or

$$\mathrm{Re}T_0^{(\lambda)}(\omega)-\frac{\omega}{\omega_0}\mathrm{Re}T_0^{(\lambda)}(\omega)$$
$$=\frac{2\omega(\omega^2-\omega_0^2)}{\pi}P\int_{\omega_a}^{\infty}\frac{d\omega'\,\mathrm{Im}T_0^{(\lambda)}(\omega')}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)},$$
$$(\text{for } \lambda=2 \text{ or } 5). \quad (50\text{b})$$

All quantities refer to the center-of-mass system and the energy variable refers to the average of the initial and final meson energies. The constant energy ω_0 may be chosen for convenience.

The S-wave dispersion relations may be written in terms of the amplitudes for particular isotopic spin states if use is made of Eqs. (2). If it is desired to relate the different S-wave processes at the same total energy, one may express the energies in terms of the energy ω_Σ of a pion accompanying a Σ particle. As in Sec. III B, the relations are: $\omega=\omega_\Sigma$ for $\pi-\Sigma$ scattering, $\omega=\omega_\Sigma+\Delta$ for $\pi-\Lambda$ scattering, and $\omega=\omega_\Sigma+\frac{1}{2}\Delta$ for the processes $\pi+\Lambda\leftrightarrows\pi+\Sigma$. For all processes the lower limit ω_a of the dispersion integral is that energy at which ω_Σ is equal to $\mu-\Delta$.

For many considerations it is convenient to choose the reference energy ω_0 to be equal to μ or some other low energy, so that the $T_0^{(\lambda)}(\omega_0)$ are essentially the scattering lengths for S-wave scattering. These scattering lengths cannot be determined from the subtracted type dispersion relations, of course. If one assumes that the odd amplitudes $M_N^{(2)}$ and $M_N^{(5)}$ approach zero as the energy gets large, one may derive unsubtracted relations which, in the static approximation, express $T_0^{(2)}(\mu)$ and $T_0^{(5)}(\mu)$ in terms of the coupling constant terms and S- and P-wave dispersion integrals.[15] Our present knowledge of the low-energy $\pi-Y$ processes is insufficient to estimate any of the scattering lengths in this manner.

In this paper we shall not attempt to relate the S-wave equations to any experimental data in order to investigate the possible behaviors of the $\pi-Y$ amplitudes. Further study is being given to this problem.

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[15] A similar procedure for the pion-nucleon scattering case is discussed in reference 12. See Eqs. (3.22) to (3.26) of this reference and the discussion following these equations.

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Radiative Pion Decay into Electrons*

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The possibility of distinguishing the pion structure-dependent radiation from the conventional inner bremsstrahlung radiation in the radiative decay of pions into electrons is discussed. Calculation of the photon energy spectrum and angular correlation shows that evidence for pion structure would be obtained if any photons of energy less that 70 Mev were detected in 180° coincidence with π-decay electrons. The probability of such events per unit solid angle is $\gtrsim 0.2\times10^{-7}$ relative to ordinary $\pi\rightarrow\mu+\nu$ decay, if the assumption of a conserved vector current is made to relate the rate of radiative decay through the weak V-interaction to the rate of $\pi^0\rightarrow 2\gamma$ decay.

I. INTRODUCTION

THE universal V-A form of the Fermi interaction has in recent years been suggested by the evidence in β and μ decay. The other weak interactions are then, in principle, consequences of strong couplings together with the universal Fermi interaction. In the decay of π mesons into electrons, where the momentum transfer is large, evidence on the decay mechanism can be obtained,[1,2] in principle, by observing the associated radiative decay $\pi\rightarrow e+\nu+\gamma$. In this paper we amplify the calculation by Vaks and Ioffe[1] and discuss the possibility of distinguishing structure-dependent effects from less interesting structure-independent effects. We supplement the electron spectrum already presented[1,3] by calculating the photon spectrum, which may be more easily observed experimentally.

The diagrams for the radiative decay are given in Fig. 1. Diagrams (a) and (b), when defined in a gauge-invariant way, give rise to the inner bremsstrahlung by a decelerated or accelerated charge or magnetic moment. The matrix element for this is proportional to $eGm/\sqrt{k}$,

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[1] V. G. Vaks and B. L. Ioffe, Nuovo cimento **10**, 342 (1958).
[2] K. Huang and F. E. Low, Phys. Rev. **109**, 1400 (1958).
[3] S. A. Bludman and M. A. Ruderman, Phys. Rev. **101**, 910 (1956).

where e and G are the electromagnetic and Fermi coupling constants, m is the electron (or muon) mass, and k the photon energy. Diagrams (c) and (d) of Fig. 1 are structure-dependent, since here the emission of a photon depends on the nature of the "black box." The matrix elements for these diagrams are proportional to $eG\sqrt{k}(\mu/M)$, where μ is the pion mass and M a mass or energy typical of the intermediate states involved in the "black box." The two processes—inner bremsstrahlung and "black box" (or structure-dependent) radiation—are coherent, but the interference term is negligible in $\pi\to e+\nu+\gamma$ decay. (In $\pi\to\mu+\nu+\gamma$ the reverse is the case: because of the small momentum transfer involved, the structure-dependent radiation is small compared with the inner bremsstrahlung, and the interference term dominates the square of the structure-dependent matrix element. For this reason radiative $\pi\to\mu$ decay, although more frequent by several orders of magnitude than radiative $\pi\to e$ decay, reveals nothing indicative of the pion decay structure.) The interesting question is not whether radiative $\pi^\pm$ decay occurs, but whether the interesting structure-dependent effects can be disentangled from the ordinary quasi-classical bremsstrahlung. We find that a unique proof of structure to the π-decay mechanism can be obtained if any photons of energy less than $k_{\max}=70$ Mev are detected in 180° correlation to the direction of the decay electron. The probability of such a decay per unit solid angle per π decay is, however, approximately 0.2×10^{-7}.

II. INNER BREMSSTRAHLUNG (IB)

The matrix element for the inner bremsstrahlung is defined as the gauge-invariant part of diagrams (a) and (b) of Fig. 1. On invariance grounds this is of the form

$$M_{\mathrm{IB}}=e(m/\sqrt{k})f_A(\mathcal{P}^2)\bar{\psi}_e[(p\cdot\epsilon/p\cdot k-\mathcal{P}\cdot\epsilon/\mathcal{P}\cdot k)+i\sigma_{\mu\nu}F_{\mu\nu}/4p\cdot k]\psi_\nu,$$

where $f_A(\mathcal{P}^2)$ is the amplitude for the nonradiative decay, $\mathcal{P}$ is the pion four-momentum, p the electron four-momentum, ϵ the photon polarization four-vector, $F_{\mu\nu}=\epsilon_\mu k_\nu-\epsilon_\nu k_\mu$, and ψ_e and ψ_ν are, respectively, the electron (or muon) and neutrino field operators. The two terms in M_{IB} correspond to emission of radiation by the accelerated charge and magnetic moment, respectively.

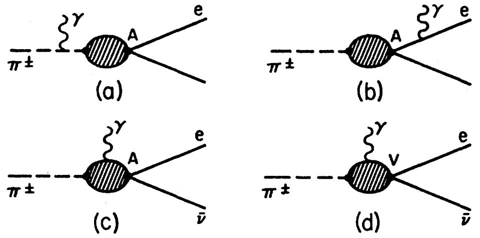

FIG. 1. The possible diagrams for the radiative electron decay of the pion.

This matrix element leads to the differential transition probability,

$$d^3W_{\mathrm{IB}}=W_{e\nu}\frac{\alpha}{(2\pi)^3}\frac{1}{\mu^2}\{\mu^2[\mathbf{p}^2-(\mathbf{p}\cdot\mathbf{k})^2/k^2]+2(Ek-\mathbf{p}\cdot\mathbf{k})(\mu k-Ek+\mathbf{p}\cdot\mathbf{k})\}\times\frac{d^3pd^3k}{(Ek-\mathbf{k}\cdot\mathbf{p})^2}\frac{\delta(\mu-E-E_\nu-k)}{EE_\nu k},\quad(1)$$

where $\alpha=e^2/4\pi$ is the fine-structure constant, E the electron energy, E_ν the neutrino energy, and $W_{e\nu}$ the nonradiative decay rate. The electron energy spectrum resulting from this expression has been given previously[1,3] and is not repeated here.

We suspect that, because of the overwhelming background of $\pi\to e+\nu$ and $\pi\to\mu\to e$ electrons, the photon radiation (or at least the hard component in which we are interested) may be more easily distinguished than the spectrum of electrons. The spectrum of photons into solid angle $d\Omega=2\pi\sin\theta d\theta$, where θ is the angle between the photon and the electron, is obtained by integrating Eq. (1) over electron energies. This relation is generally complicated by the energy condition $\mu=E+k+|\mathbf{k}+\mathbf{p}|$.

If the photon and electron emerge in opposite directions so that $\theta=180°$ and $|\mathbf{p}+\mathbf{k}|^2=(p-k)^2$, the energy condition becomes $(\mu-E-p)(\mu-E+p-2k)=0$. The photon spectrum obtained from Eq. (1) is then

$$\left(\frac{d^2W_{\mathrm{IB}}}{dkd\Omega}\right)_{\theta=180°}=W_{e\nu}\frac{\alpha\, p^2k}{\mu^2}\frac{(\mu-E-p)}{(E+p)[p\mu-k(p+E)]},\quad(2)$$

where p and E are determined by the energy condition. Now $\mu-E-p=0$ unless $k=(\mu-E+p)/2$ which lies between $(\mu-m_e)/2$ and $k_{\max}=(\mu^2-m_e^2)/2\mu$. Thus according to Eq. (2), the only inner bremsstrahlung photons at $\theta=180°$ are those of essentially maximum energy $k_{\max}$. The probability for the emission of such photons is, according to Eq. (2) (setting $k=k_{\max}$),

$$\left(\frac{d^2W_{\mathrm{IB}}}{dxd\Omega}\right)_{\theta=180°,\,k=k_{\max}}=W_{e\nu}\frac{\alpha}{(2\pi)^2}\left(1-\frac{m_e^2}{\mu^2}\right)^3=1\ \mathrm{sec}^{-1},$$

where $k=xk_{\max}$. The inner bremsstrahlung spectrum at 180° vanishes then except for photons of very near maximum energy.

For angles other than 180° (or 0°) the photon spectrum will be adequately described by setting $m_e=0$. In this approximation, we obtain for any angle θ between photon and electron

$$\frac{d^2W_{\mathrm{IB}}}{dxd\Omega}=W_{e\nu}\frac{\alpha}{2\pi^2}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\frac{(x-1)^2+1}{[2+x(\cos\theta-1)]^2}\frac{1}{x}.\quad(2a)$$

Equation (2a) is thus applicable to all angles except 0° and 180° provided $k \leq (\mu - m_e)/2 = 69.6$ Mev.

Integrating Eq. (2a) for photon energies greater than some low-energy cutoff δ, we obtain the electron-photon angular correlation,

$$dW_{\mathrm{IB}}/d\Omega = W_{e\nu}(\alpha/2)(2\pi)^{-2}\lambda^{-3} \times\{\lambda+(1-\lambda)(1-2\lambda^2)\ln(1-\lambda) +2\lambda^2(1-\lambda)[\ln(1/x_{\min}-1]\}, \quad (3)$$

where $\lambda = \sin^2\theta/2$ and $x_{\min} = 2\delta/\mu$. The rate of $\pi \to e + \nu + \gamma$ decay per unit solid angle with e and γ at 180° to each other is

$$\frac{(dW_{\mathrm{IB}}/d\Omega)_{\theta=180°}}{W_{\mu\nu}} = 1.2\times 10^{-8}/\text{steradian}. \quad (3a)$$

Equation (3) agrees with Eq. (21) of Vaks and Ioffe, since when the minimum photon energy is δ the maximum electron energy is approximately $\mu/2-\delta(1-\lambda)$, so that $y_{\max}$ in Vaks and Ioffe equals $1-(1-\lambda)x_{\min}$ above.

III. STRUCTURE-DEPENDENT (SD) RADIATION

Out of the pseudoscalar pion field operator ϕ and the electromagnetic field operator A_μ only two vectors $a\phi\tilde{F}_{\mu\nu}\mathcal{P}_\nu$ and $b\phi F_{\mu\nu}\mathcal{P}_\nu$ can be constructed in a gauge-invariant manner. Here $F_{\mu\nu} = A_\mu k_\nu - A_\nu k_\mu$, and

$$\tilde{F}_{\mu\nu} = (i/_2)\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}$$

is the tensor dual to $F_{\mu\nu}$, and a and b are functions of $\mathcal{P}\cdot k$, which must be, assuming PC invariance, relatively real. The gauge-invariant contribution of diagrams (c) and (d) of Fig. 1 must therefore be of the form,

$$\begin{aligned} M_V &= -i(\alpha)^{\frac{1}{2}}G_V a\phi\tilde{F}_{\mu\nu}\mathcal{P}_\nu\bar{\psi}_e\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_\nu, \\ M_A &= i(\alpha)^{\frac{1}{2}}G_A b\phi F_{\mu\nu}\mathcal{P}_\nu\bar{\psi}_e\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_\nu. \end{aligned} \quad (4)$$

Writing

$$q_\mu = p_\mu + p_\mu^\nu, \quad B_\mu = \bar{\psi}_e\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\psi_\nu, \quad G_{\mu\nu} = q_\mu B_\nu - q_\nu B_\mu,$$

one finds that Eqs. (4) take the form

$$\begin{aligned} M_V &= -(i/2)(\alpha)^{\frac{1}{2}}G_V a\phi\tilde{F}_{\mu\nu}G_{\mu\nu}, \\ M_A &= (i/2)(\alpha)^{\frac{1}{2}}G_A b\phi F_{\mu\nu}G_{\mu\nu}. \end{aligned} \quad (5)$$

The matrix element for the gamma decay of pseudoscalar π^0 mesons is, on invariance grounds,

$$M_{\pi^0} = -(i/2)\alpha c\phi\tilde{F}_{\mu\nu}F_{\mu\nu}, \quad (6)$$

from which the rate of π^0 decay is

$$W_{\pi^0} = (\alpha^2/4)(2\pi)^{-8}\mu^3c^2, \quad (7)$$

where c is a constant depending on the pion decay structure. In lowest-order perturbation theory (where the "black box" in Fig. 1 stands for a nucleon-antinucleon loop, each of mass M, coupled to the pion field via $(g/\sqrt{2})\bar{\psi}_N\gamma_5\boldsymbol{\tau}\psi_N\cdot\boldsymbol{\phi}$), we have[4]

$$c = a = 4(\pi)^{\frac{1}{2}}g/M. \quad (8)$$

This relation between the electromagnetic decay of the π^0 and the vector radiative decay of the $\pi^\pm$ holds, for photons of near maximum energy, to all orders if the Feynman-Gell-Mann principle of conservation of the weak vector current[5] is assumed. (Generally a is a function of k which equals c strictly only when $k=\mu/2$, the photon energy in π^0 decay. We are neglecting this possible energy dependence and setting $a=c=$ constant.) This assumption, which was also made in reference 1, determines the over-all rate of $\pi^\pm$ decay, and will be made in the remainder of this paper.

Photon Spectrum and Angular Correlation

The differential transition probability obtained from Eqs. (4) or Eqs. (5) is

$$d^3W_{\mathrm{SD}} = (2G_V^2/\alpha)(2\pi)^{-5}\mu^{-2}W_{\pi^0}k \times[(1+\gamma^2)(1-\beta\cos\theta\cos\phi)+2\gamma(\cos\phi-\cos\theta)] \times\delta(E+E_\nu+k-\mu)d^3p\,d^3k, \quad (9)$$

where β is the electron velocity, ϕ the angle between neutrino and photon and $\gamma = bG_A/aG_V$.[6]

From Eq. (9) we obtain the photon spectrum angular distribution

$$\frac{d^2W_{\mathrm{SD}}}{dx\,d\Omega} = (G_V^2/4\alpha)(2\pi)^{-4}\mu^4W_{\pi^0}\frac{x^3(1-x)^2}{[2+x(\cos\theta-1)]^4} \times\{2[(1+\gamma^2)(1+\cos^2\theta)-4\gamma\cos\theta] +x(x-2)(1+\gamma)^2(1-\cos\theta)^2\}. \quad (10)$$

In the case of $\theta=180°$ Eq. (10) reduces to

$$\left(\frac{d^2W_{\mathrm{SD}}}{dx\,d\Omega}\right)_{\theta=180°} = (G_V^2/\alpha)(4\pi)^{-4}\mu^4W_{\pi^0}(1+\gamma)^2x^3. \quad (11)$$

These equations show that, at least for $\gamma\sim 1$, structure-dependent radiation is predominantly hard and in the backwards direction. This is to be contrasted with the inner bremsstrahlung which is predominantly soft and in the forward direction. In lowest order perturbation theory $b=a$, and Eq. (10) gives for the angular distribution of photon and electron:

$$dW_{\mathrm{SD}}/d\Omega = (G^2/4\alpha)(2\pi)^{-4}\mu^4W_{\pi^0} \times\{\lambda^2/4+(1-\lambda)\lambda^{-6}(-\lambda^3+15\lambda^2 -45\lambda+35)\ln(1-\lambda)+(1-\lambda)(12\lambda^5)^{-1} \times[3\lambda^3(\lambda^3+\lambda^2+\lambda+1)+10(5\lambda^2-33\lambda+42)]\}. \quad (12)$$

[4] J. Steinberger, Phys. Rev. **76**, 1180 (1949).

[5] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[6] Equation (9) agrees with the corresponding result in reference 1 except for an over-all factor 2, but the electron spectrum obtained from Eq. (9) does not agree with that of reference 1. We obtain

$$dW_{\mathrm{SD}}/dy = (G^2/3\alpha)(2\pi)^{-3}2^{-5}\mu^4W_{\pi^0}y^2 \times[6(1-2y)(1-\gamma)^2+y^2(7(1+\gamma^2)-10\gamma)],$$

where $y=E/E_{\max}$.

Equations (3) and (12) are plotted in Fig. 2, with the lower limit[7] assumed to be 0.5×10^{16} sec^{-1} for W_{π^0}. The electron-photon angular distribution is the superposition of two noninterfering mechanisms: (a) the inner bremsstrahlung, Eq. (3), and (b) the photon emission by the intermediate states in $\pi^{\pm}$ decay, Eq. (12). In calculation of the latter the rate of radiative decay through the V interaction has been related to the rate W_{π^0} of π^0 decay by the assumption of a conserved vector current. The decay through the A interaction can be related to that through the V interaction by choosing $\gamma=(bG_A)/(aG_V)=1$ as is suggested by lowest order perturbation theory.[8]

IV. DISCUSSION

We have seen that at $\theta=180°$ no inner bremsstrahlung photons occur of energy less than $(\mu-m_e)/2$, while structure-dependent photons of all energies occur. Thus if any photons are detected of energy less than 69.6 Mev when photon and electron are in anticoincidence there is unambiguous proof of structure mediating the $\pi^{\pm}$ decay. The number of such decays per unit solid angle about $\theta=180°$ per $\pi\rightarrow\mu+\nu$ decay is calculated, according to Eq. (12), to be

$$\frac{(dW_{\mathrm{SD}}/d\Omega)_{\theta=180°}}{W_{\mu\nu}}\geqslant 0.2\times10^{-7}/\text{steradian}.$$

This conclusion rests on the assumption relating the V interaction $\pi\rightarrow e\nu\gamma$ matrix element to the rate of π^0 decay but is insensitive to the particular choice of the parameter b in the A interaction matrix element provided $b\neq -a$.

The lowest published upper limit on the rate of radiative $\pi\rightarrow e$ decay is that obtained by Cassels and collaborators.[8,9] This group measured the rate of electron-gamma production at $\theta=180°$ and used a calculated electron-photon angular correlation function to convert these measurements into a total rate of radiative $\pi\rightarrow e$ decay relative to normal $\pi\rightarrow\mu$ decay:

$$W_{e\nu\gamma}/W_{\mu\nu}\cong(3\pm5)\times10^{-6}. \tag{13}$$

In the earlier work[9] the angular correlation function was calculated assuming an ST β-decay interaction, while in the later work[7] the V-A interaction was used but the interference term between V and A was neglected.

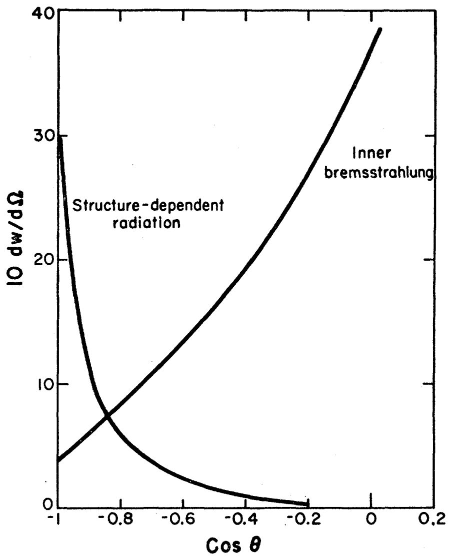

FIG. 2. Plot of the angular distribution between the photon and electron for inner bremmsstrahlung and for structure-dependent radiation.

From the rate (13) quoted in reference 9 and the fact that the relative probability for emission into the backwards direction $(1/W)dW/d(\cos\theta)$ was taken to be 3.0, we can work backwards to find

$$\frac{(dW_{e\nu\gamma}/d\Omega)_{\theta=180°}}{W_{\mu\nu}}=\frac{3.0}{2\pi}(3\pm5)\times10^{-6}$$
$$=(1.5\pm2.5)\times10^{-6}/\text{steradian}.$$

Since we calculated

$$\frac{(dW_{\mathrm{IB}}/d\Omega)_{\theta=180°}}{W_{\mu\nu}}=1.2\times10^{-8},$$

and

$$\frac{(dW_{\mathrm{SD}}/d\Omega)_{\theta=180°}}{W_{\mu\nu}}\gtrsim 2\times10^{-8}/\text{steradian}, \tag{13a}$$

the experimental sensitivity would have to be improved by two orders of magnitude to detect the interesting structure-dependent radiation. In looking for the structure-dependent radiation one should discriminate against the 70-Mev inner bremsstrahlung "line spectrum." If these photons are not observed one will have to conclude either (a) that the vector current is not conserved, and that if the pion decay structure involves baryons at all, the typical energies of the intermediate states involved probably exceed the nucleon rest mass, or (b) that the pion decay should be regarded as primary. In either case applying the idea of the universal Fermi interaction to other weak decays will have practically lost its attractiveness.

[7] G. Harris, J. Orear, and S. Taylor, Phys. Rev. **106**, 327 (1957).
[8] This result $b=a$ agrees with that obtained by N. Cabibbo (to be published). G. H. Burkhardt, J. M. Cassels, M. Rigby, A. M. Wetherall, and J. R. Wormald, Proc. Phys. Soc. (London) **72**, 144 (1958), on the contrary obtain $b=\frac{1}{3}a$. These latter authors neglect the inner bremsstrahlung contributions and the interference term between the V and A matrix elements, which contribute to the electron spectrum and to the electron-photon angular correlation.
[9] J. M. Cassels, M. Rigby, A. M. Wetherell, and J. R. Wormald, Proc. Phys. Soc. (London) **A70**, 729–734 (1957).