Null Electromagnetic Fields in General Relativity Theory*

ASHER PERES
Department of Physics, Israel Institute of Technology, Haifa, Israel
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The existence of null electromagnetic fields, within the frame of the general relativity theory, has been subject to some doubt. We here construct and investigate two solutions of the Maxwell-Einstein equations, representing null electromagnetic fields, and corresponding to unidirectional radiation flow. One of these fields is coupled with transverse plane gravitational waves, and the other one with longitudinal (not necessarily plane) gravitational waves. It is shown that not only the two electromagnetic invariants, but also the fourteen invariants of the Riemann tensor vanish. During the course of this investigation, use is made of Pirani's tetrad formalism in order to simplify Witten's spinor approach to the problem of invariants.

1. INTRODUCTION AND NOTATIONS

NULL electromagnetic fields are defined by

$$F_{mn}F^{mn}=0, \tag{1}$$

$$\eta^{mnrs}F_{mn}F_{rs}=0. \tag{2}$$

Here F_{mn} is the electromagnetic field tensor, and η^{mnrs} is defined by $\eta^{mnrs}=+1$ if *mnrs* is an even permutation of 0123, $\eta^{mnrs}=-1$ if *mnrs* is an odd permutation of 0123, and $\eta^{mnrs}=0$ if any two indices are equal. (Latin indices run from 0 to 3.)

Many examples of null electromagnetic fields (e.g., plane waves) are known within the frame of the special relativity theory. However, their existence in the general relativity theory has been subject to some doubt.[1,2] The source of these doubts was the fact that classical geometrodynamics[3] (i.e., the unified geometrical theory of gravitation and electrodynamics) can be conveniently formulated only for non-null electromagnetic fields, and breaks down in the case of null fields. It has thus even been suggested that nontrivial null fields should be ruled off by the combined Maxwell-Einstein equations.[2]

We here intend to construct and investigate two solutions of the Maxwell-Einstein equations, representing null electromagnetic fields, and corresponding to unidirectional radiation flow. One of these fields is coupled with transverse plane gravitational waves, and the other one with longitudinal (not necessarily plane) gravitational waves. It will be shown that not only the two electromagnetic invariants, but also the fourteen invariants of the Riemann tensor vanish.

During the course of this investigation, use will be made of Pirani's tetrad formalism[4] in order to simplify calculations. We shall define a tetrad of orthonormal vectors $h_{(n)}{}^m$ such that

$$g_{rs}h^r{}_{(m)}h^s{}_{(n)}=\eta_{(mn)}, \tag{3}$$

$$\eta^{(mn)}h^r{}_{(m)}h^s{}_{(n)}=g^{rs}, \tag{4}$$

where $\eta_{(mn)}$ is the Minkowski matrix, having signature $(+---)$. The indices between parenthesis are enumerators. They also take the values 0123, and repeated indices imply summation. Enumerators can be raised or lowered with the help of $\eta^{(mn)}$ or $\eta_{(mn)}$.

The physical components[4] of a tensor are obtained by contraction with the tetrad, e.g.,

$$F_{(mn)}=h^r{}_{(m)}h^s{}_{(n)}F_{rs}. \tag{5}$$

From these, one can reobtain the covariant or contravariant components, e.g.,

$$F_{mn}=h_m{}^{(r)}h_n{}^{(s)}F_{(rs)}. \tag{6}$$

Note that the physical components are scalars, but are *not invariants*, because they are defined only up to an arbitrary Lorentz transformation of the tetrad.[5]

Greek indices (dotted or undotted) will be reserved for spinor components and thus will take the values 1 or 2 only. Throughout this paper we shall use natural units:

$$c=1 \quad\text{and}\quad 8\pi G=1. \tag{7}$$

2. CONSTRUCTION OF LONGITUDINAL GRAVITATIONAL WAVES

It has recently been shown by Pirani[6] that the gravitational field of a rapidly moving mass bears a strong resemblance to gravitational radiation. In mathematical terms, the algebraic structure of the leading terms of the Riemann tensor asymptotically tends towards the second class of Petrov's classification[7,8] (radiation fields), although the complete exact Riemann tensor still belongs, of course, to the first class (non-radiative fields).

We shall now show that one can obtain true radiation fields by making a source actually move with the

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[1] G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).
[2] L. Witten, Phys. Rev. **115**, 206 (1959).
[3] C. W. Misner and J. A. Wheeler, Ann. Phys. (N. Y.) **2**, 525 (1957).
[4] F. A. E. Pirani, Acta Phys. Polon. **15**, 389 (1956); Bull. Acad. Polon. Sc. **5**, 143 (1957).
[5] D. W. Sciama, Nuovo cimento 8, 417 (1958).
[6] F. A. E. Pirani, Proc. Roy. Soc. (London) **A252**, 96 (1959).
[7] A. Z. Petrov, Uchenye Zapiski Kazan. Gosudarst. Univ. im. V.I. Ul'yanova-Lenina **114**, 55 (1954).
[8] F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957).

velocity of light (and not only almost this velocity, as in Pirani's work[6]), together with some additional manipulations.

For the sake of simplicity, we start from Marder's solution[9] for a static infinite cylinder

$$ds^2=r^{2C/(1-C)}dt^2-K^2r^{2C^2/(1-C)}dr^2-r^2d\phi^2-r^{-2C}dz^2. \quad (8)$$

Here, $C\sim 2M$, where M is the linear density of the cylinder.

Let us uniformly translate the cylinder in its own direction. This can be accomplished by the transformation

$$t=t'\cosh a+z'\sinh a, \quad (9a)$$

$$z=z'\sinh a+t'\cosh a. \quad (9b)$$

Now, let $K\to 1$, $C\to 0$, $a^2\to\infty$, but in such a way that

$$Ce^{2a}=2M'=\text{const.} \quad (10)$$

One gets

$$ds^2=dt^2-dr^2-r^2d\phi^2-dz^2-8M\ln r(dt+dz)^2, \quad (11)$$

where primes have been dropped.

This metric may be considered as produced by a pencil of light,[10] if diffraction effects are neglected. M is proportional to the light intensity. It may easily be verified that the more general metric

$$ds^2=dt^2-dx^2-dy^2-dz^2 -4\sum_n M_n \ln[(x-x_n)^2+(y-y_n)^2](dt+dz)^2, \quad (12)$$

where the M_n, x_n, and y_n are constants, also satisfies the Einstein gravitational equations *in vacuo* (except at the sources $x=x_n$, $y=y_n$). One can interpret (12) as caused by several parallel pencils of light. One thus gets a rigorous proof of a theorem first derived by Tolman,[10] according to which there is no gravitational interaction between parallel pencils of light.

Although the metric (12) belongs to Petrov's second class[7,8] it still does not look like waves. In order to get a true wave-like solution, one should perform the same operation on a metric that does not display cylindrical symmetry, e.g., one should start from the Schwarzschild line element. Unfortunately, it then turns out that the computations do not give a clearcut result, as in the previous case. Nevertheless, one can guess that the solution should have the form[11]

$$ds^2=dt^2-dx^2-dy^2-dz^2-2f(x,y,z+t)(dz+dt)^2. \quad (13)$$

This guess now has to be checked. One readily sees that the only nonvanishing independent components of the Riemann tensor are

$$R_{mrns}=f_{mn}, \quad (14)$$

where m and n take the values 1 or 2 only, while r and s take the values 3 or 0 only. It thus follows that

$$R_{00}=R_{03}=R_{33}=f_{xx}+f_{yy}, \quad (15)$$

and other components vanish. The metric (13) will therefore satisfy the Einstein gravitational equations in vacuo if, and only if, f is chosen to be a harmonic function of x and y, whatever may be its dependence on $(z+t)$. One can thus write

$$f=\mathrm{Re}[F(x+iy, z+t)], \quad (16)$$

where F is an arbitrary function of both its arguments. The metric (13) thus displays a higher degree of arbitrariness than previously known wavelike solutions of the Einstein equations, which were always either plane[12,13] or cylindrical[14] waves.

This new kind of gravitational wave has many interesting peculiarities. First, we note that the only non-Euclidean components of the metric are those associated with the z and t coordinates: these waves are therefore purely longitudinal ones. Of course, it is possible to give them a purely transverse character, by a suitable coordinate transformation. However, this would be a very awkward step, because the form (13) is extremely simple: one has both

$$g=\mathrm{Det}\,g_{mn}=-1, \quad (17)$$

and

$$g^{mn}{}_{,n}=0, \quad (18)$$

so that it may be stated that the metric (13) is already written in privileged coordinates.

3. A NULL ELECTROMAGNETIC FIELD

If $f_{xx}+f_{yy}\neq 0$, one still has

$$g^{mn}R_{mn}=0, \quad (19)$$

and

$$g^{mn}R_{mr}R_{ns}=0, \quad (20)$$

so that the source of f may be considered as a null electromagnetic field.[3] Let

$$f_{xx}+f_{yy}=-P^2. \quad (21)$$

One can easily verify that the field

$$F_{10}=F_{13}=P\cos a, \quad (22a)$$

$$F_{20}=F_{23}=P\sin a, \quad (22b)$$

where a is arbitrary, satisfies Eqs. (1), (2), and

$$R_{mn}=g^{rs}F_{mr}F_{ns}. \quad (23)$$

Let us introduce the self-dual tensor density

$$H^{mn}=g^{\frac{1}{2}}F^{mn}+\tfrac{1}{2}\eta^{mnrs}F_{rs}. \quad (24)$$

[9] L. Marder, Proc. Roy. Soc. (London) **A244**, 524 (1958).
[10] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, Oxford, England, 1934), pp. 274–277.
[11] A. Peres, Phys. Rev. Letters **3**, 571 (1959).
[12] H. Takeno, Tensor **7**, 97 (1957); **8**, 59 (1958); **9**, 76 (1959).
[13] H. Bondi, F. A. E. Pirani and I. Robinson, Proc. Roy. Soc. (London) **A251**, 519 (1959).
[14] N. Rosen, Bull. Research Council Israel **3**, 328 (1954).

Taking $g^{\frac{1}{2}}=i$, one gets

$$H^{01}=-H^{31}=iP\cos a+P\sin a=iPe^{-ia}, \quad (25a)$$

$$H^{02}=-H^{32}=iP\sin a-P\cos a=-Pe^{-ia}. \quad (25b)$$

The Maxwell equations *in vacuo* can be written

$$H^{mn}{}_{,n}=0. \quad (26)$$

It is easily found that they are satisfied if, and only if, a is function of x,y, and $(z+t)$, and, moreover

$$(Pe^{-ia})_x+i(Pe^{-ia})_y=0. \quad (27)$$

This last relation means that Pe^{-ia} is a function of $(x+iy)$. It follows that $(\ln P)$ must be a harmonic function of x and y and that a is the conjugate harmonic function.

For instance, one may take $P=A\cos k(z+t)$ and $a=b$, where A, k, and b are constants. One then gets a plane monochromatic polarized electromagnetic wave. The corresponding metric may be

$$ds^2=dt^2-dx^2-dy^2-dz^2+\tfrac{1}{2}A^2(x^2+y^2)\cos^2k(z+t). \quad (28)$$

(Many other forms for ds^2 are possible.) It may seem that this is not a plane wave solution. However, it is readily seen from (14) that the Riemann tensor displays plane symmetry, so that the gravitational waves are really plane ones.

Finally, let us note the following interesting relation[2]

$$H^{mn}=g^{\frac{1}{2}}Pe^{-ia}(L^mM^n-L^nM^m), \quad (29)$$

where L^n is the propagation null vector

$$L^n=(0,0,-1,1) \quad (30)$$

and

$$M^n=(1,i,0,0), \quad (31)$$

is another null vector, orthogonal to L^n.

4. ALGEBRAIC STRUCTURE OF THE RIEMANN TENSOR

A convenient tetrad of orthonormal vectors is[11]

$$h^m{}_{(1)}=(\cos u, \sin u, 0, 0), \quad (32a)$$

$$h^m{}_{(2)}=(-\sin u, \cos u, 0, 0), \quad (32b)$$

$$h^m{}_{(3)}=(0,0,1-f,f), \quad (32c)$$

$$h^m{}_{(4)}=(0,0,-f,1+f), \quad (32d)$$

where

$$\tan 2u=2f_{xy}/(f_{xx}-f_{yy}). \quad (33)$$

The only nonvanishing independent physical components[4] of the Riemann tensor are now

$$R_{(mrms)}=\tfrac{1}{2}(f_{xx}+f_{yy}) \pm[\tfrac{1}{4}(f_{xx}+f_{yy})^2+f_{xy}f_{yx}-f_{xx}f_{yy}]^{\frac{1}{2}}, \quad (34)$$

where r and s take the values 3 or 0 only, and m takes the value 1 or 2 only. The $+$ or $-$ sign in (34) has to be chosen according to whether $m=1$ or $m=2$, respectively. For the sake of brevity, we shall write

$$R_{(1r1s)}=M, \quad R_{(2r2s)}=N. \quad (35)$$

Following Bel,[15] we can define

$$*R_{(mnrs)}=\tfrac{1}{2}\eta_{(mnpq)}R^{(pq)}{}_{(rs)}, \quad (36a)$$

and

$$**R_{(mnrs)}=\tfrac{1}{2}*R_{(mn)}{}^{(pq)}\eta_{(pqrs)}, \quad (36b)$$

so that

$$*R_{(1m2n)}=N, \quad *R_{(2m1n)}=-M, \quad (37a)$$

and

$$**R_{(1m1n)}=N, \quad **R_{(2m2n)}=M. \quad (37b)$$

Here, in Eqs. (37), m and n take the values 3 and 0 only, and all other independent components vanish. It is readily found that

$$R_{(mnrs)}L^{(s)}=*R_{(mnrs)}L^{(s)}=**R_{(mnrs)}L^{(s)}=0, \quad (38)$$

where $L^{(s)}$ is given by Eq. (30).

One thus sees that Bel's criterion[15] for pure radiation is satisfied.

One further has

$$L_{m;n}=L_{m,n}-L_s\Gamma^s{}_{mn}=0. \quad (39)$$

The trajectories of L_s are therefore the geodetics of g_{mn}, in agreement with a general theorem of Mme. Blancheton.[16]

If $f_{xx}+f_{yy}=0$, one has $M+N=0$, and the algebraic structure of the Riemann tensor is seen to belong to the second class of Petrov's classification,[7,8] with both invariants vanishing.

5. ENERGY AND MOMENTUM

We now turn to investigate whether these longitudinal gravitational waves carry energy. The oldest expression for the energy and momentum of the gravitational field is Einstein's pseudotensor $t^m{}_n$. A straightforward computation, making use of Tolman's formulas[17] gives

$$t^m{}_n=0. \quad (40)$$

Another possible expression is the symmetric t^{mn} of Landau and Lifshitz.[18] One readily finds that also

$$t^{mn}=0. \quad (41)$$

The Einstein and Landau-Lifshitz pseudotensors have recently been generalized by Goldberg[19] who constructed a denumerable infinity of possible expressions. However, as in our case $g=-1$, all these expressions coincide, and thus also vanish.

A still larger infinity of conservation laws was then found by Komar:[20] following a suggestion of Berg-

[15] L. Bel, Compt. rend. **246**, 3015 (1958).
[16] E. Blancheton, Compt. rend. **248**, 372 (1959).
[17] R. C. Tolman, reference 10, p. 224.
[18] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Cambridge, Massachusetts, 1951), p. 317.
[19] J. N. Goldberg, Phys. Rev. **111**, 315 (1958).
[20] A. Komar, Phys. Rev. **113**, 934 (1958).

mann,[21] Komar showed that to each generator of an infinitesimal transformation there corresponds a conservation law. Unfortunately, the expression which is conserved is a current (vector) density, and thus has not the correct transformation properties to be considered as energy and momentum density. Nevertheless, Komar[20] showed that if the coordinate transformation is a rigid time-like translation, the corresponding current density is identical with the $t^m{}_0$ proposed by Møller.[22] However, a straightforward computation shows that Møller's $t^m{}_n$ also vanishes.

It should be noted that the above results hold in quasi-Galilean coordinates, and therefore cannot be imputed to some coordinate effect, as that which caused the unwarranted claims on the vanishing of the energy of Rosen's cylindrical gravitational waves.[23]

Does this mean that longitudinal gravitational waves carry no energy? Such a conclusion seems unjustified, because energy can apparently be extracted from the wave by a method first devised by Weber and Wheeler,[24] and further developed by Marder.[9] Let us consider two neighboring test particles, and let us suppose that one of them can slide on a rough rod rigidly tied to the other particle. As the curvature tensor does not vanish, the two particles will undergo some relative acceleration.[4] Heat will be generated in the friction rod, and will be available to drive some engine. It thus appears that energy can be extracted from the gravitational wave. This argument, however, is not fully convincing, because the available energy is proportional to the mass of the particles, and thus vanishes as the latter is made vanishingly small. It may thus be that the energy is not contained in the gravitational wave itself, but rather produced by some interference effect between the gravitational wave and the quasi-static field of the particles. Further considerations about this problem will be the subject of another paper.

Anyhow, one can also construct expressions for the energy density which do not vanish. For instance, Dirac's energy density[25] is, in the linear approximation

$$w=2(f_x{}^2+f_y{}^2)\geqslant 0, \tag{42}$$

where equality holds only for flat space. It should be noted, however, that the metric (13) does not satisfy Dirac's coordinate conditions.[26]

Finally, Bel's fourth rank fully symmetric tensor[27] is, in our case

$$T^{mnrs}=-2VL^mL^nL^rL^s, \tag{43}$$

where

$$V=M^2+N^2 \tag{44}$$

is Bel's energy density.[15]

6. CURVATURE INVARIANTS

As well known, the Riemann curvature tensor has, in the general case, fourteen independent algebraic invariants. An elegant method of constructing them has recently been proposed by Witten[28] who showed that the components of R_{mnrs} can be expressed as linear functions of two fourth rank spinors $\phi^{\dot\beta\dot\mu\rho\sigma}$ and $\psi^{\kappa\lambda\rho\sigma}$, the coefficients being known functions of the fundamental spinor $g^{m\dot\alpha\beta}$.

Witten's method can be greatly simplified if one takes the physical components[4] of Riemann's tensor, instead of its covariant or contravariant components. In this case, one has also to take the physical components $g^{(m)\dot\alpha\beta}$ of the fundamental spinor, which are simply the Pauli spin matrices.

Let us first compute

$$\phi^{\dot\beta\dot\mu\rho\sigma}=\tfrac{1}{4}g^{(m)\dot\beta\rho}g^{(n)\dot\mu\sigma}(R_{(mn)}-\tfrac{1}{4}g_{(mn)}R). \tag{45}$$

As

$$R_{(mn)}=-P^2L_{(m)}L_{(n)}, \tag{46a}$$

and

$$R=0, \tag{46b}$$

one gets

$$\phi^{\dot\beta\dot\mu\rho\sigma}=-P^2\phi^{\dot\beta\rho}\phi^{\dot\mu\sigma}, \tag{47}$$

where

$$\phi^{\dot\beta\rho}=\tfrac{1}{2}g^{(m)\dot\beta\rho}L_{(m)}=\tfrac{1}{2}(g^{(0)\dot\beta\rho}+g^{(3)\dot\beta\rho}). \tag{48}$$

Introducing the basic spinors

$$U^{\dot\beta}=\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad U^\rho=(1,0), \tag{49}$$

one gets

$$\phi^{\dot\beta\rho}=U^{\dot\beta}U^\rho, \tag{50}$$

so that

$$\phi^{\dot\beta\dot\mu\rho\sigma}=-P^2U^{\dot\beta}U^{\dot\mu}U^\rho U^\sigma. \tag{51}$$

Further one has, with Witten's notations[28]

$$E_{(mnrs)}=R_{(mnrs)}-{}^{**}R_{(mnrs)}, \tag{52}$$

whence

$$E_{(1p1q)}=-E_{(2p2q)}=2D^2, \tag{53}$$

and

$$\tilde{E}_{(1p2q)}=\tilde{E}_{(2p1q)}=-2iD^2, \tag{54}$$

where p and q take the values 0 or 3 only, and where

$$D^2=\tfrac{1}{2}(M-N)=(\tfrac{1}{4}P^4+f_{xy}f_{yx}-f_{xx}f_{yy})^{\frac{1}{2}}. \tag{55}$$

It follows that

$$E_{(mnrs)}+\tilde{E}_{(mnrs)}=2Q_{(mn)}Q_{(rs)}, \tag{56}$$

where

$$Q_{(1p)}=-Q_{(p1)}=D, \tag{57a}$$

$$Q_{(2p)}=-Q_{(p2)}=-iD, \tag{57b}$$

and p takes the values 0 or 3 only.

[21] P. G. Bergmann, Phys. Rev. **112**, 287 (1958).
[22] C. Møller, Ann. Phys. (N. Y.) **4**, 347 (1958).
[23] N. Rosen, Helv. Phys. Acta Suppl. IV, 171 (1956); Phys. Rev. **110**, 291 (1958).
[24] J. Weber and J. A. Wheeler, Revs. Modern Phys. **29**, 509 (1957).
[25] P. A. M. Dirac, Phys. Rev. Letters **2**, 368 (1959).
[26] P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).
[27] L. Bel, Compt. rend. **247**, 1094 (1958).
[28] L. Witten, Phys. Rev. **113**, 357 (1959).

It then follows from Eq. (22) of Witten's paper[28] that

$$\psi^{\kappa\lambda\rho\sigma}=\psi^{\kappa\lambda}\psi^{\rho\sigma}, \tag{58}$$

where

$$\psi^{\rho\sigma}=\tfrac{1}{2}g^{(m)\dot{\alpha}\rho}g^{(n)}{}_{\dot{\alpha}}{}^{\sigma}Q_{(mn)}. \tag{59}$$

A straightforward computation gives

$$\psi^{\rho\sigma}=4DU^{\rho}U^{\sigma}, \tag{60}$$

whence

$$\psi^{\kappa\lambda\rho\sigma}=16D^2U^{\kappa}U^{\lambda}U^{\rho}U^{\sigma}. \tag{61}$$

As $U^{\lambda}U_{\lambda}\equiv 0$, it follows that all fourteen Witten's invariants vanish. The metric (13) thus belongs to the third class of Witten's classification.[28] [Incidentally, this shows that Witten's assertion that there exists a one to one correspondence between his classification[28] of Einstein's spaces and that of Petrov,[7,8] is not correct. Actually, class $\mathrm{I_W}$ corresponds to class $\mathrm{I_P}$ (the indices W and P stand for Witten and Petrov, respectively) because $\mathrm{I_W}$ has four invariants, and $\mathrm{I_P}$ has at most four, while $\mathrm{II_P}$ has at most two and $\mathrm{III_P}$ has none. Class $\mathrm{II_W}$ has two independent invariants, like $\mathrm{II_P}$ and thus corresponds either to $\mathrm{II_P}$ or to $\mathrm{I_P}$, because it may occasionally occur that two of the four invariants of $\mathrm{I_P}$ vanish. Finally, $\mathrm{III_W}$ has no invariant, and thus may correspond either to $\mathrm{II_P}$ or to $\mathrm{III_P}$. $\mathrm{III_W}$ cannot correspond to $\mathrm{I_P}$, because if all the invariants of $\mathrm{I_P}$ vanish, space-time is Euclidean.]

The physical interpretation of the vanishing of all the invariants is simple: let us perform on the metric (13) a uniform translation along the z axis, i.e., a transformation such as (9). One gets

$$ds^2=dt^2-dx^2-dy^2-dz^2 \\ -2K^2f[x,y,K(z+t)](dz+dt)^2, \tag{62}$$

where K is an arbitrary positive constant. By choosing K sufficiently small, the deviation from flatness can be made arbitrarily small. In physical terms, this means that an observer traveling with a high velocity in the negative z direction will feel almost no gravitational field. This may also be seen by computing the acceleration of a test particle having velocity $v^k=dx^k/dt$. From

$$dv^k/dt=-(\Gamma^k{}_{mn}-\Gamma^0{}_{mn}v^k)v^mv^n, \tag{63}$$

one gets

$$dv^x/dt=f_x(1+v^z)^2 \\ -v^x[f'(1+v^z)^2+(f_xv^x+f_yv^y)(1+v^z)], \tag{64a}$$

with a similar expression for (dv^y/dt), and

$$dv^z/dt=-f'(1+v^z)^3-2(f_xv^x+f_yv^y)(1+v^z)^2, \tag{64b}$$

where $f'=\partial f/\partial(z+t)$. When $v^z\rightarrow -1$, all these expressions vanish, as expected.

7. TRANSVERSE WAVES

Let us now consider the metric

$$ds^2=dt^2-Adx^2-2Bdxdy-Cdy^2-dz^2, \tag{65}$$

where A, B, and C are functions of $(z+t)$ only. This metric is identical (up to coordinate transformations) to those investigated by Takeno[12] and by Bondi, Pirani, and Robinson.[13] However, the form (65) is simpler than those given by these authors.

The only nonvanishing independent components of the Riemann tensor are

$$R_{mrns}=-\tfrac{1}{2}\partial^2g_{mn}/\partial(z+t)^2+\tfrac{1}{4}g^{pq}\dot{g}_{mp}\dot{g}_{nq}, \tag{66}$$

where a dot denotes differentiation with respect to $(z+t)$, and where r and s take the values 0 or 3 only. It follows that

$$R_{00}=R_{03}=R_{33}=\tfrac{1}{2}g^{mn}\partial^2g_{mn}/\partial(z+t)^2+\tfrac{1}{4}\dot{g}^{mn}\dot{g}_{mn}. \tag{67}$$

If this expression vanishes, one has pure gravitational waves. Otherwise, one has combined plane electromagnetic and gravitational waves, as before. As the algebraic structure of the Riemann tensor is the same as in the previous case, many of our previous results also apply to these transverse waves, with at most minor corrections: for instance, Eqs. (23) and (25) still hold, but their form is unchanged only if they are written in physical components.[4] The only difference is that the Einstein and Landau-Lifshitz expressions for the energy density[17,18] do not vanish in this case: they are even different from each other, so that it is doubtful whether any meaning should be attached to them. On the other hand, Møller's expression[22] still vanishes.

8. JUMP CONDITIONS AT DISCONTINUITIES

We have hitherto supposed that the metrics (13) and (65) are regular functions of the coordinates. This assumption will now be raised. We first note that the vector

$$L_s=(z+t)_{,s} \tag{68}$$

is a null vector, so that the hypersurfaces $(z+t)=\text{const}$ are null hypersurfaces, and therefore allow discontinuities in the second derivatives[29–33] of g_{mn} with respect to $(z+t)$. It is indeed easily found that

$$L_mR_{nprs}+L_nR_{pmrs}+L_pR_{mnrs}=0, \tag{69}$$

so that Lichnerowicz's conditions[31] are automatically satisfied. Further, one has

$$L_mF_{np}+L_nF_{pm}+L_pF_{mn}=0 \tag{70}$$

and

$$L^mF_{mn}=0. \tag{71}$$

These discontinuities are thus compatible with those allowed for an electromagnetic wave front.[33]

The possibility of discontinuities in the first derivatives of g_{mn} has recently been investigated by

[29] F. K. Stellmacher, Math. Ann. **115**, 740 (1938).
[30] S. O'Brien and J. L. Synge, Communs. Dublin Inst. Advanced Studies **A9** (1952).
[31] A. Lichnerowicz, Compt. rend. **246**, 893 (1958).
[32] H. Treder, Ann. Physik (7) **2**, 225 (1958).
[33] J. Ehlers and R. K. Sachs, Z. Physik **155**, 498 (1959).

Papapetrou and Treder,[34] who found that they are allowed on null hypersurfaces if, and only if, the discontinuities of $g_{mn,s}$ can be written as

$$[g_{mn,s}]=b_{mn}L_s, \tag{72}$$

where the b_{mn} are subject to

$$(b_{mn}-\tfrac{1}{2}g_{mn}g^{rs}b_{rs})L^n=0. \tag{73}$$

In the case of longitudinal waves, one has

$$b_{mn}=-2[f']L_mL_n, \tag{74}$$

[34] A. Papapetrou and H. Treder, Math. Nachr. **20**, 53 (1959); Ann. Physik (7) **3**, 360 (1959).

so that (73) is always satisfied. However, it follows from (29) and (74) that such discontinuities are not real, in the sense that they can be transformed away by a suitable choice of the coordinates.[29,34]

On the other hand, in the case of transverse waves, one has

$$b_{mn}L^n=0, \tag{75}$$

so that (73) reduces to

$$g^{mn}b_{mn}=0, \tag{76}$$

which means that the discontinuities of the first derivatives of g_{mn} have to be such that the first derivative of $g=B^2-AC$ remains continuous.

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Wormhole Initial Conditions*

CHARLES W. MISNER†
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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Initial conditions for the source-free Einstein equations are exhibited which represent, in a singularity-free manner on a manifold with the topology of Wheeler's "wormhole," two neutral objects of equal positive masses instantaneously at rest.

At the time Wheeler first showed[1] that classical objects (geons) behaving like massive particles could be constructed theoretically from gravitational and electromagnetic fields, he suggested that charged particles could also be constructed from these fields. The existence of charged particles in the Einstein-Maxwell theory, with the charge-current density everywhere zero, requires a departure from Euclidean topology. One example of a suitable topology, the "wormhole," is shown in Fig. 1. It has been shown[2] that solutions of the Einstein-Maxwell equations actually exist which can be interpreted (in a classical idealization) as spaces containing charged, massive, particles; these examples have topologies somewhat different from the wormhole. In this note we shall show that a solution of the Einstein equations exists having the form shown in Fig. 1. The solution given here refers to the special case of a wormhole free of electromagnetic field, and therefore, the two ends or "mouths" of the wormhole behave as *neutral* concentrations of mass energy.

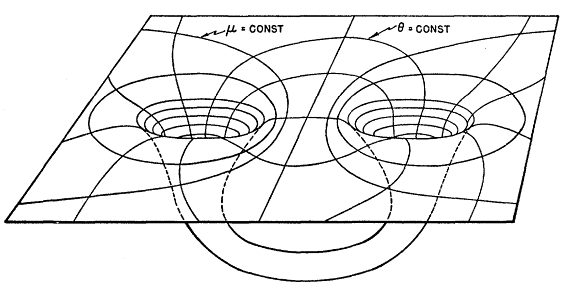

FIG. 1. A coordinate plane ($\varphi=0,\pi$) through the symmetry axis of the "wormhole" metric is sketched here imbedded in a fictitious 3-space to show the topology and curvature. At large distances all the coordinate lines, $\mu=$const and $\theta=$const, are arcs of circles.

Rather than attempt to solve the entire set of Einstein equations in the wormhole topology we restrict ourselves to the initial value equations,[2] $R_\mu{}^0-\frac{1}{2}\delta_\mu{}^0R=0$, on one fixed hypersurface $t=0$. These equations (analogous to $\nabla\cdot\mathbf{E}=0=\nabla\cdot\mathbf{H}$ in electromagnetism) and free of second time derivatives and, therefore, impose restrictions on the initial values to be specified for $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial t$. (The remaining Einstein equations serve to determine the second time derivatives.) Choosing for simplicity a time symmetric problem[3] where, initially, $\partial g_{\mu\nu}/\partial t=0$ and $g_{0\mu}=-\delta_\mu{}^0$, these equations reduce to the single condition

$$^3R=0, \tag{1}$$

where ^3R is the curvature scalar for the three-dimensional metric g_{ij} of this initial surface. Corre-

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[1] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[2] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 594 (1957).

[3] Dieter R. Brill, Ann. Phys. **7**, 466 (1959).