at high energy, and we assume no further subtraction is necessary. Hence we may simply differentiate the equations [Eqs. (17a) and (28a)] for the spin-independent amplitudes with respect to q^2 and make the static approximation in order to derive the static equations for T_{1N}.

The Born approximation terms and the unphysical continuum contributions are simplified in the no-recoil limit. The limits of integration in Eqs. (17) and (28) become $\omega_a=\mu$ (for $\pi-\Lambda$ scattering), $\omega_a=\mu-\Delta$ (for $\pi-\Sigma$ scattering), and $\omega_a'=\mu-\frac{1}{2}\Delta$, where Δ is the mass difference $\Delta=m_\Sigma-m_\Lambda$. The quantities Ω of Eqs. (16) and (27) are $\Omega_{\Sigma\Sigma}=\Omega_{\Lambda\Lambda}=0$; $\Omega_{\Sigma\Lambda}=-\Delta$, $\Omega_{\Lambda\Sigma}=\Delta$, and $\Omega'=\frac{1}{2}\Delta$. The no-recoil forms of the quantities Q and R are given in Eqs. (20) and (29).

The equations that result from the procedure described above are the static P-wave equations for $\pi-Y$ scattering. These equations may also be derived from Chew-Low theory.[3,4] We shall write the equations in terms of the amplitudes corresponding to fixed values of the total isotopic spin, rather than the amplitudes with simple symmetry properties that were used in Sec. II. The amplitudes T_j are those of Eqs. (36) and (37), where now j indicates both total angular momentum, isotopic spin, and whether Σ's or Λ's are involved. There are ten P-wave amplitudes, corresponding to $j=(2,\frac{3}{2})$, $(2,\frac{1}{2})$, $(1\Sigma,\frac{3}{2})$, $(1\Sigma,\frac{1}{2})$, $(0,\frac{3}{2})$, $(0,\frac{1}{2})$, $(1\Lambda,\frac{3}{2})$ $(1\Lambda,\frac{1}{2})$, $(1\Sigma\Lambda,\frac{3}{2})$ and $(1\Sigma\Lambda,\frac{1}{2})$, where the second index denotes the angular momentum, and the first index refers to the appropriate one of the five processes discussed at the beginning of Sec. II. In order that the equations for the processes $\Sigma\rightarrow\Sigma$, $\Lambda\rightarrow\Lambda$, and $\Lambda\rightarrow\Sigma$ may easily be combined, we write all the equations in terms of the energy variable ω_Σ, i.e., the center-of-mass energy of the pion accompanying a Σ particle. The center-of-mass energy of the pion in the process $\pi+\Lambda\rightarrow\pi+\Lambda$ is related to ω_Σ by the equation $\omega_\Lambda=\omega_\Sigma+\Delta$, and the average pion energy ω in the process $\pi+\Lambda\rightarrow\pi+\Sigma$ is given by $\omega=\frac{1}{2}(\omega_\Sigma+\omega_\Lambda)=\omega_\Sigma+\frac{1}{2}\Delta$. The static P-wave equations are:

TABLE I. Values of X_j.

j	X_j^{I}	X_j^{II}	X_j^{III}	X_j^{IV}
$(2,\frac{3}{2})$	$2f_\Lambda^2$	$2f_\Sigma^2$	0	0
$(2,\frac{1}{2})$	$-f_\Lambda^2$	$-f_\Sigma^2$	0	0
$(1\Sigma,\frac{3}{2})$	$-2f_\Lambda^2$	$2f_\Sigma^2$	0	0
$(1\Sigma,\frac{1}{2})$	f_Λ^2	$-7f_\Sigma^2$	0	0
$(0,\frac{3}{2})$	$2f_\Lambda^2$	$-4f_\Sigma^2$	0	0
$(0,\frac{1}{2})$	$-f_\Lambda^2$	$2f_\Sigma^2$	$-9f_\Lambda^2$	0
$(1\Lambda,\frac{3}{2})$	0	0	0	$2f_\Lambda^2$
$(1\Lambda,\frac{1}{2})$	0	$-3f_\Lambda^2$	0	$-f_\Lambda^2$
$(1\Sigma\Lambda,\frac{3}{2})$	0	0	$2\sqrt{2}f_\Lambda f_\Sigma$	0
$(1\Sigma\Lambda,\frac{1}{2})$	0	$3\sqrt{2}f_\Lambda f_\Sigma$	$-\sqrt{2}f_\Lambda f_\Sigma$	0

$$\begin{aligned}\mathrm{Re}T_j(\omega_\Sigma)=\frac{1}{3\mu^2}\bigg(&\frac{X_j^{\mathrm{I}}}{\omega_\Sigma-\Delta}+\frac{X_j^{\mathrm{II}}}{\omega_\Sigma}+\frac{X_j^{\mathrm{III}}}{\omega_\Sigma+\Delta}+\frac{X_j^{\mathrm{IV}}}{\omega_\Sigma+2\Delta}\bigg)\\ &+\frac{1}{\pi}P\int_{\mu-\Delta}^{\infty}d\omega_\Sigma'\bigg[\frac{\mathrm{Im}T_j(\omega_\Sigma')}{\omega_\Sigma'-\omega_\Sigma}\\ &+\sum_i\bigg(\frac{A_{ji}^{\mathrm{I}}\,\mathrm{Im}T_i(\omega_\Sigma')}{\omega_\Sigma'+\omega_\Sigma}+\frac{A_{ji}^{\mathrm{II}}\,\mathrm{Im}T_i(\omega_\Sigma')}{\omega_\Sigma'+\omega_\Sigma+\Delta}\\ &+\frac{A_{ji}^{\mathrm{III}}\,\mathrm{Im}T_i(\omega_\Sigma')}{\omega_\Sigma'+\omega_\Sigma+2\Delta}\bigg)\bigg].\end{aligned} \tag{38}$$

The X_j, expressed in terms of the coupling constants $f_\Lambda=G_\Lambda\mu/2m$ and $f_\Sigma=G_\Sigma\mu/2m$, are given in Table I. The elements of the matrix A_{ji}^{I} are nonzero only if i and j refer to $\Sigma-\pi$ elastic scattering processes; the nonzero elements are given by

$$A^{\mathrm{I}}=\frac{1}{18}\begin{array}{c}\begin{array}{cccccc}(2,\frac{3}{2}) & (2,\frac{1}{2}) & (1\Sigma,\frac{3}{2}) & (1\Sigma,\frac{1}{2}) & (0,\frac{3}{2}) & (0,\frac{1}{2})\end{array}\\ \begin{bmatrix}1 & 2 & 3 & 6 & 2 & 4\\ 4 & -1 & 12 & -3 & 8 & -2\\ 5 & 10 & 3 & 6 & -2 & -4\\ 20 & -5 & 12 & -3 & -8 & 2\\ 10 & 20 & -6 & -12 & 2 & 4\\ 40 & -10 & -24 & 6 & 8 & -2\end{bmatrix}\end{array}.$$

The finite elements of A_{ji}^{II} and A_{ji}^{III} correspond only to the process $\pi+\Lambda\rightarrow\pi+\Sigma$, and to $\pi-\Lambda$ scattering, respectively. These elements are given by

$$A^{\mathrm{II}}=\frac{1}{3}\begin{array}{c}\begin{array}{cc}(1\Sigma\Lambda,\frac{3}{2}) & (1\Sigma\Lambda,\frac{1}{2})\end{array}\\ \begin{pmatrix}-1 & -2\\ -4 & 1\end{pmatrix}\end{array},\quad A^{\mathrm{III}}=\frac{1}{3}\begin{array}{c}\begin{array}{cc}(1\Lambda,\frac{3}{2}) & (1\Lambda,\frac{1}{2})\end{array}\\ \begin{pmatrix}1 & 2\\ 4 & -1\end{pmatrix}\end{array}.$$

It should be pointed out that the integrand in Eq. (38) was expanded to lowest order in ω_Σ'/m, so that this equation is valid only if the contribution of $\mathrm{Im}T_j(\omega_\Sigma')$ can be neglected for $\omega_\Sigma'/m \gtrsim 1$.

In the case of global symmetry ($f_\Sigma^2=f_\Lambda^2=f^2$), if we neglect the $\Sigma-\Lambda$ mass difference Δ, the $\pi-Y$ P-wave equations for isotopic spin 2 and 0 reduce to the $\pi-N$ P-wave equations for isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$, respectively, i.e.,

$$\mathrm{Re}\, T_j(\omega_\Sigma)=-\frac{1}{3}\frac{X_j}{\mu^2\omega_\Sigma}+\frac{1}{\pi}P\int_\mu^\infty d\omega_\Sigma' \times\left(\frac{\mathrm{Im}T_j(\omega_\Sigma')}{\omega_\Sigma'-\omega_\Sigma}+\frac{\sum_i A_{ji}\,\mathrm{Im}T_i(\omega_\Sigma')}{\omega_\Sigma'+\omega_\Sigma}\right),$$

where

$$X_j=X_j^{\mathrm{I}}+X_j^{\mathrm{II}}+X_j^{\mathrm{III}}, \tag{39}$$

$$X_{(2,\frac{3}{2})}=4f^2,\quad X_{(2,\frac{1}{2})}=X_{(0,\frac{3}{2})}=-2f^2,$$
$$X_{(0,\frac{1}{2})}=-8f^2, \tag{40}$$

and the matrix A is given by,

$$A=\frac{1}{9}\begin{array}{c} \begin{array}{cccc}(2,\frac{3}{2}) & (2,\frac{1}{2}) & (0,\frac{3}{2}) & (0,\frac{1}{2})\end{array} \\ \begin{pmatrix} 1 & 2 & 2 & 4 \\ 4 & -1 & 8 & -2 \\ 4 & 8 & -1 & -2 \\ 16 & -4 & -4 & 1 \end{pmatrix}\end{array}.$$

The $\pi-Y$ P-wave equations for isotopic spin 1 can then be written as linear combinations of the isotopic spin 2 and 0 equation of the same total angular momentum, i.e.,

$$T_{1\Sigma}=\tfrac{1}{3}T_2+\tfrac{2}{3}T_0,$$
$$T_{1\Lambda}=\tfrac{2}{3}T_2+\tfrac{1}{3}T_0,$$
$$T_{1\Sigma\Lambda}=\sqrt{2}(\tfrac{1}{3}T_2-\tfrac{1}{3}T_0).$$

Both the $\Sigma-\Lambda$ mass splitting and the presence of additional $\bar{K}$ meson channels modify the $\pi-Y$ equations relative to the $\pi-N$ equations. However, we do not think that the static equations (with multiple meson production neglected) contain sufficient information to calculate the corrections to global symmetry caused by these effects. Instead we will find an approximate solution in the case of global symmetry, similar in nature to the solution obtained for $\pi-N$ scattering in the effective range approximation of Chew and Low.[12] According to our normalization, Eq. (37), the P-wave amplitude $T_j(\omega)$ can be written in the form

$$T_j(\omega)=e^{i\delta_j}\sin\delta_j/k^3. \tag{41}$$

We consider here only $\pi-\Sigma$ scattering for which $k_i=k_f$ and $\omega=\omega_\Sigma$; hence we have dropped the subscript Σ on the energy variable and have set $k_i=k_f=k$. For the isotopic spin 2 and 0 states $\delta_j(\omega)$ is a real function of ω below the threshold for single π-meson production. If we neglect the contribution of the $\bar{K}N$ channel as well as π-meson production, the P-wave equation becomes an integral equation for $\delta_j(\omega)$. We shall seek an approximate solution by substituting

$$\mathrm{Im}T_j(\omega)=\pi\lambda_j\delta(\omega-\omega_j), \tag{42}$$

under the integral in Eq. (38), where λ_j and ω_j are adjustable parameters and j denotes a state in which there may be a resonance. Note that λ_j must be positive since $\mathrm{Im}T_j(\omega)$ is positive in the physical region. If we assume that the contribution to the dispersion integrals of the other channels (the A_{ji} terms) is small, we get

$$\mathrm{Re}T_j(\omega)=\epsilon_j(\omega)/(\omega_j-\omega), \tag{43}$$

$$\epsilon_j(\omega)=\frac{1}{3\mu^2}\left(X_j^{\mathrm{I}}\frac{(\omega_j-\Delta)}{(\omega-\Delta)}+X_j^{\mathrm{II}}\frac{\omega_j}{\omega}+X_j^{\mathrm{III}}\frac{(\omega_j+\Delta)}{(\omega+\Delta)}\right)+\lambda_j-\frac{1}{3\mu^2}X_j, \tag{44}$$

where X_j is defined by Eq. (39). Since $\mathrm{Im}T_j$ is zero in this approximation for all energies satisfying the inequality $\omega\neq\omega_j$, we may write for such energies

$$\mathrm{Re}(1/T_j)=1/\mathrm{Re}T_j=(\omega_j-\omega)/\epsilon_j(\omega). \tag{45}$$

The approximate amplitude described by Eqs. (42) and (43) does not satisfy the unitarity condition implied by the reality of the phase shift δ_j. We shall improve our approximation by finding the amplitude that does satisfy the unitarity condition and, in addition, satisfies Eq. (45). It is seen from Eq. (41) that the unitary condition may be written in the form

$$\mathrm{Im}T_j(\omega)=\frac{k^3}{k^6+\{\mathrm{Re}[1/T_j(\omega)]\}^2}. \tag{46}$$

The amplitude defined by Eqs. (45) and (46) is the familiar resonance type amplitude, and may be expressed in the form

$$e^{i\delta_j(\omega)}\sin\delta_j(\omega)=\frac{k^3\epsilon_j(\omega)}{(\omega_j-\omega)-ik^3\epsilon_j(\omega)}, \tag{47}$$

and

$$\mathrm{Im}T_j(\omega)=\pi\epsilon_j(\omega)\left(\frac{1}{\pi}\frac{k^3\epsilon_j(\omega)}{(\omega-\omega_j)^2+[k^3\epsilon_j(\omega)]^2}\right). \tag{48}$$

In order that Eq. (48) for $\mathrm{Im}T_j(\omega)$ be consistent with the delta function approximation which we used under the integral, we want $k^3\epsilon_j(\omega)$ to be as small as possible. This implies that $\epsilon_j(\omega)$ must be small for large ω. It is seen from Eqs. (44) and (40) that this condition may be achieved for the state $j=(2,\frac{3}{2})$ if we choose

$$\lambda_{(2,\frac{3}{2})}-(1/3\mu^2)X_{(2,\frac{3}{2})}=0.$$

[12] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

For the other three states $j=(2,\frac{1}{2})$, $(0,\frac{3}{2})$, and $(0,\frac{1}{2})$,

$$\lambda_j-(1/3\mu^2)X_j>0,$$

so that the delta function approximation is not as well justified for these states.

It has been pointed out by Chew, Goldberger, Low, and Nambu,[12] in their discussion of pion-nucleon scattering that Eq. (47) is an approximate solution of the dispersion relation at low energies for an arbitrary value of the resonance energy ω_j. The position of the resonance is determined by the high-energy contribution to the integral, about which nothing is known. In the $\pi-N$ P-wave equation, this contribution is summarized by a cutoff $\omega_{\max}$ in the integral and is related to the position of the resonance by $\omega_{\max}\approx f^2\omega_j$. Because of the presence of the additional $\bar{K}$-meson channels in $\pi-Y$ scattering, it is unlikely that the same cutoff $\omega_{\max}$ can be applied to the $\pi-Y$ equations. Even if we make such an assumption, it is not true that we can neglect the contribution of the $\bar{K}N$ channel under the integral. In order to illustrate this statement we assume that the Σ and Λ parities are the same, and that the intrinsic K parity is odd (relative to the NY pair), so that a P state $\pi-Y$ system corresponds to a P state $\bar{K}-N$ system. The hydrogen bubble chamber experiments at Berkeley[13] show that the elastic and inelastic K^--proton cross sections at 400 Mev/c far exceed the maximum for S waves. Assuming that the cross sections for $l\geqq 2$ are small at this energy, we find that the P-wave cross section for $\pi-Y$ production must be greater than 13 millibarns. This energy corresponds to about 270 Mev/c pion momentum in the center of mass of the $\pi-\Sigma$ system and, by detailed balance, 13 mb should also be the approximate minimum P-wave cross section for the process $\pi+Y\rightarrow\bar{K}+N$. Such a cross section leads to an appreciable modification of the unitarity condition [Eq. (46)] and may make a large contribution to the dispersion integral.

On the other hand, it is probable that the effect of pion production is small. Single pion production by pions on nucleons is only about $\frac{1}{2}$ mb at 270 Mev/c center-of-mass momentum and is usually neglected in the pion-nucleon dispersion relations. Hence we assume the corresponding effect is small in the pion-hyperon dispersion relations.

If the intrinsic K parity is even, so that the $P_{\frac{1}{2}}$ $\pi-Y$ system corresponds to the S-state $\bar{K}-N$ system, the effect of the $\bar{K}-N$ channel on the $\pi-Y$ P-wave amplitudes is likely to be even more important than in the odd parity case. Such an effect has been discussed by Dalitz and Tuan.[2]

We have emphasized that the production of $\pi-Y$ states by P-wave $\bar{K}-N$ pairs must be large for lab K momenta in the range 300 to 400 Mev/c. The center-of-mass differential cross sections measured at Berkeley for the processes $K^-+p\rightarrow\Sigma^++\pi^-$ and $K^-+p\rightarrow\Sigma^-+\pi^+$ at lab K momenta of about 400 Mev/c appear to be smaller at 90° than in the forward or backward directions.[13] If we again assume that the K parity is odd, this may be taken as crude evidence for the presence of an appreciable $P_{\frac{3}{2}}$ state, since the angular distribution corresponding to such a state is of the form $3\cos^2\theta+1$. It is not unlikely that this P-wave pion-hyperon production is associated with a resonance in the pi-hyperon scattering.

[13] Proceedings of the Ninth Annual Conference on High-Energy Physics at Kiev (to be published).

C. S-Wave Equations

In this section we obtain the dispersion relations for the S-wave amplitudes in the static limit, using a generalization of the method applied by Oehme[14] to the pion-nucleon scattering problem. The S-wave amplitude T_0 is defined by Eq. (35). The normalization of the S amplitudes is different from that of the P amplitudes, [see Eq. (37)], i.e.,

$$T_0=R_0|k_i|^{-\frac{1}{2}}|k_f|^{-\frac{1}{2}}.$$

As a first step in deriving the static equations for T_0, we consider the relativistic amplitude $M_0^{(\lambda)}(\omega,q^2)$ in the Breit-Lorentz system, where $M_0^{(\lambda)}$ is defined in terms of the spin-independent amplitude by the equation,

$$M_0^{(\lambda)}(\omega,q^2)=[1-q^2(\partial/\partial q^2)+\tfrac{1}{2}(\omega^2-\mu^2+\mathcal{E}^2)\partial/\partial q^2]\times M_N^{(\lambda)}(\omega,q^2). \quad (49)$$

Since M_N is an analytic function of q^2 in the upper half of the ω plane for all values of q^2, it is clear that M_0 is also an analytic function of ω. The crossing relations for $M_0^{(\lambda)}$ are identical to those for $M_N^{(\lambda)}$. If M_0 diverges no more rapidly than linearly at high energy, we may derive dispersion relations of the subtracted type by integrating the function

$$M_0^{(\lambda)}(\omega')[(\omega'+\omega)+\epsilon_\lambda(\omega'-\omega)]/[(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)]$$

around the contour including the real ω axis and a semicircle of infinite radius in the upper half ω plane. The symbol ω_0 represents any constant energy. The quantities are then expressed in terms of center-of-mass variables, and the static limit is taken. In this limit, Eqs. (31) and (34c) may be used to show that $M_0^{(\lambda)}$ is independent of q^2 and is equal to the S-wave amplitude $T_0^{(\lambda)}$. The Born approximation terms vanish if one follows this procedure, and the results may be written in the form,

$$\mathrm{Re}T_0^{(\lambda)}(\omega)-\mathrm{Re}T_0^{(\lambda)}(\omega_0)=\frac{2(\omega^2-\omega_0^2)}{\pi}P\int_{\omega_a}^{\infty}\frac{\omega'd\omega'\,\mathrm{Im}T_0^{(\lambda)}(\omega')}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)},$$

$$(\text{for } \lambda=1,\ 3, \text{ or } 4), \quad (50a)$$

[14] Reinhard Oehme, Phys. Rev. **102**, 1174 (1956). See Eq. (29) of this reference.

or

$$\mathrm{Re}T_0^{(\lambda)}(\omega)-\frac{\omega}{\omega_0}\mathrm{Re}T_0^{(\lambda)}(\omega)=\frac{2\omega(\omega^2-\omega_0^2)}{\pi}P\int_{\omega_a}^{\infty}\frac{d\omega'\,\mathrm{Im}T_0^{(\lambda)}(\omega')}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)},\quad(\text{for }\lambda=2\text{ or }5).\qquad(50b)$$

All quantities refer to the center-of-mass system and the energy variable refers to the average of the initial and final meson energies. The constant energy ω_0 may be chosen for convenience.

The S-wave dispersion relations may be written in terms of the amplitudes for particular isotopic spin states if use is made of Eqs. (2). If it is desired to relate the different S-wave processes at the same total energy, one may express the energies in terms of the energy ω_Σ of a pion accompanying a Σ particle. As in Sec. III B, the relations are: $\omega=\omega_\Sigma$ for $\pi-\Sigma$ scattering, $\omega=\omega_\Sigma+\Delta$ for $\pi-\Lambda$ scattering, and $\omega=\omega_\Sigma+\frac{1}{2}\Delta$ for the processes $\pi+\Lambda\leftrightarrows\pi+\Sigma$. For all processes the lower limit ω_a of the dispersion integral is that energy at which ω_Σ is equal to $\mu-\Delta$.

For many considerations it is convenient to choose the reference energy ω_0 to be equal to μ or some other low energy, so that the $T_0^{(\lambda)}(\omega_0)$ are essentially the scattering lengths for S-wave scattering. These scattering lengths cannot be determined from the subtracted type dispersion relations, of course. If one assumes that the odd amplitudes $M_N^{(2)}$ and $M_N^{(5)}$ approach zero as the energy gets large, one may derive unsubtracted relations which, in the static approximation, express $T_0^{(2)}(\mu)$ and $T_0^{(5)}(\mu)$ in terms of the coupling constant terms and S- and P-wave dispersion integrals.[15] Our present knowledge of the low-energy $\pi-Y$ processes is insufficient to estimate any of the scattering lengths in this manner.

In this paper we shall not attempt to relate the S-wave equations to any experimental data in order to investigate the possible behaviors of the $\pi-Y$ amplitudes. Further study is being given to this problem.

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[15] A similar procedure for the pion-nucleon scattering case is discussed in reference 12. See Eqs. (3.22) to (3.26) of this reference and the discussion following these equations.

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Radiative Pion Decay into Electrons*

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The possibility of distinguishing the pion structure-dependent radiation from the conventional inner bremsstrahlung radiation in the radiative decay of pions into electrons is discussed. Calculation of the photon energy spectrum and angular correlation shows that evidence for pion structure would be obtained if any photons of energy less that 70 Mev were detected in 180° coincidence with π-decay electrons. The probability of such events per unit solid angle is $\gtrsim 0.2\times10^{-7}$ relative to ordinary $\pi\rightarrow\mu+\nu$ decay, if the assumption of a conserved vector current is made to relate the rate of radiative decay through the weak V-interaction to the rate of $\pi^0\rightarrow2\gamma$ decay.

I. INTRODUCTION

THE universal V-A form of the Fermi interaction has in recent years been suggested by the evidence in β and μ decay. The other weak interactions are then, in principle, consequences of strong couplings together with the universal Fermi interaction. In the decay of π mesons into electrons, where the momentum transfer is large, evidence on the decay mechanism can be obtained,[1,2] in principle, by observing the associated radiative decay $\pi\rightarrow e+\nu+\gamma$. In this paper we amplify the calculation by Vaks and Ioffe[1] and discuss the possibility of distinguishing structure-dependent effects from less interesting structure-independent effects. We supplement the electron spectrum already presented[1,3] by calculating the photon spectrum, which may be more easily observed experimentally.

The diagrams for the radiative decay are given in Fig. 1. Diagrams (a) and (b), when defined in a gauge-invariant way, give rise to the inner bremsstrahlung by a decelerated or accelerated charge or magnetic moment. The matrix element for this is proportional to $eGm/\sqrt{k}$,

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[1] V. G. Vaks and B. L. Ioffe, Nuovo cimento **10**, 342 (1958).

[2] K. Huang and F. E. Low, Phys. Rev. **109**, 1400 (1958).

[3] S. A. Bludman and M. A. Ruderman, Phys. Rev. **101**, 910 (1956).