Limiting Properties of Numbers of Self-Avoiding Walks

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A counter example is given to a conjecture of Fisher and Sykes on the number of self-avoiding walks arising in the excluded volume problem in the Ising model of ferromagnetism.

IN their discussion of the excluded volume problem and the Ising model of ferromagnetism, Fisher and Sykes[1] deal extensively with the quantity

$$\mu=\lim_{n\to\infty} c_{n+1}/c_n, \tag{1}$$

where c_n is the number of n-stepped self-avoiding walks on a lattice. They say the assumption that the limit in (1) exists "is by no means obvious mathematically but Hammersley has been able to justify the assumption rigorously." In a footnote to this they add "More precisely Hammersley[2] has shown that

$$\lim_{n\to\infty} n^{-1}\ln c_n=\ln\mu \tag{2}$$

exists." Since these two statements may lead to misunderstanding, it seems worth stating that the actual situation is:

(i) I proved (2), as stated by Fisher and Sykes; but
(ii) (2) is not a more precise statement than (1);
(iii) the limit in (1) does *not* exist in general; and
(iv) (2) is strictly weaker than (1), in that the assumed truth of (1) in itself implies (2) without any reference to those properties of a lattice or a crystal otherwise needed to establish (2).

It is quite easy to construct lattices for which (1) is false, if we permit the lattice to have more than one outlike class (see reference 2 for a definition of "crystal" and "outlike class": All physical lattices are crystals, though not all crystals are lattices). For example, (1) is obviously false for the decorated honeycomb lattice,[3] which has two outlike classes. However, all the lattices considered by Fisher and Sykes are lattices with the special property that each only possesses one outlike class. One might hope that this special property would secure the truth of (1); but this hope is unfulfilled. The counter example below is a lattice with only one outlike class. It may perhaps be true that (1) is true for the particular lattices considered by Fisher and Sykes; but it should be realized that this is a very open conjecture, as yet quite unsubstantiated by mathematical argument.

Counter example to (1): Consider the lattice whose points have coordinates $x=0, \pm 1, \pm 2, \cdots$ on a straight line, and whose bonds are two-way bonds (i.e., bonds which may be traversed in either direction) between nearest neighbors x and $x+1$, there being either one or two such bonds according as x is odd or even. An easy calculation shows that $c_{2n}=2.2^n$ and $c_{2n+1}=3.2^n$; so that $c_{n+1}/c_n=\frac{4}{3}$ or $\frac{3}{2}$ according as n is odd or even, and consequently (1) does not exist. Of course, (2) exists and gives $\ln\mu=\frac{1}{2}\ln 2$.

Proof that (1) implies (2): If (1) is true, then to every $\epsilon>0$ there corresponds $n_0(\epsilon)$ such that

$$|\ln c_{n+1}-\ln\mu-\ln c_n|\leqslant\epsilon, \quad n\geqslant n_0(\epsilon). \tag{3}$$

Hence, for $n>m\geqslant n_0(\epsilon)$,

$$|\ln c_n-(n-m)\ln\mu-\ln c_m|\leqslant\sum_{j=m}^{n-1}|\ln c_{j+1}-\ln\mu-\ln c_j| \leqslant(n-m)\epsilon. \tag{4}$$

Fix m, divide (4) by n, and let $n\to\infty$ with the result

$$\limsup_{n\to\infty}|n^{-1}\ln c_n-\ln\mu|\leqslant\epsilon, \tag{5}$$

and now (2) follows from (5) because ϵ is arbitrary.

[1] M. E. Fisher and M. F. Sykes, Phys. Rev. **114**, 45 (1959).
[2] J. M. Hammersley, Proc. Cambridge Phil. Soc. **53**, 642 (1957).
[3] I. Syozi, Progr. Theoret. Phys. (Kyoto) **6**, 306 (1951).