Random-Walk Interpretation and Generalization of Linear Boltzmann Equations, Particularly for Neutron Transport

E. GUTH AND E. INÖNÜ
Oak Ridge National Laboratory, Oak Ridge, Tennessee*

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The connection between linear recurrence relations which define generalized random walks and the related linear Boltzmann equations is clarified. The probability distribution $f_n(s)$ for "the state s" reached by a "random walker" after n steps satisfies the recurrence relation $f_{n+1}(t)=\int f_n(s)P(s,t)ds$, where the non-negative $P(s,t)$ is the probability for a transition from s to t. The Boltzmann distribution is given by $f(s)=\sum_{n=0}^{\infty} f_n(s)$. In general, $f_n(s)$ contains more information that $f(s)$. Moreover, $f_n(s)$ is the nth term in the iteration series solution of the Boltzmann equation and therefore can also be obtained from the solution of an associated Boltzmann equation which contains an additional parameter. As an example, the well-known integral Boltzmann equation for neutron transport in a nonmultiplying infinite medium is derived from a $P(s,t)$ which involves a transition in a seven-dimensional phase-time space. Brownian motion and Rayleigh's problem (related to neutron thermalization) may be treated similarly.

THE purpose of this note is to show the exact relation between linear Boltzmann integro-differential equations and recurrence relations which define random walks. Although many special problems of multiple scattering were treated in the past by using recurrence relations,[1] the general connection of the two methods does not seem to have been established.

A random walk consists of a succession of "elementary events" which change the state of the system and are statistically independent of each other. For random walk in the narrow sense of the word, the elementary events are the successive, uncorrelated steps. For multiple scattering of particles, the elementary event may be taken, in general, as a collision and subsequent traversal of a free path. It changes the state of the system from the one in which it was before a collision to the state in which it is before the next one.[2] In the random-walk problem we ask in general for the probability distribution $f_n(s)$ of the "distance" s reached by a "random walker" after n steps. On the other hand, for the multiple scattering of particles, the Boltzmann theory gives us a distribution function $f(s)$, regardless of the number of steps taken to reach s. More precisely

$$f_{\text{Boltzmann}}=\lim_{n\to\infty}\sum_{k=0}^{n} f_k(s). \tag{1}$$

Thus, using the Boltzmann theory, we lose some information. A more general theory which also yields the $f_k(s)$ can be formulated by generalizing the random-walk theory.

Consider as the simplest case a one-dimensional continuous random walk in an interval (a,b). Denoting by $P(t-s)$ the probability that a step changes the state s into a unit interval at t and which is assumed to depend only on $t-s$, we have the usual recurrence relation

$$f_{n+1}(t)=\int_a^b f_n(s)P(t-s)ds, \tag{2}$$

$f_0(s)$ and $P(t-s)$ are given. Naturally, $P(t-s)$ is non-negative; it is the continuous counterpart of a non-negative matrix used in discrete random processes.

From Eq. (2), by summing over n and using the definition $f(t)=\sum_{k=0}^{\infty} f_k(t)$ we obtain the Fredholm-type integral equation,

$$f(t)=f_0(t)+\int_a^b f(s)P(t-s)ds. \tag{3}$$

This equation corresponds to the Boltzmann equation for this simple case. Can $f_k(t)$ be obtained from $f(t)$? Clearly not in general! However, if instead of (3) one can solve the corresponding integral equation with an additional parameter λ,

$$f_\lambda(t)=f_0(t)+\lambda\int_a^b f_\lambda(s)P(t-s)ds, \tag{4}$$

then $f_k(t)$ will be obtained from $f_\lambda(t)$ by the relation

$$f_k(t)=(1/k!)[d^k f_\lambda(t)/d\lambda^k]_{\lambda=0}. \tag{5}$$

In the general case t may designate a point in a many-dimensional phase space (which may also include the time) and $P(s,t)$ may have any form although it remains nonnegative. We postulate the general recurrence relation

$$f_{n+1}(t)=\int f_n(s)P(s,t)ds, \tag{6}$$

where the integration is carried out over the whole many-dimensional domain which can be reached by the particles.

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[1] See, e.g., W. Bothe, Z. Physik **59**, 161 (1929); S. Goudsmit and J. L. Saunderson, Phys. Rev. **59**, 773 (1939), and **53**, 36 (1940). C. C. Grosjean, dissertation, Columbia University, 1951, (unpublished), and Physica **19**, 29 (1953). E. P. Wigner, Phys. Rev. **94**, 17 (1954).

[2] This precise formulation of the problem is taken over, almost verbatim, from Wigner's paper quoted in reference 1.

For neutron transport in a nonmultiplying infinite medium, the variable s represents a point in a seven-dimensional space $s \leftrightarrow (\mathbf{r},t,E,\mathbf{\Omega})$; where $\mathbf{\Omega}$ is a unit vector in the direction of velocity, E is the energy and $\mathbf{r},t$ are space-time variables. Taking as elementary event the collision plus subsequent free traversal, which changes the state of the particle from $(E',\mathbf{\Omega}',\mathbf{r}',t')$ before a collision to $(E,\mathbf{\Omega},\mathbf{r},t)$ before the next collision, one can write $P(s,t)$ in general as

$$P(\mathbf{r}', t', E', \mathbf{\Omega}' \rightarrow \mathbf{r}, t, E, \mathbf{\Omega}) = \sigma_s(\mathbf{r}'; E', \mathbf{\Omega}' \rightarrow E, \mathbf{\Omega}) e^{-\tau(E;\mathbf{r},\mathbf{r}')} \delta\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} - \mathbf{\Omega}\right) \times \delta\left(\frac{|\mathbf{r}-\mathbf{r}'|}{v} - (t-t')\right), \quad (7)$$

where

$$\tau(E;\mathbf{r},\mathbf{r}') = \int_0^{|\mathbf{r}-\mathbf{r}'|} \sigma_t(E; \mathbf{r}'+\rho\mathbf{\Omega}) d\rho;$$

$\sigma_s(\mathbf{r}'; E', \mathbf{\Omega}' \rightarrow E, \mathbf{\Omega})$ is the probability at $\mathbf{r}'$ of a scattering from $E',\mathbf{\Omega}'$ to $E,\mathbf{\Omega}$ per unit interval at $E,\mathbf{\Omega}$ and σ_t is the total cross section which may include absorption. As usual in the neutron transport theory, the cross sections are assumed to be independent of time. $f_n[\mathbf{r},t,E,\mathbf{\Omega})$ is the number of neutrons per "unit interval" at $(\mathbf{r},t,E,\mathbf{\Omega})$ after n events. Substituting (7) in (6), summing over n and expressing $f_0(s)$ by a source $S(\mathbf{r},t,E,\mathbf{\Omega})$, we obtain after integration over the δ functions:

$$f(\mathbf{r},t,E,\mathbf{\Omega}) = \int_0^\infty ds \left[\int dE' d\mathbf{\Omega}' f(\mathbf{r}-s\mathbf{\Omega}, t-s/v, E', \mathbf{\Omega}') \times \sigma_s(\mathbf{r}-s\mathbf{\Omega}; E', \mathbf{\Omega}' \rightarrow E, \mathbf{\Omega}) + S(\mathbf{r}-s\mathbf{\Omega}, t-s/v, E, \mathbf{\Omega}) \right] \times \exp[-\tau(E; \mathbf{r}, \mathbf{r}-s\mathbf{\Omega})], \quad (8)$$

which is the integral form of the time-dependent Boltzmann equation for neutron transport in a non-multiplying infinite medium.[3] Here again, $f_k(\mathbf{r},t,E,\mathbf{\Omega})$ cannot be obtained in general directly from $f(\mathbf{r},t,E,\mathbf{\Omega})$, but only from an f_λ which will be the solution of the equation corresponding to (4). (In the special case of constant cross sections, using the parameter $c=\sigma_s/\sigma_t$, f itself can be written as $f=\sum_{k=0}^\infty c^k f_k$ and thus yields f_k directly.) We note also that the solution of the Boltzmann equation by iteration involves using precisely the recurrence relation (6); i.e., the nth term in the iteration series satisfies the Eq. (6). By repeated application of the recurrence relation, one can express $f_n(s)$ formally as an n-fold convolution.

The problem of the reduction of this convolution to a single integral does not seem to have been considered in its full generality so far, although it is well known that in many special cases integral transforms accomplish this purpose. Obviously, the form of $P(s,t)$ will determine the usefulness of any transform in this respect.

Besides neutron transport, the above connection between recurrence relations and linear Boltzmann equations applies also among other cases to Brownian motion, Rayleigh's problem (which is related to neutron thermalization), and many other problems, discussed, for instance, in Uhlenbeck's Higgins lectures on statistical mechanics at Princeton (1954). The Schrödinger equation does not belong to this category. For here $P(s,t)=V(s)G(s,t)$ (G: Green's function for free space, $V(s)$: potential) is *not* nonnegative in general. For a comparison of "Schrödingerian motion" and "Brownian motion" we refer to a paper in preparation by one of us (E. G.) where particularly the "Einstein-Smoluchowski process" will be replaced by the "Uhlenbeck-Ornstein process" and connection established with Feynman's space-time form of quantum mechanics.

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[3] See, e.g., A. M. Weinberg and E. P. Wigner, *The Physical Theory of Neutron Chain Reactors* (The University of Chicago Press, Chicago, 1958), Chap. IX.