Papapetrou and Treder,[34] who found that they are allowed on null hypersurfaces if, and only if, the discontinuities of $g_{mn,s}$ can be written as

$$[g_{mn,s}]=b_{mn}L_s, \qquad (72)$$

where the b_{mn} are subject to

$$(b_{mn}-\tfrac{1}{2}g_{mn}g^{rs}b_{rs})L^n=0. \qquad (73)$$

In the case of longitudinal waves, one has

$$b_{mn}=-2[f']L_mL_n, \qquad (74)$$

[34] A. Papapetrou and H. Treder, Math. Nachr. **20**, 53 (1959); Ann. Physik (7) **3**, 360 (1959).

so that (73) is always satisfied. However, it follows from (29) and (74) that such discontinuities are not real, in the sense that they can be transformed away by a suitable choice of the coordinates.[29,34]

On the other hand, in the case of transverse waves, one has

$$b_{mn}L^n=0, \qquad (75)$$

so that (73) reduces to

$$g^{mn}b_{mn}=0, \qquad (76)$$

which means that the discontinuities of the first derivatives of g_{mn} have to be such that the first derivative of $g=B^2-AC$ remains continuous.

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Wormhole Initial Conditions*

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Initial conditions for the source-free Einstein equations are exhibited which represent, in a singularity-free manner on a manifold with the topology of Wheeler's "wormhole," two neutral objects of equal positive masses instantaneously at rest.

At the time Wheeler first showed[1] that classical objects (geons) behaving like massive particles could be constructed theoretically from gravitational and electromagnetic fields, he suggested that charged particles could also be constructed from these fields. The existence of charged particles in the Einstein-Maxwell theory, with the charge-current density everywhere zero, requires a departure from Euclidean topology. One example of a suitable topology, the "wormhole," is shown in Fig. 1. It has been shown[2] that solutions of the Einstein-Maxwell equations actually exist which can be interpreted (in a classical idealization) as spaces containing charged, massive, particles; these examples have topologies somewhat different from the wormhole. In this note we shall show that a solution of the Einstein equations exists having the form shown in Fig. 1. The solution given here refers to the special case of a wormhole free of electromagnetic field, and therefore, the two ends or "mouths" of the wormhole behave as *neutral* concentrations of mass energy.

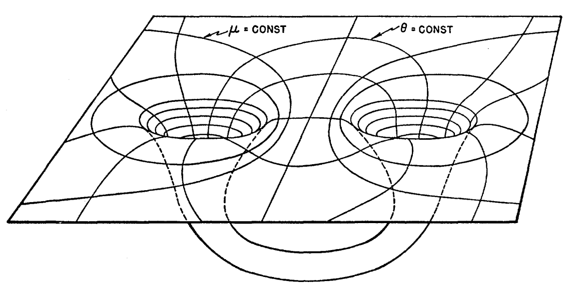

FIG. 1. A coordinate plane ($\varphi=0,\pi$) through the symmetry axis of the "wormhole" metric is sketched here imbedded in a fictitious 3-space to show the topology and curvature. At large distances all the coordinate lines, $\mu=$const and $\theta=$const, are arcs of circles.

Rather than attempt to solve the entire set of Einstein equations in the wormhole topology we restrict ourselves to the initial value equations,[2] $R_\mu{}^0-\frac{1}{2}\delta_\mu{}^0R=0$, on one fixed hypersurface $t=0$. These equations (analogous to $\nabla\cdot\mathbf{E}=0=\nabla\cdot\mathbf{H}$ in electromagnetism) and free of second time derivatives and, therefore, impose restrictions on the initial values to be specified for $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial t$. (The remaining Einstein equations serve to determine the second time derivatives.) Choosing for simplicity a time symmetric problem[3] where, initially, $\partial g_{\mu\nu}/\partial t=0$ and $g_{0\mu}=-\delta_\mu{}^0$, these equations reduce to the single condition

$$^3R=0, \qquad (1)$$

where ^3R is the curvature scalar for the three-dimensional metric g_{ij} of this initial surface. Corre-

* This research was supported by the Office of Naval Research.
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[1] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[2] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 594 (1957).
[3] Dieter R. Brill, Ann. Phys. **7**, 466 (1959).

sponding to any 3-metric g_{ij} satisfying (1), there is[4] a solution of the Einstein equations for which the metric at $t=0$ reduces to[5] $ds^2=-dt^2+g_{ij}dx^i dx^j$ and satisfies then $\partial g_{\mu\nu}/\partial t=0$; this solution is free from singularities for at least some finite time. We will obtain a wormhole solution of Eq. (1) by modifying the metric

$$ds_D{}^2=d\mu^2+(d\theta^2+\sin^2\theta d\varphi^2),\quad -\pi<\mu\leq\pi, \tag{2}$$

which represents a 3-dimensional doughnut $D=S^1\times S^2$ whose cross section ($\mu=$const) is a sphere. Part of this doughnut, near $\mu=\pi\equiv-\pi$, will become the tube connecting the wormhole mouths in Fig. 1. The antipodal part, near $\mu=0\equiv 2\pi$, must be ruptured and spread out to form the asymptotically flat space at infinity. The asymptotic form required here may be found from the metric of flat space in bispherical coordinates,

$$ds_F{}^2=(\cosh\mu-\cos\theta)^{-2}ds_D{}^2. \tag{3}$$

We limit attention to those wormhole metrics which may be written in the form

$$ds_W{}^2=\phi^4 ds_D{}^2. \tag{4}$$

It is required of ϕ that it be periodic in μ with period 2π, that as μ and θ approach 0,

$$\phi^4\sim(\cosh\mu-\cos\theta)^{-2}\sim 4(\mu^2+\theta^2)^{-2} \tag{5}$$

and that $ds_W{}^2$ satisfy Eq. (1), which in this case becomes[2]

$$\Delta_D\phi+\tfrac{1}{8}R_D\phi=-\left(\frac{\partial^2}{\partial\mu^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-\frac{1}{4}\right)\phi=0. \tag{6}$$

One solution of Eq. (6) immediately follows from the circumstance that $ds_F{}^2$ satisfies Eq. (1). From this solution the desired periodic solution is formed by addition:

$$\phi=\sum_{n=-\infty}^{\infty}[\cosh(\mu+2n\mu_0)-\cos\theta]^{-\frac{1}{2}}. \tag{7}$$

Here a slight generalization has been made by assuming that μ has period $2\mu_0$ (instead of 2π). The constant μ_0 allows for different ratios of mass to separation distance for the two "mouths." The absolute mass may be changed by any (positive) factor by multiplying ϕ^2 by this factor. The total energy or mass of the wormhole metric (4) using ϕ from Eq. (7) is found by comparison of its asymptotic form as $(\mu^2+\theta^2)\rightarrow 0$ with that of the Schwarzschild metric written in bispherical coordinates:

$$ds_S{}^2\sim[1+m(\mu^2+\theta^2)^{\frac{1}{2}}][4/(\mu^2+\theta^2)^2]ds_D{}^2. \tag{8}$$

In this way one finds

$$m_{\text{tot}}=4\sum_{n=1}^{\infty}(\sinh n\mu_0)^{-1}. \tag{9}$$

As a measure of the separation of the two "mouths" we may take the length L of the shortest closed loop through the wormhole

$$L=\int_{-\mu_0}^{\mu_0}\phi^2(\mu,\theta=\pi)d\mu. \tag{10}$$

This integral may be evaluated by noticing that $\phi(\mu,\theta=\pi)$ is an elliptic function. The result is

$$L=(4/\pi)K[(1-k^2)^{\frac{1}{2}}]E(k), \tag{11}$$

where K, E are complete elliptic integrals of the first and second kind, respectively, and the modulus k is fixed by the requirement that

$$\mu_0=\pi K(k)/K[(1-k^2)^{\frac{1}{2}}]. \tag{12}$$

The possibilities for obtaining information about the time development of this "wormhole" metric by using an electronic computer are being investigated by R. W. Lindquist.

The present metric, plus its time development, describe a two-body problem in terms of the geometrodynamics of curved empty space. Because of the symmetry of the metric under the transformation $\mu\rightarrow-\mu$, these two bodies are identical in every respect. Although they are initially at rest, the solution of the Einstein equations will show at later times $\partial g_{\mu\nu}/\partial t\neq 0$, corresponding to a motion of these bodies as well as the generation of gravitational waves due to their motion. From the work of Einstein, Infeld, and Hoffman we know that this motion is to lowest order just the Newtonian attraction of two equal masses. Because of the similarity of the wormhole tube ($\mu\simeq 0$) to the "neck" in the Schwarzschild solution,[6] we expect a singularity to develop here in a finite time T; in case the mouths are well separated ($\mu_0\gg 1$) we would have $T\simeq m_{\text{tot}}$. The gravitational field at time $t=0$ may be considered as entirely due to the bodies and containing no gravitational waves in addition, just as we would say that an electric field $\mathbf{E}=-\text{grad}\,(r_1{}^{-1}+r_2{}^{-1})$ contained no waves even though the charges were accelerated at $t=0$. For, with an appropriate choice of canonical variables for the gravitational field, it can be shown[7] that on a surface $t=0$ where $\partial g_{ij}/\partial t=0$ and $g_{0\mu}=-\delta_\mu{}^0$ the condition that g_{ij} be conformally flat is equivalent to the vanishing of all the gravitational canonical variables. (In the electromagnetic example above, the transverse parts of $\mathbf{A}$ and $\mathbf{E}$, which are canonical variables in electromagnetism, both vanish.)

[4] Y. Foures-Bruhat, Acta Math. 88, 141 (1952).

[5] We use the convention that Latin indices refer to space, Greek to space-time, and take units so that $\gamma=c=1$, where γ is the Newtonian gravitational constant.

[6] M. D. Kruskal, (to be published).

[7] R. Arnowitt, S. Deser, and C. W. Misner (to be published); see also, Phys. Rev. Letters 4, 375 (1960).