APPENDIX B. ALTERNATIVE EXPRESSION FOR $\bar{N}$

Consider the expression for Ω,

$$\Omega=\Omega_0+\sum_{n=1}^{\infty}\Omega_n, \tag{B.1}$$

where the Ω_n are given by (41). If we differentiate this with respect to μ to obtain $\bar{N}$, we see at once

$$\bar{N}=-\frac{\partial\Omega_0}{\partial\mu}-\frac{1}{\beta}\sum_n\sum_r\sum_l G_{rn}'(\zeta_l)\frac{\partial}{\partial\mu}\frac{1}{\zeta_l-\epsilon_r}, \tag{B.2}$$

since in differentiating a closed linked nth order diagram is equivalent to differentiating any of its $2n$ lines. When we sum over r, all of these give the same contribution. Therefore

$$\bar{N}=\frac{1}{\beta}\sum_r\sum_l \exp(\zeta_l 0^+)\frac{1}{\zeta_l-\epsilon_r}+\frac{1}{\beta}\sum_r\sum_l \exp(\zeta_l 0^+)\frac{G_r(\zeta_l)}{(\zeta_l-\epsilon_r)(\zeta_l-\epsilon_r-G_r(\zeta_l))}, \tag{B.3}$$

using (54) and (44). Combining

$$\bar{N}=\frac{1}{\beta}\sum_r\sum_l \exp(\zeta_l 0^+)\frac{1}{\zeta_l-\epsilon_r-G_r(\zeta_l)}. \tag{B.4}$$

This actually corresponds to the result that the mean number of particles is just the sum of the mean occupation number of each state.

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Example of a Soluble Field Theory with Finite Charge Renormalization

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A soluble field theory suggested by the Lee and Machida models is described in which coupling constant renormalization arises from a dressed boson and is finite if the contributing fermions are assumed nonrelativistic. For the unrenormalized charge to be real, the renormalized charge must satisfy a certain inequality depending on the boson and fermion mass ratios; if this inequality is violated a single boson ghost state occurs, as expected.

1. INTRODUCTION

EVER since the appearance of the Lee model[1] there has been much interest in obtaining examples of field theories wherein quantities of interest may be derived in closed form; and of those theories which have been found, several[2,3] are essentially extensions of the Lee model. A variation of Lee's procedure was discussed by Machida[4] who considered the soluble problem of a dressed boson, rather than a dressed fermion; and more recently Goldstein[5] has presented a sort of combination of the two models. In each of these theories the renormalization constants are infinite, i.e., cutoff dependent, implying an imaginary value for the unrenormalized coupling constant (charge) as the cutoff exceeds a certain critical value. For this latter situation, the analysis of Källén and Pauli[6] indicates that a ghost state is to be expected.

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Although nothing basically new is to be learned from the following discussion, it may nevertheless be of some interest to examine an exactly soluble theory with finite charge renormalization. The rather trivial remark to be made in this connection is the observation that in all such previous models infinite renormalization constants are obtained as a result of adhering to the relativistic energy-momentum relation for that particle whose momentum appears as the variable of integration in the definition of the renormalization constants. In the original Lee model, for example, replacing the boson energy $\omega(\mathbf{k})=(\mathbf{k}^2+\mu^2)^{\frac{1}{2}}$ by $\mu+\mathbf{k}^2(2\mu)^{-1}$ when integrating over the boson momentum $\mathbf{k}$, yields a finite value for Z_2^{-1}; and similarly the replacement of $E(\mathbf{p})=(\mathbf{p}^2+m^2)^{\frac{1}{2}}$ by $m+\mathbf{p}^2(2m)^{-1}$ provides a finite value for the renormalization constant Z_3^{-1} of Machida. Since one has already mutilated the physically correct interaction Hamiltonian in order to obtain a set of exactly soluble equations, little further rigor is lost in assuming nonrelativistic particles in intermediate states; and the advantage of doing this is that one may then obtain, within certain well-defined limits, a quite respectable field theory (at the possible risk, of course, of being even further removed from physical reality). These statements will be illustrated briefly by consider-

ing a slight variation of Machida's model, such that the method of presentation and the notation closely follow that of Lee and of Källén and Pauli.

2. DETAILS OF THE MODEL

Consider two distinct fermions N and V, with respective masses m_N and m_V, interacting with a boson θ, of rest mass μ and energy $\omega(\mathbf{k})=(\mathbf{k}^2+\mu^2)^{\frac{1}{2}}$, according to the reactions

$$\theta \rightleftarrows N+V.$$

In order that the boson be stable, it will be assumed that $m_N+m_V>\mu$.

The Hamiltonian for this system, composed of the unrenormalized operators $\psi_N(\mathbf{p})$, $\psi_V(\mathbf{p}')$, $a(\mathbf{k})$, their Hermitian adjoints, and the unrenormalized charge g_0, is given by

$$H=H_0+H',$$

$$H_0=\sum_{\mathbf{p}} E_N(\mathbf{p})\psi_N{}^\dagger(\mathbf{p})\psi_N(\mathbf{p})+\sum_{\mathbf{p}} E_V(\mathbf{p})\psi_V{}^\dagger(\mathbf{p})\psi_V(\mathbf{p})+\sum_{\mathbf{k}} \omega(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k}),$$

$$H'=-\sum_{\mathbf{p}\mathbf{p}'\mathbf{k}} \frac{g_0 f(\mathbf{p},\mathbf{p}';\mathbf{k})}{(2\omega V)^{\frac{1}{2}}}\{\psi_N{}^\dagger(\mathbf{p})\psi_V{}^\dagger(\mathbf{p}')a(\mathbf{k})+\psi_V(\mathbf{p}')\psi_N(\mathbf{p})a^\dagger(\mathbf{k})\}\delta_{\mathbf{p}+\mathbf{p}'-\mathbf{k}}-\sum_{\mathbf{k}}\frac{\delta\mu^2}{2\omega}a^\dagger(\mathbf{k})a(\mathbf{k}), \quad (1)$$

where $f(\mathbf{p},\mathbf{p}';\mathbf{k})$ represents a cutoff function inserted for generality, and $\delta\mu^2$ denotes the boson mass renormalization. The commutation properties of the fermion and boson operators are the familiar ones, specified for example, in reference 6.

Again, as in the original Lee model, there are two constants of the motion, which for this system are

$$\begin{aligned} n_V+n_\theta&=\text{constant},\\ n_N+n_\theta&=\text{constant}, \end{aligned} \quad (2)$$

where the individual n represent number operators for the respective fields. Equation (2) states that the total number of particles in any given situation is finite, and exact solutions may be expected of the corresponding Schrödinger equation.

3. EXACT EIGENSTATES

To determine the eigenstate $|\boldsymbol{\theta}_k\rangle$ of a physical θ particle of momentum $\mathbf{k}$, one utilizes the eigenstates of the bare particles which satisfy the relations

$$H_0|\theta_k\rangle=H_0a^\dagger(\mathbf{k})|0\rangle=\omega(\mathbf{k})|\theta_k\rangle,$$
$$H_0|N_p\rangle=H_0\psi_N{}^\dagger(\mathbf{p})|0\rangle=E_N(\mathbf{p})|N_p\rangle,$$
$$H_0|V_{p'}\rangle=H_0\psi_V{}^\dagger(\mathbf{p}')|0\rangle=E_V(\mathbf{p}')|V_{p'}\rangle.$$

Examination of the complete Hamiltonian indicates that the physical N and V states are identical to the corresponding bare states,

$$|\mathbf{N}_p\rangle=|N_p\rangle,\quad |\mathbf{V}_{p'}\rangle=|V_{p'}\rangle,$$

but that the physical θ state will have a structure of form

$$|\boldsymbol{\theta}_k\rangle=Z_3^{\frac{1}{2}}[|\theta_k\rangle+\sum_{\mathbf{p}\mathbf{p}'}\Phi(\mathbf{p},\mathbf{p}';\mathbf{k})\psi_N{}^\dagger(\mathbf{p})\psi_V{}^\dagger(\mathbf{p}')|0\rangle]. \quad (3)$$

Application of the Schrödinger equation

$$H|\boldsymbol{\theta}_k\rangle=\omega(\mathbf{k})|\boldsymbol{\theta}_k\rangle,$$

together with the orthonormality of the bare eigenstates and a normalization condition for the dressed θ state, then yields

$$\Phi(\mathbf{p},\mathbf{p}';\mathbf{k})=g_0(2\omega V)^{-\frac{1}{2}}f(\mathbf{p},\mathbf{p}';\mathbf{k})\delta_{\mathbf{p}+\mathbf{p}'-\mathbf{k}}\times[E(\mathbf{p},\mathbf{p}')-\omega(\mathbf{k})]^{-1}, \quad (4)$$

$$\delta\mu^2=-\sum_{\mathbf{p}\mathbf{p}'} g_0{}^2f^2V^{-1}\delta_{\mathbf{p}+\mathbf{p}'-\mathbf{k}}\times[E(\mathbf{p},\mathbf{p}')-\omega(\mathbf{k})]^{-1}, \quad (5)$$

$$Z_3{}^{-1}=1+\sum_{\mathbf{p}\mathbf{p}'} g_0{}^2f^2(2\omega V)^{-1}\delta_{\mathbf{p}+\mathbf{p}'-\mathbf{k}}\times[E(\mathbf{p},\mathbf{p}')-\omega(\mathbf{k})]^{-2}, \quad (6)$$

where $E(\mathbf{p},\mathbf{p}')=E_N(\mathbf{p})+E_V(\mathbf{p}')$. With a relativistic fermion energy spectrum, $E_{N,V}(\mathbf{p})=(m_{N,V}{}^2+\mathbf{p}^2)^{\frac{1}{2}}$, both $\delta\mu^2$ and $Z_3{}^{-1}$ are strongly divergent; but a nonrelativistic spectrum, $E_{N,V}(\mathbf{p})=m_{N,V}+\mathbf{p}^2(2m_{N,V})^{-1}$, ensures a convergent result for $Z_3{}^{-1}$, although $\delta\mu^2$ is still infinite (if $f=1$). As previously mentioned, this latter alternative will be employed to obtain a theory with a finite charge remormalization. It may be noted that these quantities depend on the momentum of the θ particle.[7]

Passing from the discrete sum over fermion momentum of Eq. (6) to the corresponding integral, with $f=1$ one obtains

$$Z_3{}^{-1}(\mathbf{k})=1+\frac{g_0{}^2}{8\pi}\left(\frac{m}{\omega}\right)\left[\frac{2m}{M+\mathbf{k}^2(2M)^{-1}-\omega}\right]^{\frac{1}{2}}, \quad (7)$$

where $M=m_N+m_V$ and $m^{-1}=m_N{}^{-1}+m_V{}^{-1}$. It is interesting to note that $Z_3{}^{-1}$ varies smoothly between its limiting forms: $Z_3{}^{-1}(\infty)=1$ and

$$Z_3{}^{-1}(0)=1+\frac{g_0{}^2}{8\pi}\left(\frac{m}{\mu}\right)\left[\frac{2m}{M-\mu}\right]^{\frac{1}{2}}, \quad (8)$$

indicating, for $g_0{}^2>0$, that the usual probability interpretation of Z_3 is possible for any $\mathbf{k}$ value.

To determine the significance of Eqs. (7) and (8), one may calculate the scattering amplitude for the process $N+V\rightarrow N+V$. Expressing the physical NV

[7] The same conclusion may be reached by means of a standard graphical analysis, which illustrates that the reason for the momentum dependence of Z_3 and $\delta\mu^2$ is the lack of fermion Lorentz symmetry. I am indebted to Dr. D. Pursey for this remark.

state as

$$|\mathbf{N}+\mathbf{V}\rangle=\sum_{\mathbf{p}\mathbf{p}'}F(\mathbf{p},\mathbf{p}')\psi_N{}^\dagger(\mathbf{p})\psi_V{}^\dagger(\mathbf{p}')|0\rangle+\sum_{\mathbf{k}}G(\mathbf{k})|\boldsymbol{\theta}_k\rangle,$$

one then applies the Schrödinger equation

$$H|\mathbf{N}+\mathbf{V}\rangle=E_0|\mathbf{N}+\mathbf{V}\rangle,$$

together with the condition

$$G(\mathbf{k})=-\sum_{\mathbf{p}\mathbf{p}'}F(\mathbf{p},\mathbf{p}')\langle\boldsymbol{\theta}_k|\psi_N{}^\dagger(\mathbf{p})\psi_V{}^\dagger(\mathbf{p}')|0\rangle,$$

to obtain

$$[E_0-E(\mathbf{p},\mathbf{p}')]F(\mathbf{p},\mathbf{p}') =\frac{g_0{}^2Z_3(\mathbf{p}+\mathbf{p}')}{2V\omega(\mathbf{p}+\mathbf{p}')}\cdot\frac{[E_0-\omega(\mathbf{p}+\mathbf{p}')]}{[E(\mathbf{p},\mathbf{p}')-\omega(\mathbf{p}+\mathbf{p}')]} \cdot\sum_{\mathbf{q}}\frac{F(\mathbf{q},\,\mathbf{p}+\mathbf{p}'-\mathbf{q})}{[E(\mathbf{q},\mathbf{p}+\mathbf{p}'-\mathbf{q})-\omega(\mathbf{p}+\mathbf{p}')]},$$

which can be readily solved and which contains no divergent quantities. The point of interest here is that the combination $g_0{}^2Z_3(\mathbf{p}+\mathbf{p}')$ measures the strength of the interaction, and it is therefore natural to define this quantity as the renormalized coupling constant, g^2. In the rest frame of the two fermions, $\mathbf{p}+\mathbf{p}'=0$, and $Z_3{}^{-1}$ is then given by Eq. (8), which quantity may be called the renormalization constant. In general, however, the renormalized charge is velocity dependent.[8]

Alternately, one can require that g^2 be strictly constant, and the requisite momentum dependence is then implicitly contained in $g_0{}^2$. Solving for the latter, one finds

$$g_0{}^2=g^2\left[1-\frac{g^2}{8\pi}\left(\frac{m}{\omega}\right)\left(\frac{2m}{M+\mathbf{k}^2(2M)^{-1}-\omega}\right)^{\frac{1}{2}}\right]^{-1},$$

from which it follows, assuming a constant[9] and positive g^2, that $g_0{}^2$ will also be positive if the inequality

$$\frac{g^2}{8\pi}\left(\frac{m}{\mu}\right)\left(\frac{2m}{M-\mu}\right)^{\frac{1}{2}}<1 \tag{9}$$

is satisfied. It is interesting to note that this is just the condition that an expansion of $Z_3{}^{-1}(g^2)$ in powers of g^2 converge. If this inequality is violated, g_0 is then imaginary, and as in the Lee model, the unitarity of the theory is destroyed.

[8] Inserting the factor $\delta(\mathbf{k})$ into each term of H' would remove this velocity dependence by postulating that a θ particle can only exist at rest in that frame in which the theory is defined.

[9] For this choice of constant g, H', when written in terms of g and the renormalized boson operators, has the conventional "local" form in configuration space, while the free field Hamiltonian density has the form of a spatial integral over the renormalized operators. In terms of g_0 and the original unrenormalized operators, this situation is just reversed.

4. GHOST STATES

Following the procedure of Källén and Pauli,[6] the existence of ghost states may be inferred from a consideration of the eigenvalue problem

$$H|\boldsymbol{\theta}_k\rangle=\omega_0(\mathbf{k})|\boldsymbol{\theta}_k\rangle,\quad \omega_0{}^2=\mathbf{k}^2+\mu_0{}^2,$$

where the physical θ state of momentum $\mathbf{k}$ is given by an expression similar to that of Eq. (3). The same analysis which lead to Eqs. (4) and (5) then yields

$$\Phi(\mathbf{p},\mathbf{p}';\mathbf{k})=g_0f(2\omega V)^{-1}\delta_{\mathbf{p}+\mathbf{p}'-\mathbf{k}}[E(\mathbf{p},\mathbf{p}')-\omega_0]^{-1},$$

and

$$\omega_0-\omega=g_0{}^2(2\omega V)^{-1}\sum_{\mathbf{p}\mathbf{p}'}(\omega-\omega_0)\delta_{\mathbf{p}+\mathbf{p}'-\mathbf{k}} \times[E(\mathbf{p},\mathbf{p}')-\omega_0]^{-1}[E(\mathbf{p},\mathbf{p}')-\omega]^{-1}, \tag{10}$$

where the previous expression for $\delta\mu^2$, Eq. (5), has been used, and f has been set equal to unity since the resulting sum converges. Because the quantities of interest here are those values of ω_0 such that $\omega_0\neq\omega$ (and in particular, where $\omega_0<\omega$), Eq. (10) may be rewritten as

$$-1=g_0{}^2(2\omega V)^{-1}\sum_{\mathbf{p}\mathbf{p}'}\int_0^1 dx\,\delta_{\mathbf{p}+\mathbf{p}'-\mathbf{k}}[E(\mathbf{p},\mathbf{p}')-\bar{\omega}]^{-2},\qquad \bar{\omega}=\omega_0+x(\omega-\omega_0).$$

Replacing the sum by an integral, one obtains

$$\omega-\omega_0=\frac{g_0{}^2}{4\pi}\left(\frac{m}{\omega}\right)(2m)^{\frac{1}{2}}\left\{\left[M+\frac{\mathbf{k}^2}{2M}-\omega\right]^{\frac{1}{2}} -\left[M+\frac{\mathbf{k}^2}{2M}-\omega_0\right]^{\frac{1}{2}}\right\}. \tag{11}$$

It is then evident that if $g_0{}^2$ is positive, the only solution of Eq. (11) occurs when $\omega_0=\omega$. If, however, $g_0{}^2$ is negative, one finds but one solution of Eq. (11), where $\omega_0<\omega$. In the θ rest frame this corresponds to a ghost state of rest mass

$$\mu_0=\mu-\frac{g^2}{2\pi}\left(\frac{m}{\mu}\right)[2m(M-\mu)]^{\frac{1}{2}} \times\left[\frac{g^2}{8\pi}\left(\frac{m}{\mu}\right)\left(\frac{2m}{M-\mu}\right)^{\frac{1}{2}}-1\right]^{-2}.$$

From this simple and explicit calculation one sees that the conclusion reached in reference 6, concerning the appearance of a single ghost state in the Lee model, is also valid here.

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