any graph in which the π^0 decays into a single photon, real or virtual, vanishes on the grounds of current conservation, quite apart from other symmetry principles. At present, the experimental limit on the 3γ decay mode is not strong evidence for or against such a $c=-1$ component.

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Energy Renormalization in Ordinary Wave Mechanics*

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A very simple, exactly soluble compound-particle model, proposed by Wigner and Weisskopf in 1930, is briefly re-examined from the standpoint of renormalization. It consists of postulating, in the center-of-mass system, the wave equations

$$[i(\partial/\partial t)+(1/2m)\nabla^2]\psi(\mathbf{x},t)=F(\mathbf{x})\chi(t),$$
$$[i(d/dt)-\mu]\chi(t)=\int d^3x\,F(\mathbf{x})\psi(\mathbf{x},t)$$

for two particles of separation $\mathbf{x}$ and reduced mass m, interacting through the formation and decay of an intermediate particle with a real form factor F. The analytic behavior of the S matrix is discussed in the local case $F(\mathbf{x})=C\delta(\mathbf{x})$.

I. INTRODUCTION

THE purpose of this note is to point out a very simple example, involving neither second quantization nor relativity, of a theory with energy renormalization and virtual particles. This model, which is soluble exactly, was proposed by Wigner and Weisskopf[1] in 1930, and studied again, independently and from a different point of view, by Moshinsky[2] in 1951. It will be briefly re-examined in this paper from the standpoint of renormalization. It then turns out to be closely related to the so-called one-particle sector of the Lee model.[3]

We consider the following two systems, which can decay into each other: (a) a motionless particle, located at the origin, and whose wave function[4] $\chi(t)$ depends only on time; (b) a moving particle of mass m, whose wave function $\psi(\mathbf{x},t)$ depends also on the position $\mathbf{x}$. Thus the state vector can be represented in Fock space by two components:

$$|t\rangle=|\psi(\mathbf{x},t),\chi(t)\rangle. \tag{1.1}$$

The scalar product is

$$\langle\psi_1,\chi_1|\psi_2,\chi_2\rangle=\int d^3x\,\psi_1^*\psi_2+\chi_1^*\chi_2. \tag{1.2}$$

The postulated equations of motion are

$$[i(\partial/\partial t)+(1/2m)\nabla^2]\psi(\mathbf{x},t)=F(\mathbf{x})\chi(t), \tag{1.3}$$

$$[i(d/dt)-\mu]\chi(t)=\int d^3x\,F(\mathbf{x})\psi(\mathbf{x},t). \tag{1.4}$$

They have the following features: (a) the only interaction consists of each particle acting as a source for the other; (b) F is a given real form factor and μ a given real energy, so that time-reversal invariance holds; (c) the Hamiltonian H, defined by

$$i(d/dt)|t\rangle=H|t\rangle \tag{1.5}$$

is Hermitian under the scalar product (1.2), so that probability is conserved.

The lack of translational invariance is not an essential restriction. The model equivalently deals with the formation and decay of a compound particle (considered as elementary) in the center-of-mass system, m being the reduced mass. The case of main interest is the local limit

$$F(\mathbf{x})\rightarrow C\delta(\mathbf{x}) \tag{1.6}$$

for a real coupling constant C. For simplicity we assume F to be spherically symmetric about the origin.

2. STATIONARY SCATTERING STATES AND THE S MATRIX

We define Fourier transforms by

$$F(\mathbf{x})=(2\pi)^{-3}\int d^3p\,G(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{2.1}$$

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[1] E. P. Wigner and V. Weisskopf, Z. Physik 63, 62 (1930).
[2] M. Moshinsky, Phys. Rev. 81, 347 (1951); 84, 525 (1951).
[3] G. Sandri has derived the present model as the lowest sector of a Lee Model with nonrelativistic mesons (private communication).
[4] The Schrödinger picture is used throughout.

$$\psi(x,t)=(2\pi)^{-3}\int d^3p\int_{-\infty}^{\infty}dE\,\phi(\mathbf{p},E)e^{i\mathbf{p}\cdot\mathbf{x}-iEt},\quad(2.2)$$

$$\chi(t)=\int_{-\infty}^{\infty}dE\,\lambda(E)e^{-iEt}.\quad(2.3)$$

The transformed equations (1.3), (1.4) are[1]

$$(E-p^2/2m)\phi(\mathbf{p},E)=G(\mathbf{p})\lambda(E),\quad(2.4)$$

$$(E-\mu)\lambda(E)=(2\pi)^{-3}\int d^3p\,G^*(\mathbf{p})\phi(\mathbf{p},E).\quad(2.5)$$

If $E>0$, (2.4) gives

$$\phi(\mathbf{p},E)=\frac{G(\mathbf{p})\lambda(E)}{E-(p^2/2m)+i\epsilon}+g(\mathbf{p})\delta\left(E-\frac{p^2}{2m}\right),\quad(2.6)$$

where $\epsilon\to0^+$ and where g is an arbitrary function giving the asymptotic form of the wave packet for $t\to-\infty$. Inserting (2.6) in (2.5),

$$D^*(E)\lambda(E)=(2\pi)^{-3}\int d^3p\,G^*(\mathbf{p})g(\mathbf{p})\delta\left(E-\frac{p^2}{2m}\right),\quad(2.7)$$

where

$$D(E)=E-\mu-(2\pi)^{-3}\int d^3p\frac{|G(\mathbf{p})|^2}{E-(p^2/2m)-i\epsilon}.\quad(2.8)$$

Assuming that $G(\mathbf{p})\neq0$ for all real momenta, we find that $D(E)$ has no positive zeros. [As a proof, note that $\mathrm{Im}D(E)\neq0$.] Then (2.7) has the unique solution

$$\lambda(E)=(2\pi)^{-3}\int d^3p\frac{G^*(\mathbf{p})}{D^*(p^2/2m)}g(\mathbf{p})\delta\left(E-\frac{p^2}{2m}\right).\quad(2.9)$$

Insertion in (2.6) gives

$$\phi(\mathbf{p},E)=\int d^3q\,Q(\mathbf{p},\mathbf{q})g(\mathbf{q})\delta(E-q^2/2m),\quad(2.10)$$

where

$$Q(\mathbf{p},\mathbf{q})=\delta(\mathbf{p}-\mathbf{q})+\frac{G(\mathbf{p})G^*(\mathbf{q})}{(2\pi)^3[(q^2/2m)-(p^2/2m)+i\epsilon]D^*(q^2/2m)}.\quad(2.11)$$

Equations (2.9) and (2.10) give the scattering states for an initial asymptotic wave packet $g(\mathbf{p})$ in momentum space. Only S waves are scattered.

The S matrix follows directly from (2.11). The transition amplitude between two momenta $\mathbf{p}$, $\mathbf{q}$ is

$$S(\mathbf{p}\to\mathbf{q})=\delta_{\mathbf{pq}}+\frac{2\pi\delta_T[(p^2/2m)-(q^2/2m)]|G(\mathbf{p})|^2}{iVD^*(p^2/2m)},\quad(2.12)$$

where V is the volume of quantization and where the Dirac δ-function

$$\delta_T(0)=T/2\pi,\quad(2.13)$$

T being the total scattering time.

3. STABLE BOUND STATE

If Eqs. (2.4), (2.5) have a negative-energy solution for $E=E_b<0$, we can set for simplicity

$$\phi(\mathbf{p},E)\to\phi_b(\mathbf{p}),\quad(3.1)$$

$$\lambda(E)\to1.\quad(3.2)$$

Equation (2.4) gives

$$\phi_b(\mathbf{p})=\frac{G(\mathbf{p})}{E_b-(p^2/2m)}.\quad(3.3)$$

Then (2.5) yields the condition

$$D(E_b)=0.\quad(3.4)$$

Hence each negative zero of D gives a bound state. But there can be at most one such zero, since $D(E)$ is a monotonically increasing function for $E<0$, as shown by differentiation. We find the cases

$D(0)>0$: one stable bound state,

$D(0)<0$: no stable bound state.

The norm of the momentum-space state vector

$$|\rangle_b=|\phi_b,1\rangle\quad(3.5)$$

is

$${}_b\langle|\rangle_b=1+(2\pi)^{-3}\int d^3p\frac{|G(\mathbf{p})|^2}{[E_b-(p^2/2m)]^2}\quad(3.6)$$

$$=D'(E_b).\quad(3.7)$$

The function $|\phi_b|^2$ is the probability cloud of virtual ψ particles surrounding a stable χ particle. The quantity E_b is its observed energy, while μ is its bare energy.

4. BEHAVIOR OF THE S MATRIX IN THE LOCAL LIMIT

If we take

$$F(\mathbf{x})=C\delta(\mathbf{x}),\quad(4.1)$$

i.e.,

$$G(\mathbf{p})=C,\quad(4.2)$$

then the integral in D [Eq. (2.8)] diverges linearly. The following is a convenient energy cutoff:

$$G(\mathbf{p})=CP/(p^2+P^2)^{\frac{1}{2}},\quad(4.3)$$

with $P\to\infty$ after all calculations. We obtain by elementary integration, for $E>0$,

$$D(E)=E-\mu+\frac{C^2P^3m}{2\pi(2mE+P^2)}-i\frac{C^2P^2m(2mE)^{\frac{1}{2}}}{2\pi(2mE+P^2)}.\quad(4.4)$$

For $P \rightarrow \infty$,

$$D(E)=E-\mu'-i(mC^2/2\pi)(2mE)^{\frac{1}{2}}, \qquad (4.5)$$

where

$$\mu'=\mu-C^2Pm/2\pi. \qquad (4.6)$$

The S matrix then depends on the energy as

$$[E-\mu'+i(mC^2/2\pi)(2mE)^{\frac{1}{2}}]^{-1}, \qquad (4.7)$$

which is the one-level Breit-Wigner formula.[5] If the observable quantity μ' is to be finite, we see that the bare energy μ must diverge linearly in P.

Behavior of $D(E)$ in the Complex E and $\sqrt{E}$ Planes

Equation (2.8) implies a branch cut along the positive real axis of E, such that $\mathrm{Im}\sqrt{E}<0$. Setting for convenience

$$k=(E/|\mu'|)^{\frac{1}{2}}, \qquad (4.8)$$

$$\alpha=(mC^2/4\pi)(2m/|\mu'|)^{\frac{1}{2}}, \qquad (4.9)$$

we have

$$D(E)=|\mu'|(k^2-2i\alpha k\mp 1) \quad (\mathrm{Im}k<0) \qquad (4.10)$$

if $\mu'>0(<0)$, i.e., in the unstable (stable) case. In the second ("unphysical") Riemann sheet defined by the cut we have the corresponding functions

$$D_1(E)=|\mu'|(k^2+2i\alpha k\mp 1) \quad (\mathrm{Im}k<0). \qquad (4.11)$$

Stable Case ($\mu'<0$, Lower Sign)

The function D has one zero,

$$k=-i[(\alpha^2+1)^{\frac{1}{2}}-\alpha], \qquad (4.12)$$

corresponding to

$$E=E_b=-|\mu'|[(\alpha^2+1)^{\frac{1}{2}}-\alpha]^2; \qquad (4.13)$$

similarly, D_1 has the "unphysical" zero

$$k=i[(\alpha^2+1)^{\frac{1}{2}}+\alpha]. \qquad (4.14)$$

For $E>0$ we find

$$|D(E)|^2=|\mu'|^2[(k^2+2\alpha^2+1)^2-(2\alpha^2+1)^2+1], \qquad (4.15)$$

[5] See, e.g., J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 557.

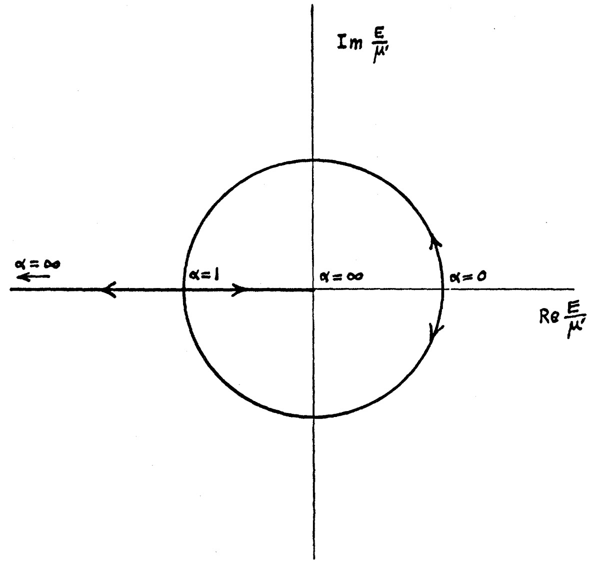

FIG. 1. The zeros of D_1 in the complex (E/μ') plane (unstable case). The arrows indicate how the zeros move when α increases. The circle has unit radius.

so that the cross section decreases monotonically as a function of the energy.

Unstable Case ($\mu'>0$, Upper Sign)

Here D has no zero, while D_1 has the zeros

$$k=i\alpha\pm(1-\alpha^2)^{\frac{1}{2}}. \qquad (4.16)$$

The position of these zeros for increasing α is shown in Fig. 1.

For $E>0$,

$$|D(E)|^2=|\mu'|^2[(k^2-1+2\alpha^2)^2+4\alpha^2(1-\alpha^2)], \qquad (4.17)$$

so that the cross section exhibits a resonance if $\alpha^2<\frac{1}{2}$.

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