Electromagnetic Structure of the Nucleon*

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Dispersion relations are proved for the electromagnetic and mesonic nucleon vertex functions considered as a function of the nucleon mass. The results are used to express the isotopic scalar and the isotopic vector electromagnetic form factors of the nucleon in terms of pion electroproduction (or photoproduction) and pion-nucleon scattering amplitudes in the $J=\frac{1}{2}$, $T=\frac{1}{2}$ state.

I. INTRODUCTION

In the past various attempts have been made to study the electromagnetic structure of the nucleon. In particular Chew, Karplus, Gasiorowicz, and Zachariasen[1] and Federbush, Goldberger, and Treiman[2] attacked the problem using dispersion relation techniques. The quantity to be investigated is the matrix element $\langle p'|j_\mu(0)|p\rangle$ where $\langle p'|$ and $|p\rangle$ are one-nucleon states of indicated four momentum and $j_\mu(0)$ is the electromagnetic current: $j_\mu(x)=-(\partial/\partial x_\nu)^2 A_\mu(x)$, where $A_\mu(x)$ is the photon field operator. In isotopic spin space $\langle p'|j_\mu(0)|p\rangle$ transforms in part like a vector and in part like a scalar. For practical reasons the above-mentioned authors were unable to study the isotopic scalar part of the nucleon structure and had to confine their calculations to the isotopic vector part. Furthermore the validity of certain analyticity properties of the theory had to be assumed.

In this paper a different approach which does not suffer from the above difficulties is proposed. It is based on the observation that the various invariant functions describing the electromagnetic structure of the nucleon depend on the three scalars in the problem: p^2, p'^2, and $q^2=(p-p')^2$. Any of the three may be chosen as the variable to be continued to complex values with the other two treated as fixed real parameters. Chew *et al.*[1] and Federbush *et al.*[2] choose to continue q^2 whereas we shall choose p^2. As a consequence we are able to study both the isotopic scalar and vector parts of the nucleon structure. Furthermore the validity of all the analyticity properties required in our approach can be proved rigorously.

Depending on the asymptotic behavior of the functions under study the dispersion relations do or do not require "subtractions." In general this asymptotic behavior is not known. For a local field such subtractions must be in the form of a polynomial in the dispersion variable with coefficients that depend on the remaining parameters in the problem. Since it is the latter dependence (i.e., dependence on q^2) that we are interested in we cannot afford, in general, to have subtracted dispersion relations. If the function under study is known for some fixed value of the dispersion variable then we can afford to have a once-subtracted dispersion relation for that function. Accordingly we shall assume that nonsubtracted dispersion relations are valid for one of the form factors and once-subtracted relations are valid for the other.

In Sec. 2 the dispersion relations in p^2 are constructed and proved. In Sec. 3 the absorptive part of the form factors is calculated in the usual approximation, i.e., only lowest mass intermediate states are considered. In this way the electromagnetic form factors are expressed in terms of amplitudes for pion electroproduction (or photoproduction) in the $J=\frac{1}{2}$, $T=\frac{1}{2}$ state, and the pion-nucleon vertex function. The latter is evaluated by the same techniques in Sec. 4 and expressed in terms of pion-nucleon scattering phase shifts in the $J=\frac{1}{2}$, $T=\frac{1}{2}$ state. Although these $J=\frac{1}{2}$, $T=\frac{1}{2}$ amplitudes and phase shifts are in principle measurable experimentally, they are at present either not known or known but poorly. A theoretical determination of these quantities is being considered.

2. DERIVATION OF DISPERSION RELATIONS FOR THE ELECTROMAGNETIC FORM FACTORS

The matrix element of the current operator taken between one-nucleon states may be written as follows:

$$\langle p's'|j_\mu(0)|ps\rangle=\frac{\bar{u}_\alpha(p's')}{(p_0'/M)^{\frac{1}{2}}}(\Gamma_\mu)_{\alpha\beta}\frac{u_\beta(ps)}{(p_0/M)^{\frac{1}{2}}}. \tag{1}$$

Here $\bar{u}_\alpha(k,r)=u_\beta^*(k,r)(\gamma_4)_{\beta\alpha}$ and $u_\beta(k,r)$ is the spinor describing a nucleon of momentum k and spin r. It is normalized according to

$$\bar{u}_\alpha(k,r)u_\alpha(k,r')=\pm\delta_{r,r'},$$

the upper sign to be used for positive energies ($r=1,2$) and the lower for negative ($r=3,4$). In the following we shall usually omit spinor indices (such as α, β above) whenever the meaning is clear. Our Dirac matrices are such that $\boldsymbol{\gamma}=i\boldsymbol{\alpha}\beta$, $\gamma_4=\beta$, $\gamma_\mu\gamma_\nu=\delta_{\mu\nu}+i\sigma_{\mu\nu}$. If a_μ and b_μ are two four vectors, then we define their scalar product as $a\cdot b=a_\mu b_\mu=\mathbf{a}\cdot\mathbf{b}-a_0b_0$. Finally M is the nucleon mass.

It follows from invariance under the Lorentz group

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[1] G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. 110, 265 (1958).

[2] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. 112, 642 (1958).

that the structure of Γ_μ must be

$$\Gamma_\mu=\sum_{j,k=0,1}(i\gamma\cdot p')^j[i\gamma_\mu A_1{}^{jk}+i\sigma_{\mu\nu}q_\nu A_2{}^{jk}+q_\mu A_3{}^{jk}](i\gamma\cdot p)^k, \quad (2)$$

where $q_\mu=p_\mu-p_\mu'$ and $A_i{}^{jk}$ are functions of the three scalars in the problem which may be chosen as q^2, $W'^2=-p'^2$ and $W^2=-p^2$. Since p' will always represent a physical nucleon we may set $W'=M$. Thus $A_i{}^{jk}\equiv A_i{}^{jk}(q^2,M^2,W^2)$. Using Eq. (2) and

$$\bar{u}(p's')(i\gamma\cdot p'+M)=0;\quad s'=1,2,$$

we may write

$$\begin{aligned}\bar{u}(p's')\Gamma_\mu&=\bar{u}(p's')\{[i\gamma_\mu F_1(W)+i\sigma_{\mu\nu}q_\nu F_2(W)+q_\mu F_3(W)]\\&\times(W-i\gamma\cdot p)/2W+[i\gamma_\mu F_1(-W)+i\sigma_{\mu\nu}q_\nu F_2(-W)\\&+q_\mu F_3(-W)](W+i\gamma\cdot p)/2W\},\quad (3)\end{aligned}$$

where the $F_i(\pm W)\equiv F_i(q^2,M,\pm W)$ are certain linear combinations of the $A_i{}^{jk}$ (by definition, M and W are positive).

The number of invariant functions may be further reduced by making use of the requirement that the theory be gauge invariant, i.e., that the vertex function satisfy the generalized Ward identity.[3] In terms of our functions $F_i(W)$ the generalized Ward identity reads

$$(M\mp W)F_1(\pm W)+q^2F_3(\pm W)=(M\mp W)e_N,\quad (4)$$

where e_N is the nucleon charge. We use Eq. (4) to eliminate $F_1(\pm W)$ from Eq. (3) and obtain

$$\begin{aligned}\bar{u}(p's')(\Gamma_\mu-i\gamma_\mu e_N)=\bar{u}(p's')\Bigg\{&\Bigg[i\sigma_{\mu\nu}q_\nu F_2(W)\\&+\left(q_\mu-\frac{q^2}{M-W}i\gamma_\mu\right)F_3(W)\Bigg]\frac{W-i\gamma\cdot p}{2W}\\&+\Bigg[i\sigma_{\mu\nu}q_\nu F_2(-W)\\&+\left(q_\mu-\frac{q^2}{M+W}i\gamma_\mu\right)F_3(-W)\Bigg]\frac{W+i\gamma\cdot p}{2W}\Bigg\}.\quad (5)\end{aligned}$$

The form factors $F_1(q^2)$ (charge structure) and $F_2(q^2)$ (magnetic moment structure) that appear in the literature[1,2] are related to our functions by

$$\begin{aligned}F_1(q^2)&=e_N+q^2F_3'(q^2,M,M),\\F_2(q^2)&=F_2(q^2,M,M),\end{aligned}\quad (6)$$

where the prime denotes differentiation with respect to W and where we have used

$$F_3(q^2,M,M)=0.\quad (7)$$

[3] Y. Takahashi, Nuovo cimento **6**, 371 (1957).

Equation (7) is a consequence of invariance of the theory under space and time inversion which requires that

$$\begin{aligned}A_{1,2}{}^{jk}(q^2,M^2,W^2)&=A_{1,2}{}^{kj}(q^2,W^2,M^2),\\A_3{}^{jk}(q^2,M^2,W^2)&=-A_3{}^{kj}(q^2,W^2,M^2).\end{aligned}$$

Using the reduction formalism[4] to "take out" the nucleon p from $\langle p's'|j_\mu(0)|ps\rangle$ we obtain

$$\bar{u}(p's')\Gamma_\mu=(p_0'/M)^{\frac{1}{2}}i\int d^4x \times e^{ip\cdot x}\theta(-x_0)\langle p's'|[j_\mu(0),\bar{\eta}(x)]|0\rangle,\quad (8)$$

where $\theta(x_0)=\frac{1}{2}(1+x_0/|x_0|)$ and $\bar{\eta}(x)=(-\gamma_\mu{}^T\partial/\partial x_\mu+M)\bar{\psi}(x)$ with $\psi(x)$ the nucleon field operator. In writing Eq. (8) we have left out an equal-time commutator which can have an effect only on the "subtraction" terms in the final dispersion relations.

Consequently

$$\begin{aligned}F_i(\pm W)=\sum_{s'=1}^{2}(p_0'/M)^{\frac{1}{2}}i\int d^4x&\\\times e^{ip\cdot x}\theta(-x_0)\langle p's'|[j_\mu(0),\bar{\eta}(x)]|0\rangle&\\\times\nu_\mu{}^i(\pm W)u(p's'),\quad i=2,3,&\quad (9)\end{aligned}$$

where the $\nu_\mu{}^i(\pm W)$ are appropriately constructed so as to project the $F_i(\pm W)$ out of Γ_μ. The actual values of $\nu_\mu{}^i(\pm W)$ are of no importance (they are given in Appendix A)—the need for them arises solely from the irrelevant (for the purposes of proving dispersion relations) fact that nucleons and photons have nonzero spin.

To establish dispersion relations we go into a special frame of reference—the rest frame of p'. In this frame we have

$$\begin{aligned}p'&=(0,M),\\p&=(\xi h,p_0),\\q&=(\xi h,\,p_0-M),\end{aligned}\quad (10)$$

and we may consider p_0 as the dispersion variable. We note that

$$p_0=(W^2+M^2+q^2)/(2M),\quad (11)$$

$$h^2=(p_0-M)^2+q^2=[(W^2-M^2)^2+2q^2(W^2+M^2)+q^4]/(4M^2),\quad (12)$$

and ξ is an arbitrary unit vector. In this frame we may write

$$F_i(\pm W)=\int_0^\infty dr\int_{-\infty}^\infty dx_0\,e^{ip_0x_0}f_i(\pm W;x_0,r),\quad r=|\mathbf{x}|.\quad (13)$$

The actual expression for the $f_i(\pm W;x_0,r)$ is given in Appendix B. The only thing that concerns us here is

[4] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 205 (1955).

the fact that the $f_i(\pm W; x_0, r)$ vanish for $x_0<0$ and $x_0<r$ [composed as they are of $\theta(x_0)$ and a commutator which vanishes for space-like x] and have no singularities as a function of W, the latter property following from the expressions given in Appendix B where the W dependence of the f_i is explicitly exhibited.

Consequently, for fixed r, the functions

$$F_i^+(p_0; r) \equiv \frac{1}{2}\int_{-\infty}^{\infty} dx_0\, e^{ip_0x_0}[f_i(W; x_0,r) + f_i(-W; x_0,r)], \quad (14)$$

and

$$F_i^-(p_0; r) \equiv \frac{1}{2W}\int_{-\infty}^{\infty} dx_0\, e^{ip_0x_0}[f_i(W; x_0,r) - f_i(-W; x_0,r)], \quad (15)$$

satisfy all the requirements necessary for the Hilbert relation

$$\mathrm{Re}F_i^{\pm}(p_0; r) = \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty} \frac{dp_0'}{p_0'-p_0}\, \mathrm{Im}F_i^{\pm}(p_0'; r), \quad (16)$$

to hold.[5] In writing Eq. (16) we have assumed that no subtractions were necessary—we shall return to this question later.

Introducing Eq. (16) into Eq. (13) and inverting the order of dp_0' and dr integrations we obtain the desired dispersion relations:

$$\mathrm{Re}F_i^{\pm}(p_0) = \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty} \frac{dp_0'}{p_0'-p_0}\, \mathrm{Im}F_i^{\pm}(p_0'). \quad (17)$$

The interchange of order of integration is permissible provided that h is never imaginary for p_0 in the range $-\infty<p_0<+\infty$. Since we are only interested in proving the dispersion relations for space-like (or light-like) q we have $q^2 \geqq 0$, and therefore $h=[(p_0-M)^2+q^2]^{\frac{1}{2}}$ indeed cannot be imaginary.

At this point it may be worthwhile to point out the difference in the derivation of dispersion relations in $W^2=-p^2$ and in q^2. In the latter case it is convenient to rewrite h as $[(q_0+M)^2+p^2]^{\frac{1}{2}}$ with q_0 as the dispersion variable; and we wish to prove the dispersion relations for p *time*-like, i.e., $p^2<0$ (in fact $p^2=-M^2$). Hence for some q_0 in the range $-\infty<q_0<+\infty$ h will be imaginary and the order of integration cannot be simply interchanged. Of course, this does not mean that dispersion relations cannot be proved but rather that the method successfully used above when the dispersion variable was p_0 fails when the dispersion variable is q_0. As yet no other method has been devised to prove rigorously the dispersion relations in q_0.

Equation (17) is a dispersion relation in p_0 (or W^2) for the functions $F_i^{\pm}(p_0)$. We shall now show that $\mathrm{Im}F_i^{\pm}(p_0)$ vanishes for $p_0<M+\mu+(q^2+\mu^2)/(2M)$ where μ is the pion mass, and this circumstance will permit us to convert Eq. (17) into a dispersion relation in W for the functions $F_i(\pm W)$.

It follows from invariance of the theory under space and time inversions that the imaginary or absorptive part of the various functions is obtained by replacing $i\theta(-x_0)$ in Eq. (8) by $\frac{1}{2}$[6]:

$$\begin{aligned}
\bar{u}(p's')\Gamma_\mu{}^A &= (p_0'/M)^{\frac{1}{2}}\frac{1}{2}\int d^4x\, e^{ip\cdot x}\langle p's'|[j_\mu(0),\bar{\eta}(x)]|0\rangle \\
&= (p_0'/M)^{\frac{1}{2}}\frac{1}{2}\int d^4x\, e^{ip\cdot x}\Big\{\sum_n\int\frac{d^3k}{(2\pi)^3} \\
&\quad\times\langle p's'|j_\mu(0)|n,k\rangle\langle n,k|\bar{\eta}(0)|0\rangle e^{-ik\cdot x} \\
&\quad-\sum_{n'}\int\frac{d^3k'}{(2\pi)^3}\langle p's'|\bar{\eta}(0)|n',k'\rangle \\
&\qquad\times\langle n',k'|j_\mu(0)|0\rangle e^{-i(p'-k')\cdot x}\Big\}, \quad (18)
\end{aligned}$$

where we have introduced a complete set of states labeled n (or n') with rest-energy M_n and momentum $k=[\mathbf{k}, +(\mathbf{k}^2+M_n{}^2)^{\frac{1}{2}}]$ and where

$$\begin{aligned}
\bar{u}(p's')\Gamma_\mu{}^A &= \bar{u}(p's')\bigg\{\bigg[i\sigma_{\mu\nu}q_\nu\,\mathrm{Im}F_2(W)+\Big(q_\mu-\frac{q^2}{M-W}i\gamma_\mu\Big) \\
&\quad\times\mathrm{Im}F_3(W)\bigg]\frac{W-i\gamma\cdot p}{2W}+\bigg[i\sigma_{\mu\nu}q_\nu\,\mathrm{Im}F_2(-W) \\
&\quad+\Big(q_\mu-\frac{q^2}{M+W}i\gamma_\mu\Big)\mathrm{Im}F_3(-W)\bigg]\frac{W+i\gamma\cdot p}{2W}\bigg\}. \quad (19)
\end{aligned}$$

For the same reasons that were outlined above we may interchange the order of d^4x and d^3k integrations and obtain

$$\begin{aligned}
\bar{u}(p's')\Gamma_\mu{}^A &= (p_0'/M)^{\frac{1}{2}}\pi\Big\{\sum_n 2p_0\theta(p_0)\delta(W^2-M_n{}^2) \\
&\quad\times\langle p's'|j_\mu(0)|n,p\rangle\langle n,p|\bar{\eta}(0)|0\rangle+\sum_{n'}2q_0\theta(-q_0) \\
&\quad\times\delta(q^2+M_{n'}{}^2)\langle p's'|\bar{\eta}(0)|n',-q\rangle \\
&\qquad\times\langle n',-q|j_\mu(0)|0\rangle\Big\}. \quad (20)
\end{aligned}$$

The state n must be a state of nucleon number one. It cannot be the one-nucleon state because[4]

$$\langle \text{one nucleon}|\bar{\eta}(0)|0\rangle=0, \quad (21)$$

[5] The symbol $\mathcal{P}$ denotes principal value.

[6] See, e.g., Appendix to the paper by R. Oehme, Phys. Rev. **100**, 1503 (1955).

but it can be a state of one nucleon and any number of pions, K-$\bar{K}$ pairs, and baryon-antibaryon pairs (photons are excluded because we are only calculating to lowest order in the electromagnetic coupling). The lightest such state is the one-nucleon plus one-pion state for which $M_n \geqq M+\mu$. On the other hand the state n' must be a state of nucleon number zero. Furthermore, the argument of the delta-function $\delta(q^2+M_{n'}^2)$ can vanish only if $M_{n'}=0$ since $q^2 \geqq 0$. The only state that can satisfy these requirements is the vacuum state and it will not contribute because of Eq. (21). Thus we conclude that the summation over n' may be ignored and consequently $\mathrm{Im}F_i(\pm W)$ vanishes for $W^2<(M+\mu)^2$.

For the magnetic moment form factor F_2 we assume that no subtractions are necessary and obtain from Eq. (17)

$$\mathrm{Re}F_2(M) =\frac{\mathcal{P}}{\pi}\int_{M+\mu}^{\infty} dW\left[\frac{\mathrm{Im}F_2(W)}{W-M}+\frac{\mathrm{Im}F_2(-W)}{W+M}\right]. \quad (22)$$

For the charge form factor F_3 we make use of Eq. (7) to write a once-subtracted dispersion relation so that

$$\mathrm{Re}F_3'(M) =\frac{\mathcal{P}}{\pi}\int_{M+\mu}^{\infty} dW\left[\frac{\mathrm{Im}F_3(W)}{(W-M)^2}+\frac{\mathrm{Im}F_3(-W)}{(W+M)^2}\right], \quad (23)$$

where the prime denotes differentiation.

As stated in the Introduction, we do not know how many subtractions are needed. Pending a study of the asymptotic behavior of these functions our choice of no subtractions for F_2 and one subtraction for F_3 must be viewed as a postulate.

3. THE ABSORPTIVE PART OF THE ELECTROMAGNETIC FORM FACTORS

The desired electromagnetic form factors [see Eq. (6)] are given by Eqs. (22) and (23) in terms of the absorptive parts $\mathrm{Im}F_i(\pm W)$. It follows from Sec. 2, particularly Eq. (20), that we have

$$\mathrm{Im}F_i(W) =\sum_{s'=1}^{2} (p_0'/M)^{\frac{1}{2}}\pi \sum_n 2p_0\theta(p_0)\delta(W^2-M_n^2) \times\langle p's'|j_\mu(0)|n,p\rangle\langle n,p|\bar{\eta}(0)|0\rangle\nu_\mu{}^i(W)u(p's'), \quad (24)$$

where the sum over n runs over all states of nucleon number one, rest-mass M_n and total four momentum p. The lowest mass such state is the one-nucleon plus one-pion state (since the one-nucleon state does not contribute), for which $M_n \geqq M+\mu$. Its contribution to Eq. (24) may be written as

$$\begin{aligned}\mathrm{Im}^{N\pi}F_i(W) &=\sum_{s'=1}^{2} (p_0'/M)^{\frac{1}{2}}\pi \sum_\tau \int d^4p''d^4l\,(2\pi)^{-3}(2p_0'') \\ &\times(2l_0)\theta(p_0'')\theta(l_0)\delta(p''^2+M^2)\delta(l^2+\mu^2) \\ &\times\delta^{(4)}(p''+l-p)\langle p's'|j_\mu(0)|(p'',l)_{in}\rangle \\ &\times\langle(p'',l)_{in}|\bar{\eta}(0)|0\rangle\nu_\mu{}^i(W)u(p's'), \quad (25)\end{aligned}$$

where $\sum_\tau$ refers to the sum over the discrete quantum numbers (spin, isospin) of the (p'',l) system and where we have chosen to take for the complete set of states n the states with the "in" convention. In going from Eq. (24) to Eq. (25) some mass-shell delta-functions were added to extend the three-dimensional integrations to four dimensions.

Because of the structure of the dispersion relations one hopes that the contribution from the lowest mass intermediate states dominates all other contributions. Accordingly we shall approximate $\mathrm{Im}F_i(W)$ by $\mathrm{Im}^{N\pi}F_i(W)$ and drop the superscript $N\pi$ in order not to complicate the notation.

The factor $\langle(p'',l)_{in}|\bar{\eta}(0)|0\rangle$ appearing in Eq. (25) is related to the pion-nucleon vertex function. We may write

$$\langle 0|\eta_\alpha(0)|(p''s'',l\lambda)_{in}\rangle =-ig(\tau_\rho\Gamma)_{\alpha\beta}\frac{u_\beta(p''s'')}{[p_0''/M]^{\frac{1}{2}}}\frac{\epsilon_\rho(\lambda)}{(2l_0)^{\frac{1}{2}}}, \quad (26)$$

where s'' describes the spin (and isotopic spin) of the nucleon p'', λ describes the isotopic spin of the pion l, τ_ρ is the usual Pauli matrix operating in isotopic spin space and $\epsilon_\rho(\lambda)$ is a unit vector in isotopic spin space. The constant g is related to the pion-nucleon coupling constant[7] and will be specified more precisely below. From invariance of the theory under the Lorentz group one has

$$\Gamma=\sum_{i,j=0,1} (i\gamma\cdot p)^i\gamma_5(i\gamma\cdot p'')^jB^{ij}, \quad (27)$$

where $p_\mu=p_\mu''+l_\mu$ and the B^{ij} are functions of the three scalars in the problem: l^2, p''^2, and p^2. Thus $B^{ij}\equiv B^{ij}(-l^2, -p''^2, -p^2)$ and we shall only need in Eq. (25) $B^{ij}(\mu^2,M^2,W^2)$. It will be convenient to write

$$\Gamma u(p''s'')=\left[\frac{W-i\gamma\cdot p}{2W}K(W)+\frac{W+i\gamma\cdot p}{2W}K(-W)\right] \times\gamma_5 u(p''s''), \quad (28)$$

where $K(\pm W)\equiv K(\mu^2,M,\pm W)$ is a certain linear

[7] The pion-nucleon coupling constant is more conventionally defined in terms of the matrix element $\langle p|J(0)|p''\rangle$ of the pion current. For a discussion showing that we are dealing here with the same constant see M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. 2, 226 (1957), Sec. IV.

combination of the B^{ij}. If we normalize $K(W)$ according to

$$K(M)=1, \tag{29}$$

then g is the Lepore-Watson coupling constant.[8] In the next section $K(\pm W)$ will be evaluated in terms of pion-nucleon scattering phase shifts.

The other factor in Eq. (25), $\langle p's'|j_\mu(0)|(p'',l)_{in}\rangle$, is related to pion electroproduction ($q^2>0$) or photoproduction ($q^2=0$). It is needed in Eq. (25) only for such values of the total energy and momentum transfer that lie in the *physical* range of these variables. Consequently it may be taken over directly from experiment. We may write

$$\begin{aligned}e_\mu{}^*\langle p's'|j_\mu(0)|(p''s'',l\lambda)_{in}\rangle\\ =\left[\frac{M^2}{2l_0p_0'p_0''}\right]^{\frac{1}{2}}\bar{u}(p's')(M_AA_\rho+M_BB_\rho+M_CC_\rho\\ +M_DD_\rho+M_EE_\rho+M_FF_\rho)u(p''s'')\epsilon_\rho(\lambda),\end{aligned} \tag{30}$$

where, as a consequence of Lorentz and gauge invariance

$$\begin{aligned}M_A&=-\tfrac{1}{2}i\gamma_5(\gamma,\gamma),\\ M_B&=i\gamma_5(p'+p'',l),\\ M_C&=-\gamma_5(\gamma,l),\\ M_D&=-\gamma_5[(\gamma,p'+p'')-\tfrac{1}{2}iM(\gamma,\gamma)],\\ M_E&=i\gamma_5(q,l),\\ M_F&=-\gamma_5(q,\gamma).\end{aligned} \tag{31}$$

Here e^* is the photon polarization four vector, $q=p''+l-p'=p-p'$ is the photon momentum four vector, and

$$(a,b)\equiv a\cdot e^*b\cdot q-a\cdot qb\cdot e^*.$$

Charge independence implies that the isotopic spin dependence of $A_\rho,\cdots F_\rho$ is given by

$$A_\rho=\tfrac{1}{2}\{\tau_3,\tau_\rho\}A^++\tfrac{1}{2}[\tau_3,\tau_\rho]A^-+\tau_\rho A^0,\text{ etc.} \tag{32}$$

With these definitions our $A^{+,-,0}$, etc., are precisely the same as the ones defined for the pion electroproduction process by Fubini, Nambu, and Wataghin.[9]

Noting that Eqs. (26), (27), and (28) imply

$$\begin{aligned}\langle(p''s'',l\lambda)_{in}|\bar{\eta}(0)|0\rangle\\ =-ig\frac{\epsilon_\rho{}^*(\lambda)}{(2l_0)^{\frac{1}{2}}}\frac{\bar{u}(p''s'')}{[p_0''/M]^{\frac{1}{2}}}\tau_\rho\gamma_5\\ \times\left[\frac{W-i\gamma\cdot p}{2W}K^*(W)+\frac{W+i\gamma\cdot p}{2W}K^*(-W)\right],\end{aligned} \tag{33}$$

and making use of

$$\sum_\lambda \epsilon_\rho(\lambda)\epsilon^*{}_{\rho'}(\lambda)=\delta_{\rho\rho'}, \tag{34}$$

we obtain after substitution of all these definitions into Eq. (25):

$$\begin{aligned}\mathrm{Im}F_i(W)\\ =-ig\sum_{s'=1}^{2}\pi\int d^4p''d^4l\,(2\pi)^{-3}\theta(p_0'')\theta(l_0)\\ \times\delta(p''^2+M^2)\delta(l^2+\mu^2)\delta^{(4)}(p''+l-p)\bar{u}(p's')\\ \times(H_\mu{}^S+\tau_3H_\mu{}^V)(M-i\gamma\cdot p'')\gamma_5\left[\frac{W-i\gamma\cdot p}{2W}K^*(W)\right.\\ \left.+\frac{W+i\gamma\cdot p}{2W}K^*(-W)\right]\nu_\mu{}^i(W)u(p's'),\end{aligned} \tag{35}$$

where

$$e_\mu{}^*H_\mu{}^S=M_AA^S+\cdots+M_FF^S,\ A^S=3A^0,\text{ etc.}, \tag{36}$$

$$e_\mu{}^*H_\mu{}^V=M_AA^V+\cdots+M_FF^V,\ A^V=A^++2A^-,\text{ etc.} \tag{37}$$

Equations (36) and (37) show that we are concerned only with those electroproduction amplitudes for which the pion-nucleon system is in an eigenstate of total isotopic spin T with the eigenvalue $T=\frac{1}{2}$.

Since $\mathrm{Im}F_i(W)$ is a Lorentz invariant the right-hand side of Eq. (35) may be evaluated in any frame of reference. The most convenient frame turns out to be the barycentric frame of the electroproduction process, which is also the rest frame of p:$p=(0,W)$. We define in this frame

$$\begin{aligned}p'&=(\mathbf{p}_1,E_1),\\ p''&=(\mathbf{p}_2,E_2),\\ x&=\cos\theta=\mathbf{p}_1\cdot\mathbf{p}_2/(|\mathbf{p}_1|\,|\mathbf{p}_2|),\end{aligned} \tag{38}$$

and the relation of these quantities to the invariants is

$$\begin{aligned}x&=(p'\cdot p''+E_1E_2)/(|\mathbf{p}_1|\,|\mathbf{p}_2|),\\ E_1&=(\mathbf{p}_1{}^2+M^2)^{\frac{1}{2}}=-p\cdot p'/W\\ &=(W^2+M^2+q^2)/(2W),\\ E_2&=(\mathbf{p}_2{}^2+M^2)^{\frac{1}{2}}=-p\cdot p''/W\\ &=(W^2+M^2-\mu^2)/(2W).\end{aligned} \tag{39}$$

The amplitudes $A,\cdots F$ are functions of three scalars (if one ignores the dependence on the masses of particles that are on their mass-shells); in the barycentric frame these may be taken as q^2, W^2, and x. To perform the integrations in Eq. (35) only the x dependence must be known and this dependence may be made explicit by a multipole expansion.

Let

$$\bar{u}(p's')H_\mu e_\mu{}^*u(p''s'')=-4\pi(W/M)\chi^*(s')\mathfrak{F}\chi(s''), \tag{40}$$

where $\chi(s)$ is a two-component Pauli spinor and[10]

[8] J. Lepore and K. M. Watson, Phys. Rev. **76**, 1157 (1949).

[9] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958).

[10] The fourth component of e^* has been eliminated by using $q\cdot e^*=0$.

$$\mathfrak{F}=i\boldsymbol{\sigma}\cdot\mathbf{e}^*\mathfrak{F}_1-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1\times\mathbf{e}^*\boldsymbol{\sigma}\cdot\mathbf{p}_2}{|\mathbf{p}_1||\mathbf{p}_2|}\mathfrak{F}_2+\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}_1\mathbf{p}_2\cdot\mathbf{e}^*}{|\mathbf{p}_1||\mathbf{p}_2|}\mathfrak{F}_3$$
$$+\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}_2\mathbf{p}_2\cdot\mathbf{e}^*}{\mathbf{p}_2^2}\mathfrak{F}_4+\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}_1\mathbf{p}_1\cdot\mathbf{e}^*}{\mathbf{p}_1^2}\mathfrak{F}_5+\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}_2\mathbf{p}_1\cdot\mathbf{e}^*}{|\mathbf{p}_1||\mathbf{p}_2|}\mathfrak{F}_6. \quad (41)$$

The $\mathfrak{F}_i$, $i=1,\cdots,6$, are certain linear combinations of the $A,\cdots,F$ defined by the above equations. The definitions are such that these $\mathfrak{F}_i$, $i=1,2,3,4$, correspond to the amplitudes $\mathfrak{F}_i$ defined by Chew, Goldberger, Low, and Nambu[11] for the photoproduction process. The multipole expansion of $\mathfrak{F}_i$, $i=1,2,3,4$, is given by[11]

$$\mathfrak{F}_1=\sum_{l=0}^{\infty}[lM_{l+}+E_{l+}]P_{l+1}'(x)+[(l+1)M_{l-}+E_{l-}]P_{l-1}'(x), \quad (42)$$

$$\mathfrak{F}_2=\sum_{l=1}^{\infty}[(l+1)M_{l+}+lM_{l-}]P_l'(x), \quad (43)$$

$$\mathfrak{F}_3=\sum_{l=1}^{\infty}[E_{l+}-M_{l+}]P_{l+1}''(x)+[E_{l-}+M_{l-}]P_{l-1}''(x), \quad (44)$$

$$\mathfrak{F}_4=\sum_{l=1}^{\infty}[M_{l+}-E_{l+}-M_{l-}-E_{l-}]P_l''(x), \quad (45)$$

and the expansion for $i=5,6$ is given by

$$\mathfrak{F}_5=-\mathfrak{F}_1-x\mathfrak{F}_3+\sum_{l=0}^{\infty}[(l+1)L_{l+}P_{l+1}'(x)-lL_{l-}P_{l-1}'(x)], \quad (46)$$

$$\mathfrak{F}_6=-x\mathfrak{F}_4+\sum_{l=1}^{\infty}[lL_{l-}-(l+1)L_{l+}]P_l'(x). \quad (47)$$

The amplitudes $M_{l\pm}$, $E_{l\pm}$, $L_{l\pm}$ refer to electroproduction due to magnetic, electric or longitudinal multipoles in which the pion-nucleon system has orbital angular momentum l and total angular momentum $l\pm\frac{1}{2}$. Their isotopic spin dependence follows from Eq. (32). They are independent of x but are functions of W and q^2. In particular for $q^2=0$ the longitudinal amplitudes $L_{l\pm}$ vanish and the magnetic and electric amplitudes go over into those of the photoproduction process.

We are now ready to perform the integrations in Eq. (37). The result is

$$\mathrm{Im}F_2^{V,S}(W)=-\frac{g}{W}\frac{|\mathbf{p}_2|}{|\mathbf{p}_1|}\left(\frac{E_2-M}{E_1+M}\right)^{\frac{1}{2}}K^*(W)\times[(W+M)M_{1-}^{V,S}(W)-(W-E_1)L_{1-}^{V,S}(W)], \quad (48)$$

$$\mathrm{Im}F_3^{V,S}(W)=-\frac{g}{W}\frac{|\mathbf{p}_2|}{|\mathbf{p}_1|}\left(\frac{E_2-M}{E_1+M}\right)^{\frac{1}{2}}\frac{W-M}{q^2}K^*(W)\times[q^2M_{1-}^{V,S}(W)+(W-E_1)(W+M)L_{1-}^{V,S}(W)]. \quad (49)$$

It is seen that only the magnetic dipole and longitudinal monopole enter as was to be expected since only these multipoles could be emitted or absorbed by a nucleon without changing its total angular momentum and parity.

The remaining functions needed in Eqs. (22) and (23) are

$$\mathrm{Im}F_2^{V,S}(-W)=-\frac{g}{W}\frac{|\mathbf{p}_2|}{|\mathbf{p}_1|}\left(\frac{E_2+M}{E_1-M}\right)^{\frac{1}{2}}K^*(-W)\times[(W-M)E_{0+}^{V,S}(W)-(W-E_1)L_{0+}^{V,S}(W)], \quad (50)$$

$$\mathrm{Im}F_3^{V,S}(-W)=-\frac{g}{W}\frac{|\mathbf{p}_2|}{|\mathbf{p}_1|}\left(\frac{E_2+M}{E_1-M}\right)^{\frac{1}{2}}\frac{W+M}{q^2}K^*(-W)\times[q^2E_{0+}^{V,S}(W)+(W-E_1)(W-M)L_{0+}^{V,S}(W)], \quad (51)$$

and here it is the electric and longitudinal dipoles that enter. (These are the only multipoles that could be emitted or absorbed by a nucleon without changing its total angular momentum but with a change in parity.)

Introducing Eqs. (48), (49), (50), and (51) into Eqs. (22) and (23) we obtain for the $F_i(q^2)$ [see Eq. (6)] the following expressions:

$$\mathrm{Re}F_1^{V,S}(q^2)=e_N^{V,S}-\frac{g}{\pi}\mathcal{P}\int_{M+\mu}^{\infty}\frac{dW}{W}\frac{|\mathbf{p}_2|}{|\mathbf{p}_1|}\left\{\left(\frac{E_2-M}{E_1+M}\right)^{\frac{1}{2}}\frac{K^*(W)}{W-M}\times[q^2M_{1-}^{V,S}+(W-E_1)(W+M)L_{1-}^{V,S}]-\left(\frac{E_2+M}{E_1-M}\right)^{\frac{1}{2}}\frac{K^*(-W)}{W+M}[q^2E_{0+}^{V,S}+(W-E_1)\times(W-M)L_{0+}^{V,S}]\right\}, \quad (52)$$

$$\mathrm{Re}F_2^{V,S}(q^2)=-\frac{g}{\pi}\mathcal{P}\int_{M+\mu}^{\infty}\frac{dW}{W}\frac{|\mathbf{p}_2|}{|\mathbf{p}_1|}\left\{\left(\frac{E_2-M}{E_1+M}\right)^{\frac{1}{2}}\frac{K^*(W)}{W-M}\times[(W+M)M_{1-}^{V,S}-(W-E_1)L_{1-}^{V,S}]+\left(\frac{E_2+M}{E_1-M}\right)^{\frac{1}{2}}\frac{K^*(-W)}{W+M}[(W-M)E_{0+}^{V,S}-(W-E_1)L_{0+}^{V,S}]\right\}, \quad (53)$$

[11] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

Here $F_i{}^S(q^2)+F_i{}^V(q^2)$ is the electromagnetic form factor of the proton, $F_i{}^S(q^2)-F_i{}^V(q^2)$ is the electromagnetic form factor of the neutron ($i=1$: charge form factor, $i=2$: magnetic moment form factor); $e_N{}^{V,S}=\frac{1}{2}e$, where e is the proton charge; $|\mathbf{p}_1|$, $|\mathbf{p}_2|$, E_1, E_2, are known functions of W and q^2 [see Eq. (39)]. The multipole moments $M_{1-}{}^{V,S}$, $L_{1-}{}^{V,S}$, $E_{0+}{}^{V,S}$, and $L_{0+}{}^{V,S}$ are functions of W and q^2 which in principle can be determined experimentally. There remains to be determined the mesonic form factor of the nucleon $K(W)$ and we address ourselves now to that problem.

4. MESONIC FORM FACTOR OF THE NUCLEON

In analogy to the electromagnetic case we shall refer to $K(W)$ as the mesonic form factor of the nucleon. According to the definitions in Sec. 3 we have

$$\langle 0|\eta(0)|(p''s'',l\lambda)_{in}\rangle$$
$$=-ig\tau_\rho\left[\frac{W-i\gamma\cdot p}{2W}K(W)+\frac{W+i\gamma\cdot p}{2W}K(-W)\right]\gamma_5 \times\frac{u(p''s'')}{(p_0''/M)^{\frac{1}{2}}}\frac{\epsilon_\rho(\lambda)}{(2l_0)^{\frac{1}{2}}}, \quad (54)$$

where $K(\pm W)\equiv K(\mu^2,M,\pm W)$, $K(M)=1$, $p_\mu=p_\mu''+l_\mu$, $p^2=-W^2$, $p''^2=-M^2$, $l^2=-\mu^2$, and g is the Lepore-Watson coupling constant.

We now apply the reduction formalism[4] to the meson l and obtain

$$\langle 0|\eta(0)|(p''s'',l\lambda)_{in}\rangle$$
$$=i\int d^4x\, e^{il\cdot x}\theta(-x_0)\langle 0|[\eta(0),J_\rho(x)]|p''s''\rangle \times\epsilon_\rho(\lambda)/[2l_0]^{\frac{1}{2}}, \quad (55)$$

where we have left out an equal-time commutator. The meson current $J_\rho(x)$ is defined by

$$J_\rho(x)=\left[-\left(\frac{\partial}{\partial x_\nu}\right)^2+\mu^2\right]\varphi_\rho(x), \quad (56)$$

where $\varphi_\rho(x)$ is the meson field operator. Combining Eqs. (54) and (55) we deduce after some straightforward manipulations that

$$K(\pm W)$$
$$=\frac{1}{3g}(Mp_0'')^{\frac{1}{2}}\sum_{s''=1}^{2}\bar{u}(p''s'')\tau_\rho\gamma_5\frac{i\gamma\cdot p\mp W}{\mu^2-(W\mp M)^2} \times\int d^4x\, e^{il\cdot x}\theta(-x_0)\langle 0|[\eta(0),J_\rho(x)]|p''s''\rangle. \quad (57)$$

The W dependence of the right-hand side of Eq. (57) is explicitly displayed by going into the rest frame of p'':

$$p''=(0,M),$$
$$l=(\xi h_l,l_0), \quad (58)$$
$$p=(\xi h_l,M+l_0),$$

with ξ an arbitrary unit vector and

$$h_l=(l_0^2-\mu^2)^{\frac{1}{2}}, \quad (59)$$
$$l_0=(W^2-M^2-\mu^2)/(2M). \quad (60)$$

We can then show that the functions $K^+(l_0)\equiv\frac{1}{2}[K(W)+K(-W)]$ and $K^-(l_0)\equiv\frac{1}{2}[K(W)-K(-W)]/W$ satisfy dispersion relations in l_0.

The proof is quite analogous to that given in Sec. 2 with one difference. The h of Sec. 2 was shown never to be imaginary for all values of interest of the remaining parameters. Here, however, h_l will be imaginary when

$$-\mu<l_0<\mu. \quad (64)$$

Nevertheless dispersion relations can be proved because the absorptive part of the various K functions vanishes for l_0 satisfying condition (61). These absorptive parts are proportional to

$$\int_{-\infty}^{\infty}dx_0\, e^{-ix_0l_0}\langle 0|[\eta(0),J_\rho(x)]|Ms''\rangle$$
$$=\sum_n\int\frac{d^3k}{(2\pi)^3}2\pi\delta(l_0+M-k_0)\langle 0|\eta(0)|n,k\rangle \times\langle n,k|J_\rho(\mathbf{x},0)|Ms''\rangle-\sum_{n'}\int\frac{d^3k'}{(2\pi)^3}2\pi\delta(l_0+k_0') \times\langle 0|J_\rho(\mathbf{x},0)|n',k'\rangle\langle n',k'|\eta(0)|Ms''\rangle, \quad (62)$$

where we have introduced a complete set of physical states n or n'. The state n must be a state of nucleon number one, hence a state of at least one nucleon and one meson, i.e., $k_0\geqq M+\mu$. Similarly n' must be a state of at least one meson, i.e., $k_0'\geqq\mu$. These inequalities combined with the delta functions show that the expression (62) vanishes for l_0 in the range (61). Subsequent integration over d^3x shows further that the absorptive part of the various K functions also vanishes for $l_0<-\mu$ and therefore our final dispersion relations may be written as[12]

$$\mathrm{Re}K(\pm W)$$
$$=\frac{\mathcal{P}}{\pi}\int_{M+\mu}^{\infty}\left[\frac{\mathrm{Im}K(W')}{W'\mp W}+\frac{\mathrm{Im}K(-W')}{W'\pm W}\right]dW', \quad (63)$$

where we assume that no subtractions are necessary.

Approximating the absorptive part of $K(W)$, in the usual way, by the contribution from the lowest mass intermediate state (i.e., the one-nucleon, one-pion state) we find from Eq. (57) by replacing $\theta(-x_0)$ by $(2i)^{-1}$

[12] This method of proof is due to R. Oehme [Nuovo cimento **4**, 1316 (1956)] and has been used by K. Symanzik [Phys. Rev. **105**, 743 (1957)] to derive the same analyticity properties that we prove in this Section. In Symanzik's treatment the inessential complications due to nucleon spin were omitted.

$$\begin{aligned}\mathrm{Im}K(W)&\\ =&-\frac{i}{3g}(Mp_0'')^{\frac{1}{2}}\pi\sum_r\int\frac{d^4l'd^4p'''}{(2\pi)^3}\\ &\times\delta^{(4)}(p-p'''-l')\delta(p'''^2+M^2)\delta(l'^2+\mu^2)\theta(p_0''')\\ &\times\theta(l_0')2p_0'''2l_0'\sum_{s''=1}^{2}\bar{u}(p''s'')\tau_\rho\gamma_5\frac{i\gamma\cdot p-W}{\mu^2-(W-M)^2}\\ &\times\langle 0|\eta(0)|(p''',l')_{in}\rangle\langle(p''',l')_{in}|J_\rho(0)|p''s''\rangle,\quad(64)\end{aligned}$$

where $\sum_r$ denotes the sum over the discrete quantum numbers (spin, isotopic spin) of the pion-nucleon system (p''',l').

The first matrix element in Eq. (64) is just the pion-nucleon vertex under study. The second matrix element is related to pion-nucleon scattering. Using invariance under Lorentz transformations one finds[13]

$$\begin{aligned}&\epsilon_\rho^*(\lambda)\langle p''s''|J_\rho(0)|(p'''s''',l'\lambda')_{in}\rangle\\ &=\left[\frac{M^2}{2l_0'p_0''p_0'''}\right]^{\frac{1}{2}}\bar{u}(p''s'')\\ &\qquad\times\left(A-i\gamma\cdot\frac{l'+l}{2}B\right)u(p'''s'''),\quad(65)\end{aligned}$$

where A and B are functions of the scalars in the scattering problem as well as of the isotopic spin indices λ,λ'. The latter dependence is limited by charge independence to be[13]

$$\begin{aligned}\begin{pmatrix}A\\B\end{pmatrix}=\epsilon_\rho^*(\lambda)\bigg\{&\tfrac{1}{3}\tau_\rho\tau_{\rho'}\begin{pmatrix}A^{\frac{1}{2}}\\B^{\frac{1}{2}}\end{pmatrix}\\ &+(\delta_{\rho\rho'}-\tfrac{1}{3}\tau_\rho\tau_{\rho'})\begin{pmatrix}A^{\frac{3}{2}}\\B^{\frac{3}{2}}\end{pmatrix}\bigg\}\epsilon_{\rho'}(\lambda'),\quad(66)\end{aligned}$$

where the superscripts $\frac{1}{2}$, $\frac{3}{2}$ refer to the total isotopic spin T of the meson-nucleon system.

With these definitions we obtain from Eq. (64)

$$\begin{aligned}\mathrm{Im}K(W)=&\frac{\frac{1}{2}\pi K(W)}{\mu^2-(W-M)^2}\int\frac{d^4l'd^4p'''}{(2\pi)^3}\delta^{(4)}(p-p'''-l')\\ &\times\delta(p'''^2+M^2)\delta(l'^2+\mu^2)\theta(p_0''')\theta(l_0')\\ &\times\mathrm{Tr}\bigg\{(W+i\gamma\cdot p)(M-i\gamma\cdot p''')\\ &\times\left(A^{\frac{1}{2}*}-i\gamma\cdot\frac{l'+l}{2}B^{\frac{1}{2}*}\right)(M-i\gamma\cdot p'')\bigg\}.\quad(67)\end{aligned}$$

The integrations in Eq. (67) are most conveniently performed in the barycentric frame for the scattering process, which is also the rest frame of p. In this frame A and B can be considered as functions of W^2 and $x=\cos\theta$ where θ is the scattering angle. The x dependence may be made explicit by a Legendre polynomial expansion.[14] In this way we find

$$\mathrm{Im}K(\pm W)=[e^{i\alpha(\pm W)}\sin\alpha(\pm W)]^*K(\pm W),\quad(68)$$

where $\alpha(+W)$ is the P-wave $J=\frac{1}{2}$, $T=\frac{1}{2}$ meson-nucleon scattering phase shift and $\beta(+W)=\alpha(-W)$ is the S-wave $J=\frac{1}{2}$, $T=\frac{1}{2}$ meson-nucleon phase shift.[15] Below threshold for meson production $(W<M+2\mu)$ these phase shifts are real. We shall take them to be real also above the threshold which is consistent with the approximation of keeping only the lowest mass intermediate state when evaluating $\mathrm{Im}K(\pm W)$.

Thus we find that we must solve the coupled linear integral equations

$$\begin{aligned}\mathrm{Re}K(\pm W)=\frac{\mathcal{P}}{\pi}\int_{M+\mu}^{\infty}\bigg[&\frac{e^{-i\alpha(W')}\sin\alpha(W')K(W')}{W'\mp W}\\ &+\frac{e^{-i\alpha(-W')}\sin\alpha(-W')K(-W')}{W'\pm W}\bigg]dW',\quad(69)\end{aligned}$$

or, equivalently,

$$\begin{aligned}K(z)=\frac{1}{\pi}\int_{M+\mu}^{\infty}\bigg[&\frac{e^{-i\alpha(W')}\sin\alpha(W')K(W')}{W'-z}\\ &+\frac{e^{-i\alpha(-W')}\sin\alpha(-W')K(-W')}{W'+z}\bigg]dW',\quad(70)\end{aligned}$$

where

$$\begin{aligned}K(W)&=\lim_{\epsilon\to0^+}K(W+i\epsilon),\\ K(-W)&=\lim_{\epsilon\to0^+}K(-W-i\epsilon).\end{aligned}\quad(71)$$

The method for solving equations of the form (70) is originally due to Muskhelishvili.[16] We deduce from Eq. (70) that $K(z)$ is a function analytic in the cut z plane, the cuts going from $+(M+\mu)$ to $+\infty$ and from $-(M+\mu)$ to $-\infty$ and the jump in $K(z)$ across these cuts is given by Eq. (68). Consider now the function $\exp[Q(z)]$ where

$$Q(z)=\frac{1}{\pi}\int_{M+\mu}^{\infty}\left(\frac{\alpha(W')}{W'-z}+\frac{\alpha(-W')}{W'+z}\right)dW',\quad(72)$$

$$\begin{aligned}Q(W)&=\lim_{\epsilon\to0^+}Q(W+i\epsilon),\\ Q(-W)&=\lim_{\epsilon\to0^+}Q(-W-i\epsilon).\end{aligned}\quad(73)$$

[13] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[14] Our functions A,B are precisely the same as defined in reference 13 and the details of the polynomial expansion may be taken over directly from that paper.

[15] The identity $\beta(+W)=\alpha(-W)$ is a special case of the more general identity which relates phase shifts belonging to the same J and T, but different orbital angular momentum l, eigenvalues. In the notation of reference 13 this identity is $\delta_{l^+}(\pm W)=\delta_{(l+1)^-}(\mp W)$.

[16] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff N. V., Groningen, 1953). See also reference 2 and Appendix to paper by S. Okubo, R. E. Marshak, and E. C. G. Sudarshan, Phys. Rev. **113**, 944 (1959).

It is readily verified that $\exp[Q(z)]$ has the same analytic properties as $K(z)$. Of course this solution is not unique and in particular, because of the homogeneous nature of Eq. (70), it may be multiplied by a constant. We make use of this freedom to insure that the normalization $K(M)=1$ is satisfied and so take for our solution

$$K(z)=\exp[Q(z)-Q(M)]. \tag{74}$$

In deriving Eq. (74) it is assumed that the behavior of the phase shifts $\alpha(\pm W)$ is such that the integral $Q(z)$ exists. The solution, Eq. (74), could still be multiplied by a polynomial $P(z)$ such that $P(M)=1$. Under certain reasonable assumptions[2] it can be shown that such a polynomial should not appear.

CONCLUSIONS

We have expressed the electromagnetic form factors of the nucleon in terms of pion electroproduction (or photoproduction) and pion-nucleon scattering amplitudes in the $J=\frac{1}{2}$, $T=\frac{1}{2}$ state. These amplitudes are functions of the total energy W in the barycentric frame and of (in the electroproduction case) q^2, the square of the momentum transfer from the electrons. They are needed for physical values of W and q^2 and therefore can, in principle, be determined experimentally. At present these amplitudes are not known very well and therefore we refrain from giving any numerical results for the form factors. A theoretical approach, based on the Mandelstam[17] representation, to determine these amplitudes is being considered.

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APPENDIX A

The quantities $\nu_\mu{}^i(\pm W)$ that appear in Eq. (9) are defined by

$$F_i(\pm W)=\mathrm{Tr}\{(2M)^{-1}(M-i\gamma\cdot p')\Gamma_\mu\nu_\mu{}^i(\pm W)\}, \tag{A-1}$$

from which we obtain by a straightforward calculation

$$\begin{aligned}\nu_\mu{}^2(\pm W)=&-(i\gamma\cdot p\mp W)(8Mh^2)^{-1}[(M\pm W)^2+q^2]^{-1}\\&\times\{3(M\pm W)[q^2 i\gamma_\mu-(M\mp W)q_\mu]\\&+[2q^2-(M\pm W)^2]i\sigma_{\mu\nu}q_\nu\},\end{aligned} \tag{A-2}$$

$$\begin{aligned}\nu_\mu{}^3(\pm W)=&(i\gamma\cdot p\mp W)(8Mh^2)^{-1}[(M\pm W)^2+q^2]^{-1}\\&\times\{8M^2h^2q^{-2}q_\mu+(M\mp W)[2(M\pm W)^2-q^2]\\&\times[i\gamma_\mu-(M\mp W)q^{-2}q_\mu]\\&+3(M^2-W^2)i\sigma_{\mu\nu}q_\nu\},\end{aligned} \tag{A-3}$$

where h^2 is defined by Eq. (12).

APPENDIX B

The functions $f_i(\pm W;x_0,r)$ appearing in Eq. (13) are defined by

$$\begin{aligned}f_i(\pm W;x_0,r)&\\=&\sum_{s'=1}^{2} ir^2\int d\Omega_{\mathbf{x}}\exp(i\boldsymbol{\xi}\cdot\mathbf{x}h)\theta(x_0)\\&\times\langle Ms'|[j(0),\bar{\eta}(\mathbf{x},-x_0)]|0\rangle\\&\times n_\mu\nu_\mu{}^i(\pm W)u(Ms'),\end{aligned} \tag{B-1}$$

where for convenience we write $j_\mu(0)=j(0)n_\mu$ with n_μ a four vector in the direction of $j_\mu(0)$ and where the $\nu_\mu{}^i(\pm W)$ are evaluated in the rest frame of p'. We recall that $\boldsymbol{\xi}$ is an arbitrary unit three vector. The fact that f_i does not depend on $\boldsymbol{\xi}$ may be made explicit by integrating over the angles of $\boldsymbol{\xi}$. In this way we get

$$\begin{aligned}f_i(\pm W;x_0,r)=&\sum_{s'=1}^{2} ir^2\int d\Omega_{\mathbf{x}}\\&\times\theta(x_0)\langle Ms'|[j(0),\bar{\eta}(\mathbf{x},-x_0)]|0\rangle\\&\times\lambda_i(\pm W;\mathbf{x})u(Ms'),\end{aligned} \tag{B-2}$$

where

$$\lambda_i(\pm W;\mathbf{x})=(4\pi)^{-1}\int d\Omega_{\xi}\exp(i\boldsymbol{\xi}\cdot\mathbf{x}h)n_\mu\nu_\mu{}^i(\pm W). \tag{B-3}$$

Quite explicitly we have

$$\begin{aligned}8M\lambda_2(W;\mathbf{x})&\\=&-2(n_0+\tfrac{1}{3}i\boldsymbol{\gamma}\cdot\mathbf{n})j_0(rh)-i\boldsymbol{\gamma}\cdot\mathbf{x}\bigg[\frac{(W-M)^2+q^2}{M}n_0\\&-(W+M)i\boldsymbol{\gamma}\cdot\mathbf{n}\bigg]\frac{j_1(rh)}{rh}-\frac{q^2+W^2-WM-2M^2}{M}\\&\times\bigg[\mathbf{n}\cdot\mathbf{x}\frac{j_1(rh)}{rh}+\frac{(W-M)^2+q^2}{2M}(\tfrac{1}{3}i\boldsymbol{\gamma}\cdot\mathbf{n}r^2\\&-i\boldsymbol{\gamma}\cdot\mathbf{x}\mathbf{n}\cdot\mathbf{x})\frac{j_2(rh)}{(rh)^2}\bigg],\end{aligned} \tag{B-4}$$

$$\begin{aligned}8M\lambda_3(W;\mathbf{x})&\\=&2(n_0+\tfrac{1}{3}i\boldsymbol{\gamma}\cdot\mathbf{n})j_0(rh)+i\boldsymbol{\gamma}\cdot\mathbf{x}\bigg[\frac{(W-M)^2+q^2}{M}n_0\\&-(W-M)i\boldsymbol{\gamma}\cdot\mathbf{n}\bigg]\frac{j_1(rh)}{rh}+\frac{q^2+W^2-WM+4M^2}{M}\\&\times\bigg[\mathbf{n}\cdot\mathbf{x}\frac{j_1(rh)}{rh}+\frac{(W-M)^2+q^2}{2M}(\tfrac{1}{3}i\boldsymbol{\gamma}\cdot\mathbf{n}r^2\\&-i\boldsymbol{\gamma}\cdot\mathbf{x}\mathbf{n}\cdot\mathbf{x})\frac{j_2(rh)}{(rh)^2}\bigg],\end{aligned} \tag{B-5}$$

where the $j_l(rh)$ are spherical Bessel functions of order l. Corresponding expressions are valid for $\lambda_i(-W;\mathbf{x})$ with W replaced by $-W$. Since $(rh)^{-l}j_l(rh)$ is an even function of rh finite at $rh=0$, the above expressions show explicitly that the functions $f_i(\pm W;x_0,r)$ have all the properties claimed for them in Sec. 2.

[17] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).