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# Meaning and Interpretation of Acoustic Momentum and Acoustic Radiation Stress

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This paper surveys viewpoints and available procedures of analysis of acoustic radiation stress and the concept of acoustic momentum.

An alternative approach is suggested based on Eckart's principle and Poynting's condition of separate observability of matter and elastic wave motion. The results are consistent with those obtained by means of the adiabatic principle but show discrepancies with the so-called Eulerian approach for transverse waves.

Radiation stress and radiation momentum are typical field concepts needed to complement the continuum description. In a sense they serve to establish consistency with the particle approach, thus filling a need which the conventional theory of elasticity does not satisfy.

The energy momentum tensor of elastic wave motion provides a unique example of a physically meaningful space-time covariance in the framework of Galilean kinematics.

## INTRODUCTION

RADIATION stress and radiation momentum in acoustics are known as controversial subjects in classical physics because of conceptual reasons rather than for lack of experimental evidence. The present paper makes an effort to bridge a gap between different lines of approach. A brief historical survey, therefore, seems appropriate to provide a suitable setting for the problem.

Poynting probably was one of the first physicists who associated a transfer of momentum, in a quite general way, with all sorts of different energy transfer. Quoting from his collected works, p. 336 reads[1]:

"There is it seems some general theorem yet to be discovered which shall relate directly the energy and momentum issuing from a radiating source. It seems possible that in all cases of energy transfer, momentum in the direction of transfer is also passed on, and therefore there is a back pressure on the source. Such pressure certainly exists in material transfer, as in the corpuscular theory. It exists, as we now know, in all wave transfer."

This statement in one of Poynting's presidential addresses to the Physical Society was a little bit too readily superseded by the era of relativity. The well-known mass energy relation was soon accepted as Poynting's theory "yet to be discovered" and disposed of the issure, in spite of the fact that the momentum following from relativity is by no means numerically identical to the momentum to be expected from Poynting's considerations.

Much of the more recent work in acoustics[2] seems to stress Poynting's momentum (energy density over phase velocity $\mathcal{E}/u$) as the physically significant concept, rather than the higher order typical relativistic momentum contribution $\mathcal{E}u/c^2$ which is due to energy "packing." The two momenta are numerically identical if the phase velocity $u$ equals the empty space light velocity $c$. This agreement for free space electromagnetics is significant in the sense that Poynting's arguments do not seem to violate any of the fundamentals of relativity. Therefore, a legitimate question seems to be whether Poynting's momentum concept still should not have a place in physics, provided that it can be brought in line with relativity all the way from electromagnetics to acoustics.

Poynting refers to a very simple proof, given by Larmor, for the existence of radiation pressure on a perfectly reflecting surface, using the Doppler principle and energy conservation.[3] This reflecting surface, however, should have very peculiar properties to give

[1] J. H. Poynting, Collected Papers (Cambridge University Press, New York, 1920).

[2] *Marseille Colloquium on Radiation Pressure, 1955* [J. Phys. radium **17**, 377–413 (1956)].

[3] E. J. Post, J. Acoust. Soc. Am. **25**, 55 (1953).

Larmor's derivation a straightforward physical meaning. We may again cite Poynting's own words (p. 336):

"It is essential, I think, to Larmor's proof that we should be able to move the reflecting surface forward without disturbing the medium except by reflecting the waves. In the case of light waves it is easy to imagine such a reflector. . . .In the case of sound waves or transverse waves in a solid it is not so easy to picture a reflector."

Please note the correspondence between the feasibility of Larmor's mental experiment for light with the agreement of Poynting's momentum concept with relativity for the case of electromagnetic waves in empty space. Poynting was so convinced that this momentum concept should have meaning beyond the realm of free space electromagnetics, that he did not refrain from depicting some very weird models of such reflectors (p. 337):

"We may perhaps think of a solid as melted by the advancing reflector, the products of melting being passed through pores in the surface and coming out to solidify at the back."

At that time Poynting could not possible anticipate that much later liquid helium experiments would provide rather convincing evidence for the feasibility of his semipermeable device, though rather different from any of the models he proposed. Furthermore it should be noted as another significant fact that the phenomenological theory of liquid helium quite independently led to the introduction of a double momentum balance (two fluid model).[4] This theory branched out into a theory of phonons, and it may be shown that Poynting's acoustical momentum is the classical counterpart of the so-called phonon momentum.

A semipermeable device in the sense as proposed by Poynting, for liquid helium, is demonstrated by the fountain effect, and its feasibility depends on the superfluid properties of the helium component in its lowest energy state. However, a single exception like this is a valid reason to consider the independent existence and necessity of radiation momentum and stress as classical realities, regardless of the typical quantum nature of a semipermeable device to show them.

A more subtle argument, independent of the physical feasibility of semipermeable devices, has been emphasized by Brillouin[5,6] on thermodynamical grounds. He shows the logical necessity of radiation stress of the Debye heat waves in isotropic solids by correlating radiation stress and thermal expansion. This gives an alternative approach to the theory of thermal expansion and leads in a straightforward way to Grüneisen's "nonlinearity" parameter $\gamma$.

Here, however, one should guard against a commonly expressed misconception. A linear lattice is not equivalent to a (linear) Hooke's continuum. A linear lattice does not exhibit thermal expansion and it can be reduced (collapsed) to one lattice point by a finite force because the force is proportional to the deviation of lattice points from their equilibrium positions. A Hooke's medium requires an infinite force for the same operation. It is easily demonstrated for the one-dimensional case that the force required becomes logarithmically infinite. Hence the lattice corresponding to a Hooke's medium should have nonlinear interaction between the lattice points and should correspondingly show thermal expansion. The conventional theory of elasticity does not account for the expansion of the Hooke's medium, and this is where a radiation stress is necessary to complete the field description and to establish consistency with the lattice picutre.

The crucial point is that radiation stress and radiation momentum are typical field concepts. It is not meaningful to discuss them in terms of particle dynamics. One may discuss particle momentum in terms of radiation momentum, as remarked by Lucas,[2] provided one uses the field concepts of wave mechanics.[7]

Brillouin[5] furthermore resolved a controversy which developed between the Rayleigh and Langevin readings of the radiation stress problem by a clear-cut distinction between the isotropic and directional components and their corresponding physical interpretations. The directional part is the flux of radiation momentum per unit area. The omnidirectional contribution depends on deviations from Hooke's law and is usually denoted $\langle\mathcal{L}\rangle$, the time average of the difference between kinetic and potential energy density. Expressed in terms of a matrix, the following form, for a traveling single frequency wave, is reasonably well agreed on for fluids[2];

$$\left(\begin{array}{c|c} \mathcal{E} & \mathcal{E}g^a \\ \hline (\mathcal{E}/\omega)k_b & [(\mathcal{E}/\omega)g^a k_b - \delta_b{}^a\langle\mathcal{L}\rangle] \end{array}\right) = \left(\begin{array}{c|c} \text{radiation energy density} & \text{radiation energy flux} \\ \hline \text{radiation momentum} & \text{radiation stress} \end{array}\right), \tag{1}$$

where $\omega$=frequency, $k_b$=wave vector, and $g^a$=group velocity, with $a,b=1,2,3$; for nondispersive, isotropic media $\omega/|k|=|g|$.

The theoretical justification of (1), however, leads to rather disconcerting difficulties of a fundamental nature.

The Eulerian equations of continuum mechanics lead to a form equivalent to (1).[8] The directional

[4] F. London, *Superfluids* (John Wiley & Sons, New York, 1954).

[5] L. Brillouin, *Les tenseurs en mécanique et élasticité* (Masson, Paris, 1938).

[6] L. Brillouin, *Wave Propagation in Periodic Structures* (McGraw-Hill Book Company, New York, 1946).

[7] The energy of a particle is $E\approx mc^2$, its phase velocity is $u\approx c^2/v$, if $v$ is the particle velocity, then the Poynting momentum equals the conventional particle momentum $E/u=mv$.

[8] Many derivations omitted, but mentioned for the sake of comparison, are to be found in the Marseille Colloquium, reference 2.

component of the radiation stress then is given by the particle momentum flux, $\rho v^2$. The momentum flux for longitudinal waves is in the direction of propagation $k_b$ and roughly equal to the energy density $\mathcal{E}$. The particle momentum flux for transverse waves, however, has its principal direction perpendicular to the direction of propagation, which happens to be at variance with (1).

There is, on the other hand, considerable evidence that the form (1) is correct for waves of any direction of polarization, i.e., longitudinal and transverse. Weinreich[9] has shown that in particular, transverse waves happen to be active in the acousto-electric effect with a momentum corresponding to (1), which automatically leads to a momentum flux in the direction of propagation. Moreover there seems to be little reason to question that phonons corresponding to transverse waves carry their momentum in the direction of propagation.

Brillouin[5] has proposed a justification of (1) on the basis of the adiabatic principle (Ehrenfest-Boltzmann principle) which indeed leads to the form (1) regardless of the direction of polarization of the waves. It actually suggests that the form (1) is meaningful for arbitrary anisotropic matter and supports the suspicion that the usual Eulerian approach leads to correct results only for gaseous and liquid matter, but not for solid matter.

One may attempt to disentangle this situation by retracing some of the steps invoked by the derivation of the adiabatic principle as compared with the usual textbook derivation of the Eulerian equations.

Brillouin[5] starting from particle dynamics, applies the theory of constraints and identifies the hidden coordinates of constraint with the macroscopic variables perturbing a single frequency periodic system. A slow perturbation in the coordinates of constraint causes an energy change $\delta E$ of the system and a corresponding frequency change $\delta\omega$. They obey the relation

$$\delta E/E=\delta\omega/\omega, \qquad (2)$$

known as the adiabatic principle.

From here on it is perfectly valid to make a transition to continuum dynamics, since $E$ represents the total energy and nothing in the relation (2) can tell whether it applies to a particle system or a continuum. This may be regarded as a criterion for a clean transition from a particle to a continuum model.

For comparison, the particle continuum transition invoked by the derivation of the Eulerian equations is still the source of much discussion and literature, sometimes referred to as eighteenth century physics and mathematics. Nevertheless, similar conceptual inconsistencies still persist in modern field theory.

Brillouin has shown that it is possible to reconcile the Eulerian and the adiabatic approaches reasonably well, though the asymmetry between longitudinal and transverse waves still persists[10] (reference 5, pp. 290–293). His analysis requires a much more delicate distinction between the stress components with reference to points fixed in the medium and the stress components with respect to points fixed in space (reference 5, p. 249). In short, the notorious Lagrange-Euler transition is carried out one step more accurately than usual.

One would expect a further improvement by a distinction between coordinate systems which are in motion with respect to each other. The so-called Lagrangian and Eulerian stress components then become members of one family if the time is treated as a coordinate. However, relativity has never really enriched this part of classical physics. There is something more at stake than just the replacement of ordinary by covariant differentiation.

In the following section it will be found useful to treat time as a coordinate. Time as a coordinate does not necessitate the consideration of relativistic aspects. It enables one to do so, if desired, though for the time being it seems better to avoid relativity.

Summarizing the situation, after this necessarily sketchy account of theoretical investigation, the following points may be stressed for any alternative approach:

(1) Radiation momentum and radiation stress are essentially field concepts.

(2) A realistic field approach requires a clear relationship between the particle and continuum models.

(3) Time as a coordinate is helpful to disentangle the Euler-Lagrangian descriptions.

A field approach by means of a Hamiltonian principle for a continuum, in the sense discussed by Eckart,[11] meets this program reasonably well. This kind of approach has been used by Zilzel[4] to coordinate the two momenta concept for helium. However, instead of postulating a two momenta concept, one may now consider whether Poynting's weird condition of separate observability of medium and wave motion gives the desired radiation momentum as a distinct field quantity. It will be shown in the next section that this indeed can be done if one allows an independent boundary variation corresponding to the perturbation of the coordinates of constraint in the derivation of the adiabatic principle.

## THE VARIATION OF THE ACTION INTEGRAL FOR INDEPENDENT PARTICLE DISPLACEMENT AND BOUNDARY DISPLACEMENT

The following deductions gain much in elegance and brevity when a four-dimensional presentation is used. The expressions then will be covariant with respect to

[9] G. Weinreich, Phys. Rev. **104**, 321 (1956).

[10] The experimental implications of this discrepancy are being discussed in a paper of the forthcoming Proceedings of the Third International Congress on Acoustics [Elsevier Publishing Company, Inc., Amsterdam (to be published)].

[11] C. Eckart, Phys. Rev. **54**, 920 (1938).

TABLE I. Physical identification of the space-time components of $T_\nu{}^\lambda$, defined by (10).

| | |
|---|---|
| $T_0{}^0$ | mass energy density |
| $T_0{}^a$ | mass energy flow |
| $T_a{}^0$ | particle momentum density |
| $T_b{}^a$ | mechanical stress |

affine transformations, which incidentally allows one to consider Lorentz covariance.

The four-vector of particle displacement is denoted by $u^\lambda$; the fouf-vector of the boundary displacement is denoted by $q^\lambda$. The Lagrangian density is a function of $u^\lambda$ and its first derivatives only, for the latter suffice to include finite deformation. The action integral then becomes

$$A=\int_f \mathfrak{L}(u^\lambda,\partial_\nu u^\lambda)df,\quad \left(\partial_\nu=\frac{\partial}{\partial x^\nu}\right), \tag{3}$$

where $df$ is an integration element of the space-time manifold. Superscripts denote contravariant transformation behavior and subscripts denote covariant behavior. The range of the indices $\lambda,\nu$ is 0,1,2,3, the zero corresponding to the time coordinate and the others to the space coordinates. When the same index occurs both as a superscript and a subscript, summation is understood.

A variation of (3) with respect to $u^\lambda$, denoted by $\delta_1 A$, leads to the familiar result

$$\delta_1 A=\int_f [\mathfrak{L}]_\lambda \delta u^\lambda df+\int_f \partial_\nu\left[\frac{\partial\mathfrak{L}}{\partial(\partial_\nu u^\lambda)}\delta u^\lambda\right]df. \tag{4}$$

The functional derivative $[\mathfrak{L}]_\lambda$ is defined by

$$[\mathfrak{L}]_\lambda=\partial\mathfrak{L}/\partial u^\lambda-\partial_\nu[\partial\mathfrak{L}/\partial(\partial_\nu u^\lambda)]. \tag{4a}$$

It vanishes for arbitrary $\delta u^\lambda$ in the domain $f$.

A variation of (3) with respect to the boundary displacement $q^\nu$, denoted by $\delta_2 A$, simply leads to

$$\delta_2 A=\int_f \partial_\nu(\mathfrak{L}\delta q^\nu)df. \tag{5}$$

Considering the fact that (4a) should vanish, the total variation

$$\delta A=\delta_1 A+\delta_2 A, \tag{6}$$

of (2) then is given by

$$\delta A=\int_f \partial_\nu\left[\frac{\partial\mathfrak{L}}{\partial(\partial_\nu u^\lambda)}\delta u^\lambda+\mathfrak{L}\delta q^\nu\right]df. \tag{7}$$

Following a procedure very similar to that in particle dynamics, one may remark that $\delta u^\lambda$ correlates the initial and final positions of the "identifiable" space time points in the medium for $\delta q^\lambda=0$. For $\delta q^\lambda\neq 0$, the actual space time variation $\Delta u^\lambda$ of the particles will be

$$\Delta u^\lambda=\delta u^\lambda+(\partial_\sigma u^\lambda)\delta q^\sigma \tag{8}$$

(see the Appendix).

The elimination of $\delta u^\lambda$ from (7) by means of (8) gives the following expression for the total variation of the action integral:

$$\delta A=\int_f \partial_\nu\left\{\frac{\partial\mathfrak{L}}{\partial(\partial_\nu u^\lambda)}\Delta u^\lambda+\left[\mathfrak{L}\delta_\sigma{}^\nu-\frac{\partial\mathfrak{L}}{\partial(\partial_\nu u^\lambda)}(\partial_\sigma u^\lambda)\right]\delta q^\sigma\right\}df. \tag{9}$$

The integrand of (9) is a divergence which consists of the sum of two tensors (tensor densities) contracted with $\Delta u^\lambda$ and $\delta q^\sigma$, respectively. These tensor densities, which incidentally have mixed co-contravariant transformation behavior, may be denoted by the symbols

$$\partial\mathfrak{L}/\partial(\partial_\nu u^\lambda)=T_\lambda{}^\nu, \tag{10}$$

$$\{\mathfrak{L}\delta_\lambda{}^\nu-[\partial\mathfrak{L}/\partial(\partial_\nu u^\sigma)]\partial_\lambda u^\sigma\}=-K_\lambda{}^\nu. \tag{11}$$

The last tensor (11) will be recognized as a tensor of a form frequently discussed in texts on field theory.[12] Both of them obey divergence relations

$$\partial_\nu T_\lambda{}^\nu=\partial\mathfrak{L}/\partial u^\lambda\quad [\text{see (4a)}], \tag{12}$$

$$\partial_\nu K_\lambda{}^\nu=-\partial_{(\lambda)}\mathfrak{L}\quad (\text{see reference 9}), \tag{13}$$

where $\partial_{(\lambda)}\mathfrak{L}$ represents the explicit derivative of $\mathfrak{L}$, i.e., for $u^\lambda$ and $\partial_\nu u^\lambda$ constant.

The physical meaning of the space-time components of $T_\nu{}^\lambda$ and $K_\nu{}^\lambda$ are given in Tables I and II. The time label is denoted by zero, and the pure space labels are given by Latin subscripts and superscripts $a,b=1,2,3$.

The identification given in Table I is the usual one encountered in texts dealing with four-dimensional continuum dynamics. The identification of the typical relativistic terms like $T_0{}^0$ is not essential, it can be evaded by choosing $u^0=0$.

The identification of $K_0{}^0$ and $K_0{}^a$ in Table II are discussed in Morse and Feshbach[13] for an isotropic elastic solid. Morse and Feshbach leave the interpretation of $K_a{}^0$ and $K_b{}^a$ open for discussion (see the end of the section "Isotropic Elastic Media," p. 323). One may, therefore, consider the justification of the

TABLE II. Physical identification of the space-time components of $K_\nu{}^\lambda$, defined by (11).

| | |
|---|---|
| $K_0{}^0$ | density of elastic radiation energy |
| $K_0{}^a$ | flux of radiation energy |
| $K_a{}^0$ | radiation momentum density |
| $K_b{}^a$ | radiation stress |

[12] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, New York, 1949), Chap. I, Sec. 2.

[13] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, 1953), Vol. I, pp. 322–323.

identification of $K_a{}^0$ and $K_b{}^a$ as the actual subject matter of the present article.

## FURTHER DISCUSSION OF THE IDENTIFICATION OF THE COMPONENTS OF $K_\lambda{}^\nu$

For a check on the identification of $K_a{}^0$ and $K_b{}^a$, one may in particular direct attention to $K_b{}^a$, for a stress is a more readily observable quantity. The radiation momentum is a necessary computational by-product to complete the balance of energy momentum [see (13)], which may be used perfectly validly in this sense (see J. E. Piercy, reference 2).

The ideal procedure is to evaluate the physical meaning of $K_b{}^a$ for a general medium, i.e., unspecified $\mathcal{L}$, except that it depends on $u^\lambda$ and its derivatives. A simple way of doing this is to consider a free elastic body such that the integration boundary coincides with the boundary particles of the body itself. This means that there is no transport of elastic radiation energy and mass into the adjacent medium (free space) and furthermore it means that the actual displacement of the particles at the boundary is identical to the boundary displacement. This is expressed by the following relations on the bounding surface $\sigma$ of the body in three-space:

$$K_0{}^a=0, \quad T_0{}^a=0, \quad \Delta u^\lambda=\delta q^\lambda. \tag{14}$$

The conditions (14) applied to the integral expression (9) plus the separation of time and space integrations and the application of Gauss' law yields

$$\delta A=\int_{t1}^{t_2} dt \int_\sigma (T_b{}^a-K_b{}^a)\delta q^b d\sigma_a + \int_\tau (T_\lambda{}^0-K_\lambda{}^0)\delta q^\lambda \Big|_{t1}^{t_2} d\tau, \tag{15}$$

where $\tau$ is the domain of three-space occupied by the body, enclosed by the bounding surface $\sigma$.

The second integral can be chosen to disappear for $\delta q^\lambda=0$ at times $t=t_1$ and $t=t_2$. For the action integral $A$ to have an extremum for arbitrary $\delta q^b$, the time average of the mechanical forces must balance the time average of the radiation forces on the free boundary $\sigma$ with normal direction $n_a$, i.e.,

$$\langle T_b{}^a n_a\rangle=\langle K_b{}^a n_a\rangle. \tag{16}$$

This is exactly what is commonly understood by radiation stress.

The reader may check that $K_\nu{}^\lambda$ contains expression (1), given in the introduction for a single frequency plane wave. This is obvious for the omnidirectional part if we take $\mathcal{L}$ as the difference between the (classical) kinetic and potential energy densities of radiation. It should be noted, however, that this is equivalent to choosing a specific reference of zero pressure. For the directional part one may consider that $T_\lambda{}^\nu$ itself occurs in the expression (11) for $K_\lambda{}^\nu$. It is sufficient, and in general a good first order approximation, to substitute only the sinusoidal contribution of $T_\lambda{}^\nu$. This gives a $\langle K_b{}^a\rangle\neq 0$, though $\langle T_b{}^a\rangle=0$ for the sinusoidal first order contribution. The general identification of $K_\lambda{}^\nu$ with expression (1) then is straightforward even for anisotropic matter, where the group velocity $g^a$ is not parallel to the wave vector $k_a$. The simplified proof runs as follows: The definition formula (11) gives

$$\langle K_b{}^0\rangle=\langle T_c{}^0\partial_b u^c\rangle=\rho\omega u_c u^c k_b=(\mathcal{E}/\omega)k_b, \tag{17}$$

because $T_c{}^0=\rho\omega u_c$ according to (10) and the time average of twice the kinetic energy density equals the energy density in a linear Hooke's medium.

It is known already that

$$\langle K_0{}^a\rangle=\langle T_c{}^a\partial_0 u^c\rangle=T_c{}^a u^c\omega=\mathcal{E}g^a. \tag{18}$$

One may prove this though by means of the formula for the group velocity

$$\partial\omega/\partial k_a=g^a. \tag{19}$$

The radiation stress according to (11) is

$$\langle K_b{}^a\rangle=\langle T_c{}^a\partial_b u^c-\mathcal{L}\delta_b{}^a\rangle=T_c{}^a u^c k_b-\langle\mathcal{L}\rangle\delta_b{}^a. \tag{20}$$

Using (18) the expression (20) can be written as

$$\langle K_b{}^a\rangle=(\mathcal{E}/\omega)g^a k_b-\langle\mathcal{L}\rangle\delta_b{}^a, \quad \text{q.e.d.} \tag{21}$$

The elastic constants did not occur explicitly in the derivation, so the results hold for isotropic as well as anisotropic matter.

A four-dimensional presentation in essence postulates the applicability of the covariance principle for the domain of space and time. The variational form leads quite naturally to this. For acoustic phenomena, however, it is perfectly valid to check the covariance of $K_\nu{}^\lambda$ on the basis of the Galilean group. The reader who is interested to do so will be rewarded with a rather exclusive example of a meaningful space-time covariance in the realms of Galilean kinematics.[2] But note that $K_\nu{}^\lambda$ has mixed co-contravariant transformation behavior, because there is no metric tensor available to raise or lower the time indices.

Lorentz covariance only adds, as to be expected, physically insignificant corrections. Similarly, the consideration of quantum aspects does not contribute really new features to this essentially classical description. Normal mode expansion and quantization yields the phonon momentum as the analogue of radiation momentum. Substitution of $\mathcal{E}=n\hbar\omega$ in (1), where $n$ is the density of phonons per unit volume, gives $K_a{}^0=n\hbar k_a$. The directional contribution of the radiation stress turns out to be the phonon momentum flux per unit area $n\hbar k_a g^b$.

## CONCLUSION

Eckart's principle and Poynting's condition of separate observability of matter and wave motion lead to a decomposition of the conventional energy momen-

tum tensor into two parts. The first part is the energy momentum of matter and the second part represents the energy momentum of elastic radiation. The latter is the true analog of the energy momentum tensor of electromagnetic radiation.

The results obtained by means of Eckart's principle and Poynting's condition of separate observability of matter and wave motion are in close agreement with results derived by means of the adiabatic principle.

The peculiar position of the Eulerian approach raises questions about the limits of applicability of the Eulerian method for transverse waves in solids.

The concepts of acoustic momentum for transverse waves in general cannot be associated in any simple manner with particle momentum; it should be accepted rather on logical grounds as a computational necessity. As such acoustic momentum has most similarity with electromagnetic momentum in matter filled space. The latter $(D\times B)$ likewise escapes an interpretation in terms of relativistic inertia energy.

The decomposition of stress, as quoted above, ceases to be meaningful for nonperiodic or quasi-static phenomena say. A change in action then corresponds to a change in potential energy, and the integrant of (15) represents the total surface force density.

Dr. R. A. Toupin (Naval Research Laboratory, Washington) informed us that the total surface stress in (15), (which in our present terminology is mechanical stress minus radiation stress), indeed can be identified with the stress expressions for finite strain as discussed in Truesdell's memoir.[14]

The problem of radiation stress then seems to reduce to a question of singling out the physically relevant component in the total stress for dynamic phenomena.

The tensor features of radiation stress in fluids have been neatly demonstrated by Herrey.[15] A similar independent check for transverse waves in solids would be desirable to conclude whether the formal, unified approach suggested in this paper is physically justified. Some of the new semi fluids, which behave like high frequency solids, might make this sort of an experiment a feasible one.

### ACKNOWLEDGMENTS

Many discussions with Dr. Gabriel Weinreich greatly contributed to delineate the concept of acoustic momentum. Dr. H. P. Kramer and Dr. G. Raisbeck who reread the whole manuscript are gratefully acknowledged for very substantial help which led to a smoother and more concise presentation.

[14] C. Truesdell, J. Rational Mech. Analysis **1**, 125–300 (1952).
[15] E. M. Herrey, J. Acoust. Soc. Am. **27**, 891 (1955).

### APPENDIX

Formula (8) is strictly speaking correct for a scalar field only, if one insists on general covariance in four-space. A correct field approach for a vector $u^\lambda$ requires the form

$$\Delta u^\lambda = \delta u^\lambda + \mathfrak{L}_{\delta q} u^\lambda, \tag{22}$$

where

$$\mathfrak{L}_{\delta q} u^\lambda = \delta q^\sigma(\partial_\sigma u^\lambda) - u^\sigma(\partial_\sigma \delta q^\lambda) \tag{23}$$

is known as the Lie differential[16] of $u^\lambda$ with respect to $\delta q^\sigma$; $\Delta u^\lambda$ then is still a true vector for general coordinate transformations.

The extra term $-u^\sigma(\partial_\sigma \delta q^\lambda)$ gives rise to an additional integral term in (9), i.e.,

$$+\int_f \partial_\nu[T_\lambda{}^\nu u^\sigma(\partial_\sigma \delta q^\lambda)]df. \tag{24}$$

Partial integration gives

$$\begin{aligned} &+\int_f \partial_\nu\partial_\sigma(T_\lambda{}^\nu u^\sigma \delta q^\lambda)df, \\ &-\int_f \partial_\nu(\delta q^\lambda \partial_\sigma T_\lambda{}^\nu u^\sigma)df. \end{aligned} \tag{25}$$

The first integral in (25) should vanish, because it is a double divergence (in three-space it would reduce to a contour integral which can be shrunk to a point). The weakest condition to do this is to impose

$$T_\lambda{}^{\nu u \sigma} = \tfrac{1}{2}(T_\lambda{}^\nu u^\sigma - T_\lambda{}^\sigma u^\nu) = T_\lambda{}^{[\nu} u^{\sigma]}. \tag{26}$$

Admissible Lagrangian functions should satisfy (26). The antisymmetric form (26) can be associated with the angular momentum density as discussed in Wentzel.[12]

The second integral then leads to an additional term for $K_\lambda{}^\nu$;

$$\partial_\sigma(T_\lambda{}^{[\nu} u^{\sigma]}). \tag{27}$$

The divergence of (27) is identically zero, so it does not contribute in the balance of linear momentum. It vanishes altogether if $T^{\lambda\nu} = T^{\nu\lambda}$.

[16] J. A. Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin, 1954), second edition, Chap. II, Sec. 10, and Chap. VII.