TABLE II. Values of carbon polarization and beam polarization obtained for different average beam energies. Standard deviations include all known sources of error.

Energy, Mev	P_{carbon} (15°)	P_{beam}
124±1	0.474±0.020	0.447±0.034
128±1	0.502±0.040	0.498±0.051

The polarization is a rapidly increasing function of energy. Measurements were made with different cutoff energies in the neutron counter, and the results are displayed in Table II. As a check on the method, it is interesting to compare the value of P_c obtained with the p-carbon[9–12] and n-carbon[2] polarizations measured at other laboratories. Figure 8 shows this comparison. All errors have been increased to include the uncertainty in beam polarization. The n-carbon result of Harding has been shown as originally reported, and as corrected for a higher beam polarization.[13]

It is also clear that neutron shielding can be designed with accuracy. Calculations based on inelastic cross sections have been shown to be adequate.

[9] J. M. Dickson and D. C. Salter, Nuovo cimento, **6**, 235 (1957).
[10] R. Alphonce, A. Johansson, and G. Tibell, Nuclear Phys. **3**, 185 (1957).
[11] E. M. Hafner, Phys. Rev. **111**, 297 (1958).
[12] O. Chamberlain, E. Segrè, R. D. Tripp, C. Wiegand and T. Ypsilantis, Phys. Rev. **102**, 1659 (1956).

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[13] Richard Wilson, Phys. Rev. **114**, 260 (1959).

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Electromagnetic Waves in Gravitational Fields

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The scattering of plane electromagnetic waves by the gravitational field of an isolated physical system is studied. On the level of the geometrical optics approximation the general theory of light rays is formulated. In particular, the generalized formula for the Einstein deflection of light rays is obtained. On the level of the vectorial optics the problem of polarization is examined in detail. The formula obtained, describing a rotation of the plane of polarization due to the presence of the gravitational field, admits a direct geometrical interpretation. The theory is applied to the rotating body and a system of point masses. The physical results established concerning the asymptotic behavior of the electromagnetic waves are independent of the coordinate system used in the computations.

1. INTRODUCTION

THE aim of this paper is to investigate the scattering of electromagnetic plane waves due to the gravitational field of a general isolated physical system, e.g., to the field of a rotating body or the field of a system of masses carrying out the motion according to Newton's laws (a double star, for instance). Generally, by "gravitational field of an isolated system" we understand the metric tensor $g_{\alpha\beta}$ in an arbitrary coordinate system, this metric being induced by matter in motion which during the motion is concentrated in a somewhat finite 3-region Ω of the spatial coordinates x^a. We assume the deviations of these quantities from Galilean values $\eta_{\alpha\beta}$ ($\eta_{00}=1$, $\eta_{0a}=0$, $\eta_{ab}=-\delta_{ab}$) defined as $\Delta g_{\alpha\beta}=g_{\alpha\beta}-\eta_{\alpha\beta}$ to vanish at infinity together with their derivatives at least as $O(r^{-1})$, $r=(x^ax^a)^{\frac{1}{2}}$.

We will also assume that the influence of the electromagnetic field on the metric field can be neglected. When the intensities of the waves scattered by the g field are small, the last assumption is certainly physically correct. We should like to mention here that the scattering of electromagnetic waves by the field of a rotating body has been recently examined independently by Skrotskii,[1] and Balazs.[2] In the center of interest of this paper, however, is the behavior of electromagnetic waves in the presence of a gravitational field induced by an isolated system when this field is as general as possible, consistently with it being physically reasonable. The solution of our general scattering problem, how-

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[1] G. B. Skrotskii, Doklady Akad. Nauk S.S.S.R. **114**, 73 (1957) [translation: Soviet Phys.-Doklady **2**, 226 (1957)].
[2] N. L. Balazs, Phys. Rev. **110**, 236 (1958).

ever, will provide us also with a new approach to the interesting case of the rotating body.

2. MAXWELL'S EQUATIONS IN GENERAL RELATIVITY THEORY AS ELECTRODYNAMICS IN A MACROSCOPIC MEDIUM

The basic equations of our problem are of course:

$$f^{\alpha\beta}{}_{;\beta}=(4\pi/c)j^{\alpha},\quad f_{[\alpha\beta;\gamma]}=0, \tag{2.1}$$

where the skew-symmetric $f_{\alpha\beta}$ is the electromagnetic field tensor, j_{α} stands for 4-current vector, [ ] denotes alternation symbol, and semicolon stands for covariant differentiation. When $j^{\alpha}=0$ the conventional energy-momentum tensor,

$$\mathcal{E}^{\alpha\beta}=\frac{1}{4\pi c}(-f^{\alpha\sigma}f^{\beta}{}_{\sigma}+\tfrac{1}{4}g^{\alpha\beta}f^{\rho\sigma}f_{\rho\sigma}), \tag{2.2}$$

fulfills in virtue of (2.1):

$$\mathcal{E}^{\alpha\beta}{}_{;\beta}=0. \tag{2.3}$$

For our purpose, however, it will be convenient to rewrite these basic equations in another form (i.e., in a noncovariant notation), in which they will be formally equivalent to the equations of electrodynamics in a macroscopic medium in the case of the flat space-time.[3] As is well known, (2.1) are equivalent to

$$[(-g)^{\frac{1}{2}}f^{\alpha\beta}]_{,\beta}=\frac{4\pi}{c}(-g)^{\frac{1}{2}}j^{\alpha},\quad f_{[\alpha\beta,\gamma]}=0. \tag{2.4}$$

(A comma stands for ordinary differentiation.) Therefore, if we introduce

$$E_a=f_{a0},\quad B_a=\tfrac{1}{2}\epsilon_{abc}f_{bc},\quad D_a=(-g)^{\frac{1}{2}}f^{0a},\quad H_a=\tfrac{1}{2}\epsilon_{abc}(-g)^{\frac{1}{2}}f^{bc}, \tag{2.5}$$

(ϵ_{abc} is the three-dimensional Levi-Civita symbol), we can rewrite (2.4) as

$$\begin{gathered}-D_{a,0}+\epsilon_{abc}H_{c,b}=\frac{4\pi}{c}i_a,\quad B_{a,0}+\epsilon_{abc}E_{c,b}=0,\\ D_{a,a}=4\pi\rho,\quad B_{a,a}=0,\end{gathered} \tag{2.6}$$

where $i_a=(-g)^{\frac{1}{2}}j^a$, $\rho=(-g)^{\frac{1}{2}}j^0/c$. The relations between D, B and E, H (corresponding to the material equations of the Lorentz theory) can be obtained from

$$(-g)^{\frac{1}{2}}f^{\alpha\beta}=(-g)^{\frac{1}{2}}g^{\alpha\nu}g^{\beta\mu}f_{\nu\mu},\quad f_{\nu\mu}=(-g)^{-\frac{1}{2}}g_{\nu\alpha}g_{\mu\beta}(-g)^{\frac{1}{2}}f^{\alpha\beta}. \tag{2.7}$$

A simple computation (in which it is convenient to use the properties of the 3-dimensional metric $e_{ab}=-g_{ab}+g_{a0}g_{b0}/g_{00}$) shows that those relations have the simple form,

$$\begin{aligned}D_a&=E_a+\epsilon_{ab}E_b+\epsilon_{abc}g_bH_c,\\ B_a&=H_a+\epsilon_{ab}H_b-\epsilon_{abc}g_bE_c,\end{aligned} \tag{2.8}$$

where

$$\epsilon_{ab}=-\frac{(-g)^{\frac{1}{2}}}{g_{00}}g^{ab}-\delta^{ab},\quad g_a=\frac{g_{a0}}{g_{00}}, \tag{2.9}$$

At infinity, where $g_{\alpha\beta}$ tends to $\eta_{\alpha\beta}$, both ϵ_{ab} and g_a vanish so that there D coincides with E and, respectively, B with E.

The "mixed" energy-momentum density $(-g)^{\frac{1}{2}}\mathcal{E}_{\beta}{}^{\alpha}$ can be easily expressed in terms of our "macroscopic" quantities as

$$\begin{aligned}4\pi c(-g)^{\frac{1}{2}}\mathcal{E}_0{}^0&=\tfrac{1}{2}(D_sE_s+B_sH_s),\\ 4\pi c(-g)^{\frac{1}{2}}\mathcal{E}_0{}^a&=\epsilon_{abc}E_bH_c,\\ 4\pi c(-g)^{\frac{1}{2}}\mathcal{E}_a{}^0&=\epsilon_{abc}D_bB_c,\end{aligned} \tag{2.10}$$

$$4\pi c(-g)^{\frac{1}{2}}\mathcal{E}_b{}^a=-\tfrac{1}{2}\delta_b{}^a(D_sE_s+B_sH_s)+D_aE_b+B_aH_b,$$

and (if $j_\alpha=0$, i.e., $\rho=0$, $i_a=0$) the equations

$$[(-g)^{\frac{1}{2}}\mathcal{E}_{\alpha}{}^{\beta}]_{;\beta}\equiv[(-g)^{\frac{1}{2}}\mathcal{E}_{\alpha}{}^{\beta}]_{,\beta}-\tfrac{1}{2}g_{\nu\mu,\alpha}(-g)^{\frac{1}{2}}\mathcal{E}^{\nu\mu}=0 \tag{2.11}$$

are the conservation theorems in a convenient form.

The equivalence of (2.1) and the "macroscopic formulation" described by (2.6) and (2.8), where ϵ_{ab} and g_a are given by (2.9), holds in any coordinate system. The analogy with macroscopic electrodynamics makes sense, however, only if the coordinates used correspond to the Cartesian ones. In this paper we will limit ourselves just to such coordinates.

Now, we are sufficiently prepared to formulate our scattering problem more precisely. Let us treat the $g_{\alpha\beta}$ field and therefore ϵ_{ab}, g_a as given functions in coordinates corresponding to Cartesian coordinates, those quantities being induced by the matter moving in some finite region of x^a, say Ω. Suppose now that at "time" $x^0=-\infty$ a plane wave with given wavelength, direction, and polarization is propagated towards the gravitational field due to the matter in Ω. When $x^0=+\infty$ our wave certainly might be observed by observers at infinity "on the other side of Ω." The question arises as to what differently located observers can measure? How will the direction of the wave, and its polarization, be changed after scattering due to the gravitational field? The aim of this paper is to give a general answer to these questions.

3. QUANTITIES ϵ_{ab} AND g_a IN PRACTICAL APPLICATIONS

The general feature of the g's induced by real matter in motion is that its deviations from Euclidean values are extremely small, provided that we use a physically reasonable coordinates system. Indeed, even in Ω (assuming a continuous distribution of matter) the $\Delta g_{\alpha\beta}=g_{\alpha\beta}-\eta_{\alpha\beta}$ are of order $O(R_g/L)$, $R_g=kM/c^2$, M

[3] The possibility of such an interpretation was first pointed out by J. E. Tamm, J. Russ. Phys.-Chem. Soc. 56, 2–3, 284 (1924).

being total mass of the system, k the gravitational constant, L a constant of the order of magnitude of the radius of Ω. Because of the smallness of k/c^2 in all practical applications $R_g/L \ll 1$. Moreover, as is well known, the theory involves an other parameter of smallness, viz., $\beta = v/c$, v being a characteristic velocity of the matter, The "oa" and "ab" components of the matter tensor are, respectively, of the type $O(\beta)$, $O(\beta^2)$ and therefore the terms induced by them in $\Delta g_{\alpha\beta}$ in the case of small β are much smaller than these induced by the "oo" components. Therefore, in all practical applications, it is certainly reasonable and justified to take for $g_{\alpha\beta}$ (which we need in order to know ϵ_{ab} and g_a), some approximate values given by an approximation method which takes into account the smallness of the parameters mentioned above.

In the case of small $\beta = v/c$, we can certainly use for the g's the expressions of lowest orders of the E–I–H procedure,[4–6] according to which

$$g_{00} = 1 - \frac{2k}{c^2}\int d_3x'\, T^{00}(\mathbf{x}',x^0)\frac{1}{|\mathbf{x}-\mathbf{x}'|} + O(\lambda^4),$$

$$g_{0a} = \frac{4k}{c^2}\int d_3x'\, T^{0a}(\mathbf{x}',x^0)\frac{1}{|\mathbf{x}-\mathbf{x}'|} + O(\lambda^5), \quad \lambda = c^{-1}, \quad (3.1)$$

$$g_{ab} = -\delta_{ab}\left(1 + \frac{2k}{c^2}\int d_3x'\, T^{00}(\mathbf{x}',x^0)\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) + O(\lambda^4).$$

The matter tensor $T^{\alpha\beta}$ which occurs here is that which arises from the original $T^{\alpha\beta}$ by the substitution $g_{\alpha\beta} \to \eta_{\alpha\beta}$ in the latter.

Using (3.1) in (2.9), we easily get

$$\epsilon_{ab} = -\Delta g_0 \delta_{ab} + O(\lambda^4), \quad g_a = \Delta g_a + O(\lambda^5), \quad (3.2)$$

where

$$\Delta g_0 = -\frac{2k}{c^2}\int d_3x'\, T^{00}(\mathbf{x}',x^0)\frac{1}{|\mathbf{x}-\mathbf{x}'|},$$
$$\Delta g_a = \frac{4k}{c^2}\int d_3x'\, T^{0a}(\mathbf{x}',x^0)\frac{1}{|\mathbf{x}-\mathbf{x}'|}. \quad (3.3)$$

In the case of relativistic velocities, i.e., relatively large values of β, we can use for $g_{\alpha\beta}$ the lowest approximation terms in the framework of the fast motion approximation method which is essentially an expansion in powers of k/c^2 (see Bertotti,[7] Havas,[8] and Plebanski and Bertotti[9]). According to this method we have

$$\mathfrak{g}^{\alpha\beta} = (-g)^{\frac{1}{2}} g^{\alpha\beta} = \eta^{\alpha\beta} + \frac{4k}{c^2}\int d_4x'\, D_{\mathrm{ret}}(x-x')T^{\alpha\beta}(x') + O(k^2/c^4), \quad (3.4)$$

where $D_{\mathrm{ret}}(x) = \delta(x^0 - |\mathbf{x}|)/|\mathbf{x}|$. Using it in (2.9), we get

$$\epsilon_{ab} = -\Delta g_0'\delta_{ab} + \Delta'\epsilon_{ab} + O(k^2/c^4), \quad g_a = \Delta g_a' + O(k^2/c^4), \quad (3.5)$$

where

$$\Delta g_0' = -\frac{2k}{c^2}\int d_4x'\, D_{\mathrm{ret}}(x-x')[T^{00}(x') + T^{ss}(x')],$$

$$\Delta'\epsilon_{ab} = -\frac{4k}{c^2}\int d_4x'\, D_{\mathrm{ret}}(x-x')T^{ab}(x'), \quad (3.6)$$

$$\Delta g_a' = \frac{4k}{c^2}\int d_4x'\, D_{\mathrm{ret}}(x-x')T^{a0}(x').$$

[Of course, if we neglect in (3.5) and (3.6) terms of higher order in $\lambda = c^{-1}$ we would return to (3.2) and (3.3).]

Finally, one may try to use [as was done in references 1 and 2] Landau's formula for the g's,[10] valid for large distances from physical system:

$$g_{00} = 1 - \frac{2k}{c^2}\frac{M}{r}, \quad g_{0a} = -\frac{2k}{c^3}\frac{1}{r^3}\epsilon_{abc}x^b J^c,$$
$$g_{ab} = -\delta_{ab}\left(1 - \frac{2k}{c^2}\frac{M}{r}\right), \quad (3.7)$$

where M denotes the total mass of the system considered and J^a its classical angular momentum vector. Using (3.7), however, it is important to remember that the g_{0a} have here the meaning of time averages of the original g_{0a} over the period of the motion of the matter,[11] and that (3.7) is valid for large r only.

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[4] A. Einstein, L. Infeld and B. Hoffmann, Ann. Math. **39**, 66 (1938).
[5] A. Einstein and L. Infeld, Can. J. Math. **1**, 209 (1949).
[6] L. Infeld and J. Plebanski Warsaw, 1960 (to be published). The details concerning E.I.H. method in the case of continuous $T^{\alpha\beta}$ are given here.
[7] B. Bertotti, Nuovo cimento **4**, 898 (1956).
[8] P. Havas, Phys. Rev. **108**, 1351 (1957).
[9] J. Plebanski and B. Bertotti (to be published).

[10] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Cambridge, 1951), p. 328.
[11] (3.7) can be easily derived from (3.1). Substituting, in (3.1), $T^{00} = \sum m_a \delta_3(\mathbf{x}-\mathbf{a})$, $T^{0a} = \sum m_a \delta_3(\mathbf{x}-\mathbf{a})a^a{}_{,0}$ (the matter is supposed to be distributed in the form of material points) and taking the expression so obtained for $r = |\mathbf{x}| \gg |\mathbf{a}|$, we get

$$g_{00} = 1 - \frac{2k}{c^2}\frac{\sum m_a}{r} - \frac{2k}{c^2}\frac{x^a}{r^3}\sum m_a a^a + O(r^{-3}),$$

$$g_{0a} = \frac{4k}{c^2}\frac{\sum m_a a^a{}_{,0}}{r} + \frac{2k}{c^3}\frac{x^b}{r^3}\sum m_a(a^b a^a{}_{,0} - a^a a^b{}_{,0}) + \frac{d}{dx^0}\frac{2k}{c^2}\frac{x^b}{r^3}\sum m_a a^a a^b + O(r^{-3}).$$

Now, choosing the origin of x^s coordinates properly, (so that $\sum m_a a^a = 0$) and taking the time average value of g_{0a}, we get (3.7), where $M = \sum m_a$, $J^a/c = \epsilon_{abc}\sum m_a a^b a^c{}_{,0}$ (witht he assumption, of course, that in consequence of the equations of motion $J^a = \text{const}$). The difference in signs between (3.7) and Landau's original formulas arises from the adaptation of the latter to the signature $(+ - - -)$ which we are using in this paper.

Using (3.7) in (2.9), one gets

$$\epsilon_{ab}=-\delta_{ab}\Delta g_0''+O(r^{-3}),\quad g_a=\Delta g_a''+O(r^{-3}),\quad (3.8)$$

where

$$\Delta g_0''=-\frac{2kM}{c^2}\frac{1}{r},\quad \Delta g_a''=-\frac{2k}{c^3}\frac{1}{r^3}\epsilon_{abc}x^bJ^c.\quad (3.9)$$

From the discussion given in this section an important conclusion can be drawn: ϵ_{ab} and g_a in all practical cases are very small. Thus, terms involving ϵ_{ab} and g_a in (2.6), i.e., those describing the influence of the gravitational field on the development in time of E and H, are in practice extremely small. This implies that the physically most important gravitational corrections will be of the first order in ϵ_{ab} and g_a. Therefore, from the physical point of view we can be interested in solutions of (2.6)–(2.8) only with an accuracy up to terms $O(\epsilon_{ab})$, $O(g_a)$ inclusive, which of course remarkably simplifies our mathematical problem. Moreover, as we have seen in this section, the most important contributions to ϵ_{ab} are diagonal (when $\beta\ll 1$). Hence, for a large class of physical phenomena it will suffice to adopt ϵ_{ab} and g_a in the form

$$\epsilon_{ab}=-\delta_{ab}\Delta g_0,\quad g_a=\Delta g_a,\quad (3.10)$$

where the Δg_α together with their derivatives are to be treated as very small.

4. THE LIGHT RAYS THEORY

The study of light rays in a general (weak) gravitational field which will be given here, should be considered from the point of view of Maxwell's equations, (2.1), as a study of curves orthogonal to the characteristic surfaces. In the last sense, the results of this section will be later applied on the level of the vectorial optics.

As is well known, the fundamental eikonal equation has the form,

$$g^{\alpha\beta}S_{,\alpha}S_{,\beta}=0.\quad (4.1)$$

Now, according to the considerations of Sec. 3 let us assume that[12]

$$\mathfrak{g}^{\alpha\beta}=(-g)^{\frac{1}{2}}g^{\alpha\beta}=\eta^{\alpha\beta}+\Delta\mathfrak{g}^{\alpha\beta},\quad (4.2)$$

where $\Delta\mathfrak{g}^{\alpha\beta}$ is small in comparison with 1. Using it in (4.2), one gets

$$(\eta^{\alpha\beta}+\Delta\mathfrak{g}^{\alpha\beta})S_{,\alpha}S_{,\beta}=0.\quad (4.3)$$

This equation, however, can be easily solved with accuracy up to $O(\Delta\mathfrak{g}^{\alpha\beta})$ inclusive. Indeed, let

$$S=\eta_{\alpha\beta}k^\alpha(x^\beta-r^\beta)+\Delta S,\quad (4.4)$$

where r^β are arbitrary constants and k^β is an arbitrary "minkowskian" null vector:

$$\eta_{\alpha\beta}k^\alpha k^\beta=k_0^2-\mathbf{k}^2=0.\quad (4.5)$$

[12] In the first part of this section, it is more convenient to use $\Delta\mathfrak{g}^{\alpha\beta}$ instead of ϵ_{ab}, g_a. Later we shall return to those quantities.

Substituting it into (4.3) and neglecting terms of the type $O(\Delta^2)$, we get

$$k^\alpha\Delta S_{,\alpha}+\tfrac{1}{2}\Delta\mathfrak{g}^{\nu\mu}k_{\bar\nu}k_{\bar\mu}=0.\quad (4.6)$$

(The indices with bars are indices lowered by η metric, e.g., $k_{\bar\mu}=\eta_{\mu\beta}k^\beta$.)

Now, introducing

$$n^\alpha=\frac{1}{k_0}k^\alpha=\left[1,\frac{k^a}{(k^sk^s)^{\frac{1}{2}}}\right]=[1,n^a],\quad n^an^a=1,\quad (4.7)$$

one can easily check that

$$\Delta S(x^0,x^s)=-\frac{k_0}{2}\int_{r^0}^{x^0}dx'^0\times \Delta\mathfrak{g}^{\nu\mu}(x'^0,\ x^s-n^s(x^0-x'^0))n_{\bar\nu}n_{\bar\mu},\quad (4.8)$$

is the solution of (4.6) which vanishes for $x^0=r^0$.[13] Hence, we have at our disposition the approximate solution of the eikonal equation,

$$S=k_{\bar\nu}(x^\nu-r^\nu)-\frac{k_0}{2}\int_{r^0}^{x^0}dx'^0\times \Delta\mathfrak{g}^{\nu\mu}(x'^0,\ x^s-n^s(x^0-x'^0))n_{\bar\nu}n_{\bar\mu},\quad (4.9)$$

which depends on the three arbitrary parameters k^a [keeping in mind that $k_0=(k^sk^s)^{\frac{1}{2}}$, $n^a=k^a(k^sk^s)^{-\frac{1}{2}}$; terms connected with r^α have an additive character, so that they are not important from the point of view of the Hamilton-Jacobi formalism. On the other hand, they are so chosen that if $x^0=r^0$, S is $k_{\bar\nu}(x^\nu-r^\nu)$ coinciding therefore with the classical phase of the plane wave].

Having the explicit expression for the S, equating $\partial S/\partial k^a$ to constants (we choose these constants to be zero), we will get equations determining light rays as $x^a=x^a(x^0)$. As easily seen, these equations are:

$$x^a=r^a+n^a\left(x^0-r^0-\tfrac{1}{2}\int_{r^0}^{x^0}dx'^0\,\Delta'\mathfrak{g}^{\nu\mu}n_{\bar\nu}n_{\bar\mu}\right)+(\delta^{ab}-n^an^b)\left[\tfrac{1}{2}\int_{r^0}^{x^0}dx'^0(x^0-x'^0)\Delta'\mathfrak{g}^{\nu\mu}{}_{,b}n_{\bar\nu}n_{\bar\mu}+\int_{r^0}^{x^0}dx'^0\,\Delta'\mathfrak{g}^{\nu b}n_{\bar\nu}\right],\quad (4.10)$$

where the primes attached to $\Delta\mathfrak{g}^{\alpha\beta}{}_{,b}$ and $\Delta\mathfrak{g}^{\alpha\beta}$ mean that in these quantities the arguments x^0, x^s should be

[13] Indeed: $\Delta S_{,0}=-\frac{1}{2}k_0\Delta\mathfrak{g}^{\nu\mu}(x^0,x^s)n_{\bar\nu}n_{\bar\mu}+\frac{1}{2}k_0\int_{r^0}^{x^0}dx'^0\Delta\mathfrak{g}^{\nu\mu}{}_{,m}(x'^0,\ x^s-n^s(x^0-x'^0))n^mn_{\bar\nu}n_{\bar\mu}.$
On the other hand
$$\Delta S_{,m}=-\frac{k_0}{2}\int_{r^0}^{x^0}dx'^0\Delta\mathfrak{g}^{\nu\mu}{}_{,m}(x'^0,\ x^s-n^s(x^0-x'^0))n_{\bar\nu}n_{\bar\mu}.$$
Substituting it into (4.6) and remembering the notation (4.7), we verify that (4.8) is really a solution of (4.6).

replaced by the arguments x'^0, $x^s-n^s(x^0-x'^0)$, respectively [as in Eq. (4.9)].

This equation has the form $x^a=r^a+n^a(x^0-r^0)+O(\Delta)$. Therefore, preserving accuracy up to $O(\Delta)$ inclusive, we can substitute in place of x^s, where these enter as internal arguments of $\Delta\mathfrak{g}$'s, simply $r^s+n^s(x^0-r^0)$. Doing so, we get our light rays in the final form,

$$x^a=x^a(x^0)=r^a+n^a\left(x^0-r^0-\tfrac{1}{2}\int_{r^0}^{x^0}dx'^0\right.$$
$$\left.\times\Delta\mathfrak{g}^{\nu\mu}(x'^0,\ r^s+n^s(x'^0-r^0))n_{\bar\nu}n_{\bar\mu}\right)$$
$$+(\delta^{ab}-n^an^b)\left[\tfrac{1}{2}\int_{r^0}^{x^0}dx'^0(x^0-x'^0)\right.$$
$$\times\Delta\mathfrak{g}^{\nu\mu}{}_{,b}(x'^0,\ r^s+n^s(x'^0-r^0))n_{\bar\nu}n_{\bar\mu}$$
$$\left.+\int_{r^0}^{x^0}dx'^0\ \Delta\mathfrak{g}^{\nu b}(x'^0,\ r^s+n^s(x'^0-r^0))n_{\bar\nu}\right].\quad(4.11)$$

Into this formula there enter as arbitrary quantities the vector $n^\alpha=[1,n^a]$and the parameters r^0, r^s. One can easily find their meaning. First of all, we have obviously $x^a(r^0)=r^s$, so that r^s is just the initial position of the ray at the "time" r^0. On the other hand, as follows from (4.11),

$$\frac{dx^a}{dx^0}(x^0)=n^a[1-\tfrac{1}{2}\Delta\mathfrak{g}^{\nu\mu}(x^0,\ r^s+n^s(x^0-r^0))n_{\bar\nu}n_{\bar\mu}]$$
$$+(\delta^{ab}-n^an^b)\left[\tfrac{1}{2}\int_{r^0}^{x^0}dx'^0\times\right.$$
$$\Delta\mathfrak{g}^{\nu\mu}{}_{,b}(x'^0,\ r^s+n^s(x'^0-r^0))n_{\bar\mu}$$
$$\left.+\Delta\mathfrak{g}^{\nu b}(x^0,\ r^s+n^s(x^0-r^0))\right]n_{\bar\nu},\quad(4.12)$$

so that the unit vector $t^a(x^0)$ tangent to the ray is given as

$$t^a(x^0)=n^a+\text{the second line of (4.12)}.\quad(4.13)$$

Denoting the initial direction of $t^a(x^0)$, $t^a(r^0)$ as ${}^0t^a$, we find from (4.13) that

$$n^a={}^0t^a-(\delta^{ab}-{}^0t^a\,{}^0t^b)\Delta\mathfrak{g}^{\nu b}(r^0,r^s)\,{}^0t_{\bar\nu},\quad{}^0t^\alpha=[1,{}^0t^a],\quad(4.14)$$

which explains the geometrical meaning of n^a.

Eliminating n^a in (4.13) with the help of (4.14), we obtain:

$$t^a(x^0)={}^0t^a+(\delta^{ab}-{}^0t^a\,{}^0t^b)\left[\Delta\mathfrak{g}^{\nu b}(x^0,\ r^s+{}^0t^s(x^0-r^0))\,{}^0t_{\bar\nu}\right.$$
$$-\Delta\mathfrak{g}^{\nu b}(r^0,r^s){}^0t_{\bar\nu}+\tfrac{1}{2}\int_{r^0}^{x^0}dx'^0\times$$
$$\left.\Delta\mathfrak{g}^{\nu\mu}{}_{,b}(x'^0,\ r^s+{}^0t^s(x'^0-r^0))\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}\right].\quad(4.15)$$

This formula enables us to find the relation connecting the initial and final directions of a light ray which "enters" into the gravitational field at "time" $x^0=-\infty$ from infinity, and "leaves" it at $x^0=+\infty$. In order to get such a relation by a limiting process let us decompose the initial position r^a into the parallel and orthogonal parts with respect to ${}^0t^a$:

$$r^a={}^0t^ar_{\parallel}+r_\perp{}^a,\quad{}^0t^ar_\perp{}^a=0.\quad(4.16)$$

Now, keeping the $r_\perp{}^a$ constant (but arbitrary) let us put $r_\parallel=r^0$. According to (4.11) and (4.14) our ray has now the form, $x^a={}^0t^ax^0+r_\perp{}^a+O(\Delta)$. On the other hand, the equation $x^a={}^0t^ax^0+r_\perp{}^a$ just describes a classical ray having the direction ${}^0t^a$ at $x^0=-\infty$ characterized by the "impact vector" $r_\perp{}^a$. We will get the direction of the corresponding "relativistic" ray by substituting (4.16) into (4.15), where $r_\parallel$ is understood as r^0 and tends in the so-obtained expression with r^0 to $-\infty$. This direction is given as

$$t^a(x^0)={}^0t^a+(\delta^{ab}-{}^0t^a\,{}^0t^b)\left[\Delta\mathfrak{g}^{\nu b}(x^0,\ r_\perp{}^s+{}^0t^sx^0)\,{}^0t_{\bar\nu}\right.$$
$$\left.+\tfrac{1}{2}\int_{-\infty}^{x^0}dx'^0\ \Delta\mathfrak{g}^{\nu\mu}{}_{,b}(x'^0,\ r_\perp{}^s+{}^0t^sx'^0)\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}\right].\quad(4.17)$$

[The limit $\lim_{r^0\to-\infty}\Delta\mathfrak{g}^{\nu b}(r^0,\ r_\perp{}^s+r^0\,{}^0t^s)\,{}^0t_{\bar\nu}$ vanishes because when $(x^sx^s)^{\frac{1}{2}}\to\infty$ according to our assumptions $\Delta\mathfrak{g}^{\alpha\beta}(x^0,x^s)\to0$.] Now, letting x^0 here tend to $+\infty$ and denoting $t^a(+\infty)=t^a{}_f$, we get

$$t^a{}_f={}^0t^a+\tfrac{1}{2}(\delta^{ab}-{}^0t^a\,{}^0t^b)\int_{-\infty}^{+\infty}dx^0\times$$
$$\Delta\mathfrak{g}^{\nu\mu}{}_{,b}(x^0,\ r_\perp{}^s+{}^0t^sx^0)\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}.\quad(4.18)$$

This formula, which solves the problem of Einstein's deflection of light in the general case, has a very simple structure. First of all, it is evident that only the components $\Delta\mathfrak{g}^{00}$, $\Delta\mathfrak{g}^{0a}\,{}^0t^a$, $\Delta\mathfrak{g}^{ab}\,{}^0t^a\,{}^0t^b$ are active in the process of deflection. (If the coordinate system is so chosen that ${}^0t^a=[1,0,0]$ the "active" combination of components of $\Delta\mathfrak{g}^{\alpha\beta}$ is $\Delta\mathfrak{g}^{\nu\mu}\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}=\Delta\mathfrak{g}^{00}-2\Delta\mathfrak{g}^{01}+\Delta\mathfrak{g}^{11}$.) Secondly, the ray in approximation of classical physics is $x^a=r_\perp{}^a+{}^0t^ax^0$; therefore, in order to get the final direction one has to add to the initial direction ${}^0t^a$ that part of the gradient of the "active" combination of $\Delta\mathfrak{g}$'s which is orthogonal to ${}^0t^a$, taken along the classical ray and integrated over the whole history of the ray.

The formula (4.18) is invariant under coordinate transformations of the type $x'^\alpha=x^\alpha+\Delta a^\alpha(x^\beta)$, where $\Delta a^\alpha{}_{,\beta}$ vanishes for $(x^sx^s)^{\frac{1}{2}}\to\infty$. [Thus, one can show that (4.18) is invariant with respect to the substitution:

$$\Delta\mathfrak{g}^{\nu\mu}\to\Delta\mathfrak{g}^{\nu\mu}+\eta^{\mu\rho}\Delta a^\nu{}_{,\rho}+\eta^{\nu\rho}\Delta a^\mu{}_{,\rho}-\eta^{\nu\mu}\Delta a^\rho{}_{,\rho};$$

the contribution from the Δa's form an integral of total differential which vanishes because $\lim\Delta a^\alpha{}_{,\beta}=0$ when $(x^sx^s)^{\frac{1}{2}}\to\infty$.]

The "impact vector" $r_\perp{}^a$ entering into (4.18) is in some sense "the vectorial parameter of collision" with respect to the center of coordinates which we are using (see Fig. 1). In applications the origin of quasi-Cartesian coordinates is supposed to have been chosen conveniently, e.g., as identical in the classical approximation with the center of mass.

The formulas obtained in this section are often too exact to some extent. Namely, in a large class of applications we have

$$g_{00}\cong 1+\Delta g_0,\quad g_{0a}\cong\Delta g_a,\quad g_{ab}\cong-\delta_{ab}(1-\Delta g_0),\tag{4.19}$$

so that formulas for ϵ_{ab} and g_a have form (3.10). But if (4.19) is true, then

$$\Delta\mathfrak{g}^{00}\cong-2\Delta g_0,\quad \Delta\mathfrak{g}^{0a}\cong\Delta g_a,\quad \Delta\mathfrak{g}^{ab}\cong 0,\tag{4.20}$$

so that

$$\Delta\mathfrak{g}^{\nu\mu}\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}\cong-2(\Delta g_0+\Delta g_a\,{}^0t^a)=-2\Delta g_\rho\,{}^0t^\rho\tag{4.21}$$

(this explains, incidentally, why it makes sense to introduce the formal "vector" $\Delta g_\rho=[\Delta g_0,\Delta g_r]$). Therefore, in this case we have, instead of (4.18),

$$t^\alpha{}_f={}^0t^a-(\delta^{ab}-{}^0t^a\,{}^0t^b)\int_{-\infty}^{+\infty}dx^0\,{}^0t^\rho\Delta g_{\rho,b}(x^0,\,r_\perp{}^s+{}^0t^sx^0).\tag{4.22}$$

At the same time, Eq. (4.17), which is of importance subsequently, takes the form,

$$t^a(x^0)={}^0t^a-(\delta^{ab}-{}^0t^a\,{}^0t^b)\bigg[\int_{-\infty}^{x^0}dx'^0\,{}^0t^\rho$$
$$\times\Delta g_{\rho,b}(x'^0,\,r_\perp{}^s+{}^0t^sx'^0)-\Delta g_b(x^0,\,r_\perp{}^s+{}^0t^sx^0)\bigg].\tag{4.23}$$

5. THE "VECTORIAL" OPTICS

Throughout this section we shall treat E_a, H_a as fundamental field variables.

Let us assume that ϵ_{ab} and g_a have the form (3.10). Under this assumption we can write instead (2.6)–(2.8):

$$\begin{aligned}
-[(1-\Delta g_0)E_a+\epsilon_{abc}\Delta g_bH_c]_{,0}+\epsilon_{abc}H_{c,b}&=(4\pi/c)i_a,\\
[(1-\Delta g_0)E_a+\epsilon_{abc}\Delta g_bH_c]_{,a}&=4\pi\rho,\\
[(1-\Delta g_0)H_a-\epsilon_{abc}\Delta g_bE_c]_{,0}+\epsilon_{abc}E_{c,b}&=0,\\
[(1-\Delta g_0)H_a-\epsilon_{abc}\Delta g_bE_c]_{,a}&=0.
\end{aligned}\tag{5.1}$$

We are interested in solutions of (5.1) [when $i_a=0=\rho$] which when Δg_α tends to zero correspond to a plane wave with well-defined polarization; we will take such an "unperturbed" wave in the form

$$\begin{aligned}
\mathcal{E}_a&=\mathrm{Re}\ \mathcal{E}_a{}^{(0)}\exp(ik\phi_0)\\
&=\mathrm{Re}\ (au^a\mp ibv^a)\exp(ik\,{}^0t_{\bar\nu}x^\nu),\\
\mathcal{H}_a&=\mathrm{Re}\ \mathcal{H}_a{}^{(0)}\exp(ik\phi_0)\\
&=\mathrm{Re}\ (av^a\pm ibu^a)\exp(ik\,{}^0t_{\bar\nu}x^\nu),
\end{aligned}\tag{5.2}$$

where ${}^0t^\alpha=[1,{}^0t^a]$, ${}^0t^a$ is a unit vector indicating the direction of propagation of our wave; u^a, v^a are unit vectors forming together with ${}^0t^a$ the "dreibein" (${}^0t^a=\epsilon_{abc}u^bv^c$, $u^a=\epsilon_{abc}v^b\,{}^0t^c$, $v^a=\epsilon_{abc}\,{}^0t^bu^c$); a and b are the axes of the "electrical ellipse" (the upper sign corresponds to the "right" elliptical polarization); $k=\omega c$, where ω is the frequency of the wave.[14]

The most obvious method of attacking Eqs. (5.1) would consist (from a physical point of view) in an adaptation of the idea of "Born's approximation." In other words, one can decompose the solution of (5.1) into the unperturbed and the scattered wave [of order $O(\Delta)$]:

$$E_a=\mathcal{E}_a+\Delta\mathcal{E}_a,\quad H_a=\mathcal{H}_a+\Delta\mathcal{H}_a.\tag{5.3}$$

Substituting (5.3) into (5.1) and neglecting terms of order $O(\Delta^2)$, one gets linear inhomogeneous equations for $\Delta\mathcal{E}_a$, $\Delta\mathcal{H}_a$ with known inhomogeneity. These last equations can be easily solved in the framework of standard methods.[15]

Solutions of this type might be interesting in the case of small k. However, in the optical domain (k very big) the mathematical form of solutions obtained in the way discussed above is unnecessarily involved from the point of view of physical interpretation: a much more convenient approach consists in the application of the method of "vectorial optics."[16] According to this method we consider solutions (5.1) of the form,

$$\begin{aligned}
E_a&=\mathrm{Re}\ (\mathcal{E}_a{}^{(0)}+\Delta E_a)\exp(ik[\phi_0+\Delta\phi]),\\
H_a&=\mathrm{Re}\ (\mathcal{H}_a{}^{(0)}+\Delta H_a)\exp(ik[\phi_0+\Delta\phi]),
\end{aligned}\tag{5.4}$$

where $\mathcal{E}_a{}^{(0)}$, $\mathcal{H}_a{}^{(0)}$, ϕ_0 correspond to the unperturbed solution, complex ΔE_a, ΔH_a are analytic in $1/ik$, $\Delta\phi$ is real and independent of k. When k is very big

$$\Delta E_a\approx\Delta\mathcal{E}_a{}^{(0)}=\lim_{k\to\infty}\Delta E_a,\quad \Delta H_a\approx\Delta\mathcal{H}_a{}^{(0)}=\lim_{k\to\infty}\Delta H_a.$$

The expression (5.4) with ΔE_a, ΔH_a replaced by $\Delta\mathcal{E}_a{}^{(0)}$, $\Delta\mathcal{H}_a{}^{(0)}$ form just the field in the "vectorial optics"

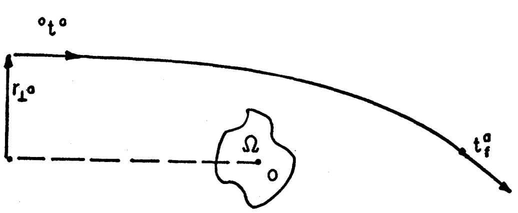

FIG. 1. A schematic illustration of the meaning of vectors $r_\perp{}^a$, ${}^0t^a$, $t_f{}^a$.

[14] In this section the gravitational constant will never enter into our formulas explicitly, so that the same symbol for ωc and the gravitational constant cannot lead to any misunderstanding.

[15] Particularly convenient would be the method of "Hertz vectors." Namely, one can easily show that

$$\Delta\mathcal{E}_a=\epsilon_{abc}\Delta Z_{c,b0}{}^{(2)}+\Delta Z_{a,00}{}^{(1)}-\Delta Z_{b,ba}{}^{(1)},$$
$$\Delta\mathcal{H}_a=-\epsilon_{abc}\Delta Z_{c,b0}{}^{(1)}+\Delta Z_{a,00}{}^{(2)}-\Delta Z_{b,ba}{}^{(2)},$$

where the "Hertz vectors" $\Delta Z_a{}^{(1)}$, $\Delta Z_a{}^{(2)}$ are solutions of

$$\Box\Delta Z_a{}^{(1)}=-\Delta g_0\,\mathcal{E}_a+\epsilon_{abc}\Delta g_b\mathcal{H}_c,\quad \Box\Delta Z_a{}^{(2)}=-\Delta g_0\mathcal{H}_0-\epsilon_{abc}\Delta g_b\,\mathcal{E}_c.$$

The retarded solutions of these equations should be taken.

[16] A similar method was used in reference 1.

approximation. In practical applications this approximation is fully justified.[17]

Substituting (5.4) into (5.1), neglecting terms of order $O(\Delta^2)$, and omitting the symbol "Re," one gets after simple computation:

$$\begin{aligned}
&\text{(a)}\quad ik(\Delta E_a+\epsilon_{abc}\,{}^0t^b\Delta H_c+{}^0t^a\Delta t_b\mathcal{E}_b{}^{(0)}\\
&\qquad+\mathcal{E}_a{}^{(0)}\Delta Q)+\Delta E_{a,0}-\epsilon_{abc}\Delta H_{c,b}\\
&\qquad\qquad-{}^0t^\rho\Delta g_{\rho,0}\mathcal{E}_a{}^{(0)}+{}^0t^a\Delta g_{b,0}\mathcal{E}_b{}^{(0)}=0,\\
&\text{(b)}\quad ik(\Delta H_a-\epsilon_{abc}\,{}^0t^b\Delta E_c+{}^0t^a\Delta t_b\mathcal{H}_b{}^{(0)}\\
&\qquad+\mathcal{H}_a{}^{(0)}\Delta Q)+\Delta H_{a,0}+\epsilon_{abc}\Delta E_{c,b}\\
&\qquad\qquad-{}^0t^a\Delta g_{\rho,0}\mathcal{H}_a{}^{(0)}+{}^0t^a\Delta g_{b,0}\mathcal{H}_b{}^{(0)}=0,\qquad(5.5)\\
&\text{(c)}\quad -ik({}^0t^a\Delta E_a+\Delta t_a\mathcal{E}_a{}^{(0)})+\Delta E_{a,a}-\Delta g_{0,a}\mathcal{E}_a{}^{(0)}\\
&\qquad\qquad+(\Delta g_{b,a}-\Delta g_{a,b})\,{}^0t^a\mathcal{E}_b{}^{(0)}=0,\\
&\text{(d)}\quad -ik({}^0t^a\Delta H_a+\Delta t_a\mathcal{H}_a{}^{(0)})+\Delta H_{a,a}-\Delta g_{0,a}\mathcal{H}_a{}^{(0)}\\
&\qquad\qquad+(\Delta g_{b,a}-\Delta g_{a,b})\,{}^0t^a\mathcal{H}_b{}^{(0)}=0,
\end{aligned}$$

where we have denoted

$$\Delta Q={}^0t^\rho\Delta\phi_{,\rho}-{}^0t^\rho\Delta g_\rho,\qquad(5.6)$$

$$\Delta t_a=\Delta g_a-\Delta\phi_{,a}+{}^0t^a(\Delta g_0-\Delta\phi_{,0}).\qquad(5.7)$$

Before proceeding with expansion in powers of $1/ik$, we shall deduce from (5.5) an important equation (exact from the point of view of the parameter k). First of all, let us decompose

$$\begin{aligned}
&\Delta E_a=\Delta E_{a\perp}+{}^0t^a\Delta E_{\shortparallel},\quad \Delta E_{a\perp}\,{}^0t^a=0,\quad \Delta E_0\,{}^0t^a=\Delta E_{\shortparallel},\\
&\Delta H_a=\Delta H_{a\perp}+{}^0t^a\Delta H_{\shortparallel},\quad \Delta H_{a\perp}\,{}^0t^a=0,\quad \Delta H_a\,{}^0t^a=\Delta H_{\shortparallel}.
\end{aligned}\qquad(5.8)$$

In a similar way one can decompose the left-hand members of (5.5) (a)–(b). Their parallel and orthogonal parts must vanish separately. Now, replace in the orthogonal part of the left-hand member of (5.5a) the free index a by d. Next, multiply the left-hand member of (5.5b) by $\epsilon_{dra}\,{}^0t^r$. The difference between the two expressions so formed must vanish because of (5.5). A simple explicit computation therefore gives

$$\begin{aligned}
&2ik\mathcal{E}_a{}^{(0)}\Delta Q+{}^0t^\rho(\Delta E_{a\perp}-\epsilon_{abc}\,{}^0t^b\Delta H_{c\perp})_{,\rho}\\
&\qquad-2\mathcal{E}_a{}^{(0)}\Delta g_{\rho,0}\,{}^0t^\rho+\epsilon_{abc}\,{}^0t^b\Delta H_{\shortparallel,c}\\
&\qquad\qquad-(\delta^{ab}-{}^0t^a\,{}^0t^b)\Delta E_{\shortparallel,b}=0.\qquad(5.9)
\end{aligned}$$

Now, let us consider Eqs. (5.5), (5.9) from the point of view of the dependence on k. First of all, Eq. (5.9) has two important consequences: because ΔE_a and ΔH_a are analytic in $1/ik$, the vanishing, in (5.9), of coefficients of terms of order $O(k)$ and $O(k^0)$ implies that $\Delta Q=0$,

$$\begin{aligned}
&{}^0t^\rho(\Delta\mathcal{E}_{a\perp}{}^{(0)}-\epsilon_{abc}\,{}^0t^b\Delta\mathcal{H}_{c\perp}{}^{(0)})_{,\rho}-2\mathcal{E}_a{}^{(0)}\Delta g_{\rho,0}\,{}^0t^\rho\\
&\qquad+\epsilon_{abc}\,{}^0t^b\Delta\mathcal{H}_{\shortparallel,c}{}^{(0)}-(\delta^{ab}-{}^0t^a\,{}^0t^b)\Delta\mathcal{E}_{\shortparallel,b}{}^{(0)}=0.\qquad(5.10)
\end{aligned}$$

On the other hand, the terms of order $O(k)$ in Eq. (5.5) (c)–(d) imply that

$$\Delta\mathcal{E}_{\shortparallel}{}^{(0)}=-\Delta t_a\mathcal{E}_a{}^{(0)},\quad \Delta\mathcal{H}_{\shortparallel}{}^{(0)}=-\Delta t_a\mathcal{H}_a{}^{(0)}.\qquad(5.11)$$

Moreover, considering in (5.5) (a)–(b) terms of order $O(k)$ and taking into account $\Delta Q=0$ and (5.11), we see that

$$\Delta\mathcal{E}_{a\perp}{}^{(0)}+\epsilon_{abc}\,{}^0t^b\Delta\mathcal{H}_{c\perp}{}^{(0)}=0,\quad \Delta\mathcal{H}_{a\perp}{}^{(0)}-\epsilon_{abc}\,{}^0t^b\Delta\mathcal{E}_{c\perp}{}^{(0)}=0.\qquad(5.12)$$

The set of Eqs. (5.10)–(5.12) solves our problem in the approximation of "vectorial optics." Indeed, the fundamental equation,

$$\Delta Q={}^0t^\rho\Delta\phi_{,\rho}-{}^0t^\rho\Delta g_\rho=0,\qquad(5.13)$$

is essentially equivalent to Eq. (4.6) for the correction to the eikonal [$k\Delta\phi$ should be identified with ΔS, moreover in the present assumptions (4.21) is valid; ${}^0t^\rho$ in (5.13) corresponds to n^α in (4.6) defined as k^α/k^0]. It can be solved in the identical way[18]:

$$\Delta\phi=\int_{r^0}^{x^0}dx'^0\,{}^0t^\rho\Delta g_\rho(x'^0,\,x^s-{}^0t^s(x^0-x'^0)).\qquad(5.14)$$

Substituting (5.14) into (5.7), one gets

$$\Delta t_a=(\delta^{ab}-{}^0t^a\,{}^0t^b)\left[\Delta g_b-\int_{r^0}^{x^0}dx'^0\,{}^0t^\rho\times\Delta g_{\rho,b}(x'^0,\,x^s-{}^0t^s(x^0-x'^0))\right].\qquad(5.15)$$

Therefore $\Delta\mathcal{E}_{\shortparallel}{}^{(0)}$, $\Delta\mathcal{H}_{\shortparallel}{}^{(0)}$ defined by (5.11) can also be treated as known. As far as the orthogonal components are concerned, we have Eqs. (5.12) and the second of Eqs. (5.10). This last equation can be considerably simplified. Eliminating $\Delta\mathcal{H}_{a\perp}{}^{(0)}$ in it with the help of (5.12) and using for the parallel components the expressions (5.11), where Δt_a is given by (5.7), it follows that it reduces to[19]

$${}^0t^\rho\Delta\mathcal{E}_{a\perp,\rho}=-\Delta\Gamma\mathcal{E}_a{}^{(0)}+\Delta P\epsilon_{abc}\,{}^0t^b\mathcal{E}_c{}^{(0)},\qquad(5.16)$$

where

$$\begin{aligned}
&\Delta\Gamma=-\Delta g_{\rho,0}\,{}^0t^\rho+\tfrac{1}{2}(\delta^{ab}-{}^0t^a\,{}^0t^b)(\Delta g_{a,b}-\Delta\phi_{,ab}),\\
&\Delta P=\tfrac{1}{2}\,{}^0t^a\epsilon_{abc}\Delta g_{c,b}.
\end{aligned}\qquad(5.17)$$

Integrating (5.16) in a similar way to that for (5.13), we get[20]

[17] The validity of this approximation in some practical cases is evident in Sec. 7.

[18] The general solution of the homogeneous equation ${}^0t^\rho\Delta\phi_{,\rho}=0$, if ${}^0t^a$ is identified with the direction of the z axis, has the form $\Delta\phi=\Delta F(x^0-z,\,x,\,y)$ where $\Delta F(x^0,x,y)$ is an arbitrary function. If for $x^0=r^0$ the solution of (5.13) has to vanish for every z, one must accept it in the form (5.14).

[19] It is simplest to compute the last two terms involving parallel components in the second of Eqs. (5.10) by using explicit coordinates.

[20] The solution is so choosen that it vanishes for $x^0=r^0$.

$$\Delta\mathcal{E}_a{}^{\perp}=-\int_{r^0}^{x^0} dx'^0\,\Delta\Gamma(x'^0,\,x^s-{}^0t^s(x^0-x'^0))\mathcal{E}_a{}^{(0)}$$
$$+\epsilon_{abc}\,{}^0t^b\mathcal{E}_c{}^{(0)}\int_{r^0}^{x^0} dx'^0\,\Delta P(x'^0,\,x^s-{}^0t^s(x^0-x'^0)), \tag{5.18}$$

and finally, using (5.18) in (5.12), we obtain

$$\Delta\mathcal{H}_a{}^{\perp}=-\int_{r^0}^{x^0} dx'^0\,\Delta\Gamma(x'^0,\,x^s-{}^0t^s(x^0-x'^0))\mathcal{H}_a{}^{(0)}$$
$$+\epsilon_{abc}\,{}^0t^b\mathcal{H}_c{}^{(0)}\int_{r^0}^{x^0} dx'^0$$
$$\times\Delta P(x'^0,\,x^s-{}^0t^s(x^0-x'^0)). \tag{5.19}$$

Collecting our results, we can write the field in the approximation of "vectorial optics" as

$$\begin{aligned} E_a&=\mathrm{Re}\,(\mathcal{E}_a{}^{(0)}+\epsilon_{abc}\Delta\Omega_b\mathcal{E}_c{}^{(0)})\\ &\quad\times\exp[ik({}^0t_\nu x^\nu+\Delta\phi)-\Delta\bar{\Gamma}]+O(1/k,\,\Delta^2),\\ H_a&=\mathrm{Re}\,(\mathcal{H}_a{}^{(0)}+\epsilon_{abc}\Delta\Omega_b\mathcal{H}_c{}^{(0)})\\ &\quad\times\exp[ik({}^0t_\nu x^\nu+\Delta\phi)-\Delta\bar{\Gamma}]+O(1/k,\,\Delta^2), \end{aligned} \tag{5.20}$$

where $\Delta\phi$ is given by (5.14). $\Delta\bar{\Gamma}$ stands for

$$\Delta\bar{\Gamma}=\int_{r^0}^{x^0} dx'^0\,\Delta\Gamma(x'^0,\,x^s-{}^0t^s(x^0-x'^0)), \tag{5.21}$$

and

$$\Delta\Omega_a=\Delta P_a+\Delta\omega_a=\Delta\bar{P}\,{}^0t^a+\epsilon_{abc}\,{}^0t^b\Delta t_c, \tag{5.22}$$

$$\Delta\bar{P}=\tfrac{1}{2}\int_{r^0}^{x^0} dx'^0\,{}^0t^a\epsilon_{abc}\Delta g_{c,b}(x'^0,\,x^s-{}^0t^s(x^0-x'^0)); \tag{5.23}$$

Δt_a is given by (5.15). For $x^0=r^0$ this solution reduces to the unperturbed wave (5.2).

All quantities appearing in (5.20) have a direct physical interpretation. First of all, let us observe that the amplitudes,

$$\mathcal{E}_a'=\mathcal{E}_a{}^{(0)}+\epsilon_{abc}\Delta\Omega_b\mathcal{E}_c{}^{(0)},\quad \mathcal{H}_a'=\mathcal{H}_a{}^{(0)}+\epsilon_{abc}\Delta\Omega_b\mathcal{H}_c{}^{(0)}, \tag{5.24}$$

fulfill

$$\mathcal{E}_a'+\epsilon_{abc}t^b\mathcal{H}_c'=0=\mathcal{H}_0'-\epsilon_{abc}t^b\mathcal{E}_c', \tag{5.25}$$

where

$$t^a={}^0t^a+\Delta t_a={}^0t^a+(\delta^{ab}-{}^0t^a\,{}^0t^b)$$
$$\times\left[\Delta g_a-\int_{r^0}^{x^0} dx'^0\,{}^0t^\rho\Delta g_{\rho,b}(x'^0,\,x^s-{}^0t^s(x^0-x'^0))\right]. \tag{5.26}$$

The relations (5.25) are characteristic of amplitudes of a plane wave with direction vector t^a. Therefore t^a from (5.26) must be identified with the direction of the perturbed wave of event x^0, x^s. The characteristic term "$\epsilon_{abc}\Delta\Omega_b\ldots$" appearing in the expressions for perturbed amplitudes has a simple meaning: geometrically speaking these terms describe the fact that $\mathcal{E}_a'$, $\mathcal{H}_a'$ are derived from $\mathcal{E}_a{}^{(0)}$, $\mathcal{H}_a{}^{(0)}$ through the small rotation $\Delta\Omega_b$. This small rotation is the sum of two independent rotations: the rotation caused by $\Delta\omega_a=\epsilon_{abc}\,{}^0t^b\Delta t_c$ must be interpreted as due to the change of direction of the propagation vector (under this rotation ${}^0t^a$ transforms into t^a). The rotation $\Delta P_a=\Delta\bar{P}\,{}^0t^a\cong\Delta\bar{P}t^a$ should be interpreted as the rotation around the propagation vector t^a by the angle $\Delta\bar{P}$. Thus, $\Delta\bar{P}$ is the angle of rotation of the plane of polarization. The $\Delta\phi$ quantity describes the correction to the phase; the real factors $\exp[-\Delta\Gamma]$ describes a sort of "damping" or "absorption" of electromagnetic waves by the gravitational field. Physically speaking, this is due to the nonclassical form of the laws of conservation in the gravitational field [see Eqs. (2.10)–(2.11)].

Our solution (5.20) is valid for arbitrary values of r^0. In order to establish the correspondence of the results of this section with our theory of light rays we must go to the limit $r^0=-\infty$. When $r^0=-\infty$ in (5.20), the solution (5.20) for $x^0\to-\infty$ tends to the unperturbed wave.[21]

Moreover, our solution with $r^0=-\infty$ has an important property. When the argument x^s is divided into $x^s=x_\perp{}^s+{}^0t^sx_{\parallel}$, $x_\perp{}^s$ and x^0 are kept constant but arbitrary, and $x_{\parallel}\to-\infty$, the expressions (5.20) again tend to the classical unperturbed wave [i.e., the operation $\lim_{x_{\parallel}\to-\infty}\lim_{r^0\to-\infty}$ performed with respect to $\Delta\phi$, $\Delta\bar{\Gamma}$, $\Delta\Omega_a$ gives in all cases 0[22]]. This property can be interpreted as meaning that our solution (5.20) satisfies just the correct boundary conditions: on the plane at infinity ($x_{\parallel}=-\infty$) which is orthogonal to the "unperturbed" direction ${}^0t^a$ (5.20) reduces to the "unperturbed" wave.

Now, when our wave arrives from the plane $x_{\parallel}=-\infty$ at x^s at the time x^0, then according to (5.26) it has the direction

$$t^a(x^0,x^s)={}^0t^a+(\delta^{ab}-{}^0t^a\,{}^0t^b)\Bigg[\Delta g_b(x^0,x^s)$$
$$-\int_{-\infty}^{x^0} dx'^0\,{}^0t^\rho\Delta g_{\rho,b}(x'^0,\,x^s-{}^0t^s(x^0-x'^0))\Bigg], \tag{5.27}$$

and, according to (5.23) its plane of polarization in comparison with the initial situation at $x''=-\infty$ is turned around t^a through the angle

$$\Delta\bar{P}(x^0,x^s)=\tfrac{1}{2}\int_{-\infty}^{x^0} dx'^0\,{}^0t^a\epsilon_{abc}\Delta g_{c,b}(x'^0,\,x^s-{}^0t^s(x^0-x'^0)). \tag{5.28}$$

[21] One should mention, however, that there exists a difficulty here; namely, because in practical applications for big $r=(x^sx^s)^{\frac{1}{2}}$, $\Delta g_0\sim r^{-1}$ the limit of $\Delta\phi$ from (5.14) (i.e., $\lim_{r^0\to-\infty}\Delta\phi$) is divergent. This difficulty being rather of mathematical, nonphysical origin, can be avoided in a formal way, e.g., by introducing some kind of convergence factor into (5.14). The physically much more interesting quantities such as $\Delta\bar{P}$, $\Delta\bar{\Gamma}$, Δt_a behave, however, in a regular way, e.g., $\lim_{x^0\to-\infty}\lim_{r^0\to-\infty}\Delta\bar{P}=0$.

[22] $\Delta\phi$ is understood here as involving some "convergence factor" securing the existence of $\lim_{r^0\to-\infty}\Delta\phi$.

At the same time the "absorption factor," $\exp[-\Delta\bar{\Gamma}]$, is

$$\exp\left[-\int_{-\infty}^{x^0} dx'^0\, \Delta\Gamma(x^0, x^s - {}^0t^s(x^0 - x'^0))\right]. \quad (5.29)$$

Now, let us study the quantities along the light ray starting from the plane $x_{\parallel} = -\infty$ at $x^0 = -\infty$ in the direction ${}^0t^a$ and with "impact vector" $r_\perp{}^a$. Such a ray is given by $x^a = {}^0t^a x^0 + r_\perp{}^a + O(\Delta)$. Substitution of it into (5.27) gives

$$t^a(x^0) = {}^0t^a + (\delta^{ab} - {}^0t^a\, {}^0t^b)\left[\Delta g_b(x^0, {}^0t^s x^0 + r_\perp{}^a) - \int_{-\infty}^{x^0} dx'^0\, {}^0t^\rho \Delta g_{\rho,b}(x'^0, r_\perp{}^s + {}^0t^s x'^0)\right], \quad (5.30)$$

which is identical with (4.23), as it should be. Therefore, all previous conclusions from Sec. 4 concerning the deflection of the direction of a geometrical ray can also be interpreted as true with respect to the direction of the wave.[23]

Taking $\Delta\bar{P}$ along the ray, we have

$$\Delta\bar{P}(x^0) = \tfrac{1}{2}\int_{-\infty}^{x^0} dx'^0\, {}^0t^a \epsilon_{abc} \Delta g_{c,b}(x'^0, r_\perp{}^s + {}^0t^s x'^0). \quad (5.31)$$

When the ray leaves the gravitational field the plane of polarization is therefore turned through (in comparison with the initial situation at $x'' = -\infty$)

$$\Delta P_f = \tfrac{1}{2}\int_{-\infty}^{+\infty} dx^0\, {}^0t^a \epsilon_{abc} \Delta g_{c,b}(x^0, r_\perp{}^s + {}^0t^s x^0). \quad (5.32)$$

Let us examine (5.32) from the point of view of coordinate transformations. Under the transformation $x^\alpha \to x^\alpha + \Delta a^\alpha(x^\beta)$ the quantities $\Delta g_a \approx \Delta g_{0a}$ transform into $\Delta g_a + \Delta a_{a,0} + \Delta a_{0,a}$, where $\Delta a_\alpha = \eta_{\alpha\rho}\Delta a^\rho$. The gradient $\Delta a_{0,a}$ cannot contribute to (5.32) because Δg_a enters in (5.32) in the form of a rotation. However, the formula (5.32) is based on the assumption that the metric has the form (4.19). Transformations induced by Δa_0 preserve this form, while those induced by Δa_a do not. Therefore one cannot demand from (5.32) the invariance with respect to transformations generated by arbitrary Δa^α which produce off-diagonal Δg_{ab}.

One can, however, generalize (5.32) to the case of a metric of the more general form (i.e., the case of ϵ_{ab} having in general off diagonal elements). The generalized formula must fulfill two conditions: (1) When the metric has the form (4.19), it must go over into (5.32); (2) it must be invariant under general coordinate transformations.

We claim that

$$\Delta P_f = \tfrac{1}{2}\int_{-\infty}^{+\infty} dx^0\, {}^0t^a \epsilon_{abc} \Delta \mathfrak{g}^{\nu c}{}_{,b}(x^0, r_\perp{}^s + {}^0t^s x^0)\, {}^0t_{\bar{\nu}} \quad (5.33)$$

fulfills these conditions. Indeed, when the metric has the form (4.27) we have, according to (4.28), $\Delta\mathfrak{g}^{\nu b}\, {}^0t_{\bar{\nu}} = \Delta g_b$. On the other hand, under $x^\alpha \to x^\alpha + \Delta a^\alpha$ the $\Delta\mathfrak{g}^{\nu\mu}$ quantities transform according to

$$\Delta\mathfrak{g}^{\nu\mu} \to \Delta\mathfrak{g}^{\nu\mu} + \eta^{\mu\rho}\Delta a^\nu{}_{,\rho} + \eta^{\nu\rho}\Delta a^\mu{}_{,\rho} - \eta^{\nu\mu}\Delta a^\rho{}_{,\rho}.$$

Thus, under the influence of the coordinate transformation ΔP_f from (5.33) changes into ΔP_f plus

$$\tfrac{1}{2}\, {}^0t^a \epsilon_{abc}\int_{-\infty}^{+\infty} dx^0 [\eta^{c\rho}\Delta a^\nu{}_{,\rho b}(x^0, r_\perp{}^s + {}^0t^s x^0) + \eta^{\nu\rho}\Delta a^c{}_{,\rho b}(x^0, r_\perp{}^s + {}^0t^s x^0) - \eta^{\nu c}\Delta a^\rho{}_{,\rho b}(x^0, r_\perp{}^s + {}^0t^s x^0)]\, {}^0t_{\bar{\nu}}. \quad (5.34)$$

The first term here vanishes as it is the rotation of a gradient. The last term vanishes because $\eta^{\nu c}\, {}^0t_{\bar{\nu}} = {}^0t^c$ and $\epsilon_{abc}\, {}^0t^a\, {}^0t^c = 0$. The remaining term vanishes because

$$dx^0 \eta^{\nu\rho}\Delta a^c{}_{,\rho b}(x^0, r_\perp{}^s + {}^0t^s x^0)\, {}^0t_{\bar{\nu}} = dx^0\left(\frac{d}{dx^0}\right)\Delta a^c{}_{,b}(x^0, r_\perp{}^s + {}^0t^s x^0)$$

forms a perfect differential; $\Delta a^c{}_{,b}$ must vanish for $(x^s x^s)^{\frac{1}{2}} \to \infty$ in the case of a transformation tending to the identity at infinity. Therefore all terms in (5.34) vanish.

Summarizing, (5.33) must be regarded as the formula describing the rotation of the plane of polarization in the general case, the case of arbitrary small $\Delta g_{\alpha\beta}$. If there exists a coordinate system such that the metric has the form (4.27), the ΔP_f reduces to (5.32).

For the sake of completeness we will find also the final value of $\Delta\bar{\Gamma}$ taken along our ray. This value, $\Delta\Gamma_f$, obtained from (5.29) is

$$\Delta\Gamma_f = \int_{-\infty}^{+\infty} dx^0\, \Delta\Gamma(x^0, r_\perp{}^s + {}^0t^s x^0), \quad (5.35)$$

where $\Delta\Gamma$ is defined by (5.17). Under the assumptions (1) that Δg_α vanishes when $(x^s x^s)^{\frac{1}{2}} \to \infty$, (2) that the d'Alembertian of Δg_α, $\Delta g_{\alpha'}{}^{\bar{\nu}}{}_{,\nu}$, vanishes or is negligible along the path of the ray,[24] and (3) that the coordinate condition $2\Delta g_{0,0} - \Delta g_{a,a} = 0$ is valid, the expression (5.35) can be computed to be

$$\Delta\Gamma_f = -\int_{-\infty}^{+\infty} dx^0 [\Delta g_{0,0}(x^0, r_\perp{}^s + {}^0t^s x^0) + \tfrac{3}{2}\, {}^0t^r \Delta g_{r,0}(x^0, r_\perp{}^s + {}^0t^s x^0)]. \quad (5.36)$$

[23] The direction of a wave is here understood to be the direction of the Poynting vector associated with it.

[24] According to any reasonable approximation procedure $\Box\Delta g_a$ is proportional to the $T^{0\alpha}$ components of the tensor of matter—this assumption means that the ray does not enter into the region Ω where matter is present.

6. GEOMETRICAL INTERPRETATION OF THE ROTATION OF THE PLANE OF POLARIZATION[25]

In this section we shall give some geometrical interpretation of the formulas (5.32), (5.33).

Let us consider a light ray starting at the time r^0 from the point r^a with initial direction ${}^0t^a$. Such a ray is given explicitly by (4.11) where n^a is given by (4.14). Now, at time r^0 let another ray start from the point r^a+dr^a with the same direction ${}^0t^a$; dr^a is assumed to be orthogonal to ${}^0t^a$. Let us call this second ray $x'^a(x^0)$.

Consider the difference $\delta x^a(x^0)=x'^a(x^0)-x^a(x^0)$. Using (4.11), (4.14) one can easily show that

$$\delta x^a(x^0)=dr^a-(\delta^{ab}-{}^0t^a\,{}^0t^b)\Delta\mathfrak{g}^{\nu b}{}_{,c}(r^0,r^s)\,{}^0t_{\bar\nu}(x^0-r^0)dr^c$$
$$-\tfrac{1}{2}\,{}^0t^a\int_{r^0}^{x^0}dx'^0\,\Delta\mathfrak{g}'^{\nu\mu}{}_{,c}\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}dr^c$$
$$+\tfrac{1}{2}(\delta^{ab}-{}^0t^a\,{}^0t^b)\int_{r^0}^{x^0}dx'^0(x^0-x'^0)$$
$$\times\Delta\mathfrak{g}'^{\nu\mu}{}_{,bc}\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}dr^c+(\delta^{ab}-{}^0t^a\,{}^0t^b)$$
$$\times\int_{r^0}^{x^0}dx'^0\,\Delta\mathfrak{g}'^{\nu b}{}_{,c}\,{}^0t_{\bar\nu}dr^c,\quad(6.1)$$

where the prime over symbols $\Delta\mathfrak{g}$ indicates that they should be taken with the arguments $x^0\to x'^0$, $x^s\to r^s+{}^0t^s(x'^0-r^0)$. According to (4.15) the unit vector tangential to $x^a(x^0)$ can be rewritten in the form

$$t^a(x^0)={}^0t^a+\Delta t_a(x^0)={}^0t^a+(\delta^{ab}-{}^0t^a\,{}^0t^b)$$
$$\times\Bigg[\Delta\mathfrak{g}^{\nu b}(x^0,r^s+{}^0t^s(x^0-r^0))\,{}^0t_{\bar\nu}-\Delta\mathfrak{g}^{\nu b}(r^0,r^s)\,{}^0t_{\bar\nu}$$
$$+\tfrac{1}{2}\int_{r^0}^{x^0}dx'^0\,\Delta\mathfrak{g}'^{\nu\mu}{}_{,b}\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}\Bigg].\quad(6.2)$$

Now, let us decompose the vector $\delta x^a(x^0)$ joining the fundamental ray with the second ray into two parts, respectively, orthogonal and parallel to $t^a(x^0)$:

$$\delta x^a(x^0)=\delta x_\perp{}^0(x^0)+t^a(x^0)\delta x_{\shortparallel}(x^0),\qquad \delta x_\perp{}^a(x^0)t^a(x^0)=0.\quad(6.3)$$

Remembering that dr^a is orthogonal to ${}^0t^a$, one can easily find that

$$\delta x_{\shortparallel}(x^0)=[\Delta\mathfrak{g}^{\nu b}(x^0,r^s+{}^0t^s(x^0-r^0)/$$
$$-\Delta\mathfrak{g}^{\nu b}(r^0,r^s)]\,{}^0t_{\bar\nu}dr^b.\quad(6.4)$$

This enables us to compute $\delta x_\perp{}^a=\delta x^a-t^a\delta x\|$ as

$$\delta x_\perp{}^a(x^0)=dr^a-{}^0t^a\,\Delta t_c(x^0)dr^c+\tfrac{1}{2}(\delta^{ab}-{}^0t^a\,{}^0t^b)(\delta^{cd}-{}^0t^c\,{}^0t^d)$$
$$\times\Bigg[\int_{r^0}^{x^0}dx'^0(x^0-x'^0)\Delta\mathfrak{g}'^{\nu\mu}{}_{,bd}\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}$$
$$+2\int_{r^0}^{x^0}dx'^0\,\Delta\mathfrak{g}'^{\nu b}{}_{,d}\,{}^0t_{\bar\nu}$$
$$-2(x^0-r^0)\Delta\mathfrak{g}^{\nu b}{}_{,d}(r^0,r^s)\,{}^0t_{\bar\nu}\Bigg]dr^d,\quad(6.5)$$

where Δt_a is defined by (6.2). This formula has the structure $\delta x_\perp{}^a=dr^a+\Delta\omega^{ab}dr^b$. Let us decompose $\Delta\omega^{ab}$ into symmetrical and shew-symmetrical parts. A straightforward computation enables us to rewrite (6.5) in the form,

$$\delta x_\perp{}^a(x^0)=dr^a+\epsilon_{abc}\Delta R_b(x^0)dr^c+\Delta D_{ab}dr^b,\quad(6.6)$$

where

$$\Delta R_b(x^0)=\epsilon_{bpq}\,{}^0t^p\Delta t_q(x^0)+{}^0t^b\,\Delta T(x^0),\quad(6.7)$$

$$\Delta T(x^0)=\tfrac{1}{2}\int_{r^0}^{x^0}dx'^0\,{}^0t^a\epsilon_{abc}\Delta\mathfrak{g}'^{\nu c}{}_{,b}\,{}^0t_{\bar\nu}$$
$$-\tfrac{1}{2}(x^0-r^0)\,{}^0t^a\epsilon_{abc}\Delta\mathfrak{g}^{\nu c}{}_{,b}(r^0,r^s)\,{}^0t_{\bar\nu},\quad(6.8)$$

$$\Delta D_{ab}=\tfrac{1}{2}(\delta^{ac}-{}^0t^a\,{}^0t^c)(\delta^{bd}-{}^0t^b\,{}^0t^d)$$
$$\times\int_{r^0}^{x^0}dx'^0(x^0-x'^0)\,{}^0t_{\bar\nu}\Bigg[{}^0t_{\bar\mu}\Delta\mathfrak{g}'^{\nu\mu}{}_{,cd}$$
$$+\frac{d}{dx'^0}(\Delta\mathfrak{g}'^{\nu c}{}_{,d}+\Delta\mathfrak{g}'^{\nu d}{}_{,c})\Bigg].\quad(6.9)$$

These formulas, together with (6.4), explain entirely the geometrical situation: the part of $\delta x^a(x^0)$ orthogonal to the tangent to $x^a(x^0)$ is according to (6.6) the result of the deformation (due to the symmetric ΔD_{ab}) and of the small rotation around the direction ΔR_b through the angle $(\Delta R_b\Delta R_b)^{\frac{1}{2}}$ performed on the initial value of $\delta x_\perp{}^a(x^0)$, i.e., dr^a. This rotation can be considered as the superposition of two independent rotations: The first rotation characterized by $\epsilon_{bpq}\,{}^0t^p\Delta t_q(x^0)$ is due to the change of direction of the tangent to the ray (under this rotation ${}^0t^a$ goes over into $t^a={}^0t^a+\Delta t_a$). The second rotation ${}^0t^a\Delta T\cong t^a\Delta T$ is a rotation around the tangent $t^a(x^0)$ through the angle ΔT.

Now, let us examine the ΔT quantity when the initial point r^a is decomposed as $r^a=r_\perp{}^a+{}^0t^ar^0$, ${}^0t^ar_\perp{}^a=0$, and r^0 tends to $-\infty$. In other words, we will consider the case when our two rays start from the plane at infinity $x_{\shortparallel}=-\infty$ at the time $r^0=-\infty$. In this case, according to (6.8), $\Delta T(x^0)$ is given by

$$\Delta T(x^0)=\tfrac{1}{2}\int_{-\infty}^{x^0}dx'^0\,{}^0t^a\epsilon_{abc}\Delta\mathfrak{g}^{\nu c}{}_{,b}(x'^0,r_\perp{}^s+{}^0t^sx'^0)\,{}^0t_{\bar\nu}$$
$$+-\tfrac{1}{2}\lim_{r^0\to-\infty}(x^0-r^0)\,{}^0t^a\epsilon_{abc}$$
$$\times\Delta\mathfrak{g}^{\nu c}{}_{,b}(r^0,r_\perp{}^s+{}^0t^sr^0)\,{}^0t_{\bar\nu}.\quad(6.10)$$

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[25] The author owes the idea of the application of the methods of Sec. 4 to the problem of this section to Professor John A. Wheeler.

However, because in the case of the gravitational field of an isolated system deviations from Euclidicity behave at infinity, like $f(x^0-r)/r$ [where $f(x^0)$ is bounded], the derivatives $\Delta\mathfrak{g}^{\alpha c}{}_{,b}(r^0, r_\perp{}^s+{}^0t^s r^0)$ will behave for large r^0 as $f^{\alpha c}[r^0-(r_\perp{}^2+r_0{}^2)^{\frac{1}{2}}](r_\perp{}^b+{}^0t^b r^0)/(r_\perp{}^2+r_0{}^2)$.

The factor ${}^0t^a\epsilon_{abc}\cdots$, however, kills here the contributions $\sim{}^0t^b$. Hence, the limit in (6.10), which can be written as

$$-\tfrac{1}{2}\lim_{r^0\to-\infty}(x^0-r^0)\,{}^0t^a\epsilon_{abc}\times f^{\alpha c}[r^0-(r_\perp{}^2+r_0{}^2)^{\frac{1}{2}}]\frac{r_\perp{}^b}{r_\perp{}^2+r_0{}^2}, \quad (6.11)$$

vanishes. Thus

$$\Delta T(x^0)=\tfrac{1}{2}\int_{-\infty}^{x^0}dx'^0\,{}^0t^a\epsilon_{abc}\Delta\mathfrak{g}^{\nu c}{}_{,b}(x'^0, r_\perp{}^s+{}^0t^s x'^0)\,{}^0t_{\bar\nu}. \quad (6.12)$$

Now, the final angle of rotation around the tangent when the rays leave the gravitational field follows on setting $x^0=+\infty$ in (6.12), i.e.,

$$\Delta T_f=\tfrac{1}{2}\int_{-\infty}^{+\infty}dx^0\,{}^0t^a\epsilon_{abc}\Delta\mathfrak{g}^{\nu c}{}_{,b}(x^0, r_\perp{}^s+{}^0t^s x^0)\,{}^0t_{\bar\nu}. \quad (6.13)$$

One can also mention here that the corresponding limiting transition in (6.4) gives

$$\lim_{x^0\to+\infty}\lim_{r^0\to-\infty}\delta x_{\shortparallel}(x^0)=0. \quad (6.14)$$

Therefore at the end of the history of our two rays δx^a is orthogonal to the final direction of the tangent to the fundamental ray. The formula (6.13) is evidently identical with (5.33), guessed from (5.32) by the use of the condition of invariance with respect to coordinate transformations.

It follows that the total rotation of the plane of polarization of a wave observed along the light ray x^a is identical with the rotation of the infinitesimal vector δx^a joining x^a and a second ray x'^a when the tangents to x^a and x'^a are initially parallel to δx^a initially normal to both rays.

7. APPLICATIONS

In this section we shall apply the general results of the previous sections, that is the general formulas (4.18) and (5.33) for the deflection of a ray and the rotation of the plane of polarization along the ray, in a few concrete cases.

Before doing so we would like to mention, however, that the substitution into (4.18) and (5.33) of $\Delta\mathfrak{g}^{\alpha\beta}$ in the form (3.4) leads to some general results. Namely, substituting $\Delta\mathfrak{g}^{\alpha\beta}$ into (4.18), which according to (3.4) can be written in the form,

$$\Delta\mathfrak{g}^{\alpha\beta}(x^0,x^s)=\frac{4k}{c^2}\int d_3y\frac{T^{\alpha\beta}(x^0-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}, \quad (7.1)$$

and interchanging the order of partial differentiation $\partial/\partial x^s$ and integration d_3y, one gets:

$$t^{fa}={}^0t^a-\frac{2k}{c^2}\int_{-\infty}^{+\infty}dx^0\int d_3y(r_\perp{}^a-y_\perp{}^a)\times\left[\frac{T_{,0}(x^0-q,\mathbf{y})}{q^2}+\frac{T(x^0-q,\mathbf{y})}{q^3}\right], \quad (7.2)$$

where

$$T=T^{\nu\mu}\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu},\quad y_{\shortparallel}={}^0t^a y^a,\quad y_\perp{}^a=y^a-{}^0t^a y_{\shortparallel},$$
$$q=[(x^0-y_{\shortparallel})^2+(\mathbf{r}_\perp-\mathbf{y}_\perp)^2]^{\frac{1}{2}}. \quad (7.3)$$

However, observing that

$$\frac{d}{dx^0}\left\{\left(1-\frac{x^0-y_{\shortparallel}}{q}\right)T(x^0-q,\mathbf{y})\right\}=(\mathbf{r}_\perp-\mathbf{y}_\perp)^2\left[\frac{T_{,0}(x^0-q,\mathbf{y})}{q^2}+\frac{T(x^0-q,\mathbf{y})}{q^3}\right], \quad (7.4)$$

we can rewrite (7.2) as:

$$t^a{}_f={}^0t^a-\frac{2k}{c^2}\int_{-\infty}^{+\infty}dx^0\int d_3y\frac{r_\perp{}^a-y_\perp{}^a}{(\mathbf{r}_\perp-\mathbf{y}_\perp)^2}\frac{d}{dx^0}\times\left\{\left(1+\frac{x^0-y_{\shortparallel}}{q}\right)T(x^0-q,\mathbf{y})\right\}. \quad (7.5)$$

Now, if the integral over d_3y is uniformly convergent one can interchange the order of d_3y integration and the differentiation d/dx^0, so that the integration over dx^0 can be performed. Again in the assumption of uniform convergence we can interchange the order of $\lim_{x^0\to\pm\infty}$ and d_3y integration. But because in the upper limit the factor $1+(x^0-y_{\shortparallel})/q$ is equal to 2 and for $x^0\to-\infty$ vanishes, and because $\lim_{x^0\to+\infty}(x^0-q)=y_{\shortparallel}$, we get simply

$$t^a{}_f={}^0t_a-\frac{4k}{c^2}\int_{-\infty}^{+\infty}dx^0\int d_2y_\perp\frac{r_\perp{}^a-y_\perp{}^a}{(\mathbf{r}_\perp-\mathbf{y}_\perp)^2}\times T^{\nu\mu}(x^0, y_\perp{}^s+{}^0t^s x^0)\,{}^0t_{\bar\nu}\,{}^0t_{\bar\mu}. \quad (7.6)$$

In the particular case of a point singularity at rest, we have $T^{00}=m\delta_3(\mathbf{x})$, $T^{0a}=0$, $T^{ab}=0$, so that our formula gives

$$t^a{}_f={}^0t^a-\frac{4km}{c^2}\frac{1}{r_\perp}\frac{r_\perp{}^a}{r_\perp}, \quad (7.7)$$

which coincides with Einstein's formula for the deflection of a ray in the case of Schwarzschild's metric.

Now, substituting $\Delta\mathfrak{g}^{\alpha\beta}$ in the form (7.1) into (5.33) and using exactly the same arguments as previously

[e.g., Eq. (7.4)], one can find that[26]

$$\Delta P_f=-\frac{4k}{c^2}{}^0t^a\epsilon_{abc}\int_{-\infty}^{+\infty}dx^0\int d_2y_\perp\frac{r_\perp{}^b-y_\perp{}^b}{(\mathbf{r}_\perp-\mathbf{y}_\perp)^2}\times T^{c\nu}(x^0,y_\perp{}^s+{}^0t^sx^0)\,{}^0t_\nu.\quad(7.8)$$

Now, let the gravitational field be induced by a rotating body; the speed of rotation we assume to be small. It therefore makes sense to assume that the deformation caused by rotation is negligible and that the body has spherical symmetry. For simplicity we will also assume that the density of our rotating sphere of radius l is constant and that the body rotates uniformly with angular velocity ω^a. The terms due to pressure in the energy momentum tensor as proportional to c^{-2} can be neglected. Hence, the energy momentum tensor of the rotating body can be taken in the form,

$$T^{\alpha\beta}=\rho v^\alpha v^\beta\quad\text{for }(x^ax^a)^{\frac{1}{2}}\leqslant l,\qquad T^{\alpha\beta}=0\quad\text{otherwise},\quad(7.9a)$$

where

$$\rho=\text{const},\quad v^\alpha=[1,v^a],\quad v^a=(1/c)\epsilon_{abc}\omega^bx^c.\quad(7.9b)$$

Since we are dealing in this case with small velocities, we should limit ourselves to the approximate formulas (4.22), (5.32) in which the quantities Δg_α are to be taken simply in the form (3.3). An elementary computation gives

$$\Delta g_0=-\frac{2km}{c^2}\frac{1}{|\mathbf{x}|}-\frac{2km}{c^2}\left(\frac{3}{2}\frac{1}{l}-\frac{1}{2}\frac{|\mathbf{x}|^2}{l^3}-\frac{1}{|\mathbf{x}|}\right)\times\theta(l-|\mathbf{x}|),\quad(7.10a)$$

$$\Delta g_a=-\frac{2km}{c^3}\epsilon_{abc}\frac{x^b}{|\mathbf{x}|^3}J^c-\frac{2km}{c^3}\epsilon_{abc}x^bJ^c\times\left(\frac{5}{2}\frac{1}{l^3}-\frac{3}{2}\frac{|\mathbf{x}|^2}{l^5}-\frac{1}{|\mathbf{x}|^3}\right)\theta(l-|\mathbf{x}|),\quad(7.10b)$$

where $\theta(u)$ is defined as 1 for $u\geqslant0$ and equal to 0 for $u<0$, $m=\int d_3x\rho$ is the total mass,

$$J^a=\int d_3x\,\rho\epsilon_{abc}x^bv^c$$

is the angular momentum ($J^a=m\omega^a\times\frac{2}{5}l^2$). These formulas coincide in the external region $|\mathbf{x}|>l$ with Landau's formulas (3.9), so that the results which we are going to get by applying (7.10) will cover also Landau's approximation. Substituting (7.10) into (4.22) and (5.32) and performing integrations, one gets

$$\begin{aligned}t^a{}_f={}&{}^0t^a-\frac{4km}{c^2}\frac{r_\perp{}^a}{r_\perp{}^2}\left\{1-\theta(l-r_\perp)\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{3}{2}}\right\}\\&-\frac{8k}{c^3}\frac{r_\perp{}^a}{r_\perp{}^3}{}^0t^b\epsilon_{bcd}\frac{r_\perp{}^c}{r_\perp}J^d\left\{1-\theta(l-r_\perp)\times\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{3}{2}}\left(1+\frac{3}{2}\frac{r_\perp{}^2}{l^2}\right)\right\}-\frac{4k}{c^3}\frac{\epsilon_{abc}{}^0t^bJ^c}{r_\perp{}^2}\\&\times\left\{1-\theta(l-r_\perp)\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{5}{2}}\right\},\end{aligned}\quad(7.11a)$$

$$\Delta P_f=\frac{10k}{c^3}\frac{{}^0t^aJ^a}{l^2}\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{3}{2}}\theta(l-r_\perp)\equiv4\,{}^0t^a\omega^a\frac{km}{c^3}\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{3}{2}}\theta(l-r_\perp).\quad(7.11b)$$

In order to find the physical interpretation of this result, let us introduce the unit direction $u^a=r_\perp{}^a/r_\perp$ and another unit vector $w^a=\epsilon_{abc}u^b\,{}^0t^c$. The triple ${}^0t^a$, u^a, w^a forms a "dreibein"; let $J^a=\alpha\,{}^0t^a+\beta w^a+\gamma u^a$. Now, (7.11a)–(7.11b) can be rewritten

$$\begin{aligned}t_f{}^a={}&{}^0t^a-u^a\left\{\frac{4km}{c^2}\frac{1}{r_\perp}\left[1-\theta(l-r_\perp)\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{3}{2}}\right]\right.\\&\left.-\frac{4k\beta}{c^3r_\perp{}^2}\left[1-\theta(l-r_\perp)\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{3}{2}}\left(1+4\frac{r_\perp{}^2}{l^2}\right)\right]\right\}\\&+w^a\frac{4k\gamma}{c^3r_\perp{}^2}\left[1-\theta(l-r_\perp)\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{5}{2}}\right],\end{aligned}\quad(7.12a)$$

$$\Delta P_f=\frac{10k\alpha}{c^3l^2}\left(1-\frac{r_\perp{}^2}{l^2}\right)^{\frac{3}{2}}\theta(l-r_\perp).\quad(7.12b)$$

Therefore, in the deflection of a ray only the "orthogonal" part of J^a (components β, γ) are active; the polarization can be influenced only by the component of J^a which is parallel to ${}^0t^a(\alpha)$.

The term is (7.12a) which is proportional to m describes the Einstein deflection of light; this deflection, of course, is a deflection in the plane of the ${}^0t^a$, $r_\perp{}^a$ vectors. The component of angular momentum which is orthogonal to both ${}^0t^a$, $r_\perp{}^a$, i.e., βw^a, causes some correction to this deflection. If $r_\perp>l$ the angle of this deflection is given as $\Delta\varphi=4km/r_\perp c^2-4k\beta/r_\perp{}^2c^3$. The component of angular momentum along $r_\perp{}^a$, however, induces a new deflection in the direction $w^a=\epsilon_{abc}u^b\,{}^0t^c$. The corresponding angle (for $r_\perp>l$) is $\Delta\vartheta=4k\gamma/r_\perp{}^2c^3$; see Fig. 2.

For the values of $r_\perp<l$, (7.12a) describes the deflection of the ray which in a part of its path penetrates *through* the rotating body. Of course, it does not make

[26] Both formulas (7.5), (7.8) hold only under the assumptions mentioned concerning the correctness of interchanging the order of integration, differentiation and taking limits when $x^0\to\pm\infty$.

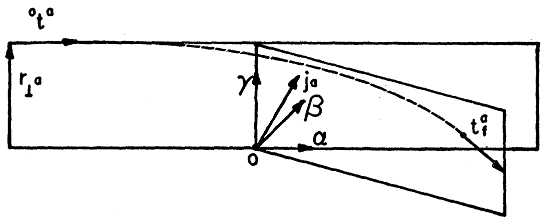

FIG. 2. An illustration of the meaning of the angles $\Delta\varphi$, $\Delta\vartheta$.

any sense in the case of a star. However, in the case of a rotating cloud of interstellar matter with very low density, a ray which penetrates through the cloud will be only partly absorbed, so that the problem of its deflection due to the gravitational field of the cloud makes sense.

In the limit $r_\perp^a \to 0$ (a ray going through the center of the cloud), the formula (7.11a) gives

$$\lim_{r_\perp \to 0} t^a{}_f = {}^0t^a - \frac{10k}{c^3}\epsilon_{abc}\, {}^0t^b J^c \frac{1}{l^2} \equiv {}^0t^a - \frac{4km}{c^3}\epsilon_{abc}\, {}^0t^b \omega^c \frac{1}{l^2}. \quad (7.13)$$

The ray is therefore deflected in the plane of ${}^0t^a$ and the direction orthogonal to ${}^0t^a$ and J^a through angle $10kJ/c^3l^2$.

As far as polarization is concerned, (7.12b) says that in the case of $r_\perp > l$ the polarization of the wave of the end of the history of the ray remains unchanged in comparison to the initial one.[27] However, for $r_\perp < l$, in the case a "rotating cloud," (7.12b) gives nontrivial ΔP_f, which depends on $r_\perp$ by the interesting factor $(1-r_\perp^2/l^2)^{\frac{3}{2}}$. Of course, the polarization of a wave going through the cloud will be affected by the interaction with matter. Our ΔP_f can be only treated as a small correction to "classical" rotation of the plane of polarization due to this interaction.

As the next application of our formulas, let us consider the deflection of light rays and the rotation of the plane of polarization due to gravitational field of a system of stars in their motion. Because the motion of stars in astronomical practice certainly can be treated as "slow" (v/c small), we should again use the formulas (4.22), (5.32) where Δg_α are to be taken simply in the form (3.3). Assuming in (3.3)

$$T^{00} = \sum_{a=1}^{N} m_a \delta_3(\mathbf{x} - \mathbf{a}(x^0)),$$
$$T^{0a} = \sum_{a=1}^{N} m_a \delta_3(\mathbf{x} - \mathbf{a}(x^0)) a^a{}_{,0}(x^0), \quad (7.14)$$

where m_a denotes mass of the ath star ($a = 1, 2\cdots, N$) which motion is given as $a^a = a^a(x^0)$, we get

$$t_f{}^a = {}^0t^a - \frac{2k}{c^2}\sum_{a=1}^{N} m_a \int_{-\infty}^{+\infty} dx^0 \frac{(r_\perp{}^a - a_\perp{}^a(x^0))}{[(x^0 - a_{\shortparallel}(x^0))^2 + r_\perp{}^2]^{\frac{3}{2}}} \times (1 - 2a_{\shortparallel}(x^0)_{,0}), \quad (7.15a)$$

$$\Delta P_f = -\frac{2k}{c^2}\sum_{a=1}^{N} m_a \int_{-\infty}^{+\infty} dx^0 \times \frac{{}^0t^a \epsilon_{abc}(r_\perp{}^b - a_\perp{}^b(x^0)) a^c{}_{\perp,0}(x^0)}{[(x^0 - a_{\shortparallel}(x^0))^2 + r_\perp{}^2]^{\frac{3}{2}}}, \quad (7.15b)$$

where $a_{\shortparallel}(x^0) = {}^0t^a a^a(x^0)$, $a_\perp{}^a(x^0) = a^a(x^0) - {}^0t^a a_{\shortparallel}(x^0)$. The origin of coordinates in these formulas is supposed to be identical with the classical center of mass of the system. In the case of a double star (direction ${}^0t^a$ orthogonal to the plane of motion), ΔP_f is different from zero.

It might be of some theoretical interest also to investigate the case of a rotating body when its angular velocity is so big that terms of higher order in c^{-1} are important. The formulas (7.6), (7.8) valid for arbitrary velocities of matter could be applied in this case. The energy momentum tensor can be taken here accordingly with the results of Salzman and Taub.[28]

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[27] This result differs from the corresponding result of reference 2. [According to reference 2 $\Delta P_f \neq 0$ for $r_\perp > l$ in the case of the rotating body.] One can check the correctness of the statement that ΔP_f vanishes for $r_\perp > l$ simply by substituting (3.7) into (5.32); the corresponding integral vanishes.

[28] G. Salzman and A. H. Taub, Phys. Rev. 95, 1659 (1954).