Level Structure of Nuclear Matter and Liquid He^3

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Using the K matrix as computed in the study of nuclear matter and liquid He^3 as the effective interaction at the Fermi surface, the possible superfluidity of these systems has been investigated. A theory of the cooperative phenomenon valid for particle-particle interaction in states of arbitrary angular momentum has been developed following the methods of Bardeen, Cooper, and Schrieffer. It is found for states of relative angular momentum other than $l=0$ that the particle pairs must be correlated with respect to an arbitrary direction in the medium. As a result the change of structure of the Fermi surface is angularly dependent. An energy gap does not occur other than for $l=0$, the particle excitation energy vanishing for certain orientations of the momentum. It is also shown that the specific heat shows a discontinuity at the transition temperature, but somewhat different from the case of $l=0$.

Application of these results to liquid He^3 shows that the cooperative effects arise from the interaction in the state with $l=2$, and that the transition temperature is at about 0.1°K. In nuclear matter the $^1S_0$ interaction is very weak and probably repulsive at the Fermi surface, and the attractive $^1D_2$ interaction gives a negligible energy shift. The $^3S_1$ interaction is attractive and in nuclear matter gives a few-tenths of an Mev energy gap. These results suggest that in finite nuclei, with pairing of identical nucleons in the same shell, the cooperative effects are not strictly analogous to those in nuclear matter but instead are closely associated with the finite level spacing.

I. INTRODUCTION

SINCE the Bardeen-Cooper-Schrieffer[1] theory of superconductivity was proposed, there have been several attempts to extend their methods to nuclear matter and liquid He^3. The applications of the original BCS formalism or its modification by Bogoljubov[2] to the many-body fermion systems[3] has been complicated, however, by the presence of strong short range repulsion in the nucleon-nucleon or He^3-He^3 interaction, this repulsion being the primary origin of the saturation phenomenon. We wish to discuss the effective interaction in those systems, using unpublished results from the work of Brueckner and Gammel,[4] and then to develop an extension of the BCS formalism appropriate to this interaction.

II. THE EFFECTIVE INTERACTION

In the study of nuclear matter and He^3, which has been made by Brueckner and Gammel,[4] the energy is determined from the K matrix defined by the integral equation

$$K_{ij,kl}=v_{ij,kl}+\sum_{\substack{mn\\ p_m>p_F\\ p_n>p_F}} v_{ij,mn}\frac{1}{E_k+E_l-E_m^*-E_n^*}K_{mn,kl}. \quad (2.1)$$

In this equation E_k and E_l are self-consistent energies for particles moving in the Fermi gas, and E_m^* and E_n^* are energies appropriate to virtual excitations above the Fermi surface. In determining the virtual excitation energies, the effects of excitation and other particles of the medium have, in the calculations of Brueckner and Gammel, been represented by introduction of a mean excitation energy taken to be the excitation energy of the Fermi gas. The virtual energies so defined differ at the Fermi surface by an energy gap which prevents the occurrence of very small energy denominators in Eq. (2.1). In addition, for numerical reasons, the values of momentum closest to the Fermi surface ($0.9p_F$ and $1.1p_F$) taken in solving Eq. (2.1) differed appreciably so that this also prevented the occurrence of very small energy denominators in Eq. (2.1). One consequence of this procedure was that the K matrices computed in the various states of angular momentum were nonsingular for both nuclear matter and He^3. It is, therefore, reasonable to take the K matrix as the effective interaction upon which to base a specific examination of the level structure. This approximation is equivalent to a separate treatment of transitions between states very near to Fermi surface, these being omitted in the evaluation of the K matrix as the effective interaction.[5]

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957). (We refer to this as BCS).

[2] N. N. Bogoljubov, V. V. Tolmachev, and D. V. Shirkov, New Method in the Theory of Superconductivity, Publication of the Academy of Science, USSR (in Russian) [translation: New York Consultants Bureau, New York, 1959].

[3] A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110**. 936 (1958); C. DeDominicis and P. C. Martin, Bull. Am. Phys. Soc. **3**, 224 (1958); L. N. Cooper, R. L. Mills, and A. M. Sessler, Phys. Rev. **114**, 1377 (1959).

[4] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958); **109**, 1040 (1958).

[5] A discussion of this approximation in the theory of finite nuclei has also been given by H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

The angular momentum reduction of the K matrix[6] is given for central forces only by the equation

$$(\mathbf{k}'|K|\mathbf{k})=\sum_l(2l+1)(k'|K_l|k)P_l(\hat{k}',\hat{k}). \quad (2.2)$$

In nuclear matter, the strong noncentral forces are important, and the K matrix is defined by Eq. (61) of Brueckner and Gammel[4]:

$$(\mathbf{k}'sm_s'|K|\mathbf{k}sm_s)$$
$$=\sum_J\int d\mathbf{r}\sum_{l=J-1}^{J+1}(2l+1)(-i)^l\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}j_l(k'r) \quad (2.3)$$
$$\times F_l{}^{Jm_sm_s}(\hat{k}',\hat{r})C(Jm_s';l0;sm_s')V(r)\sum_{l'=J-1}^{J+1}$$
$$\times(2l'+1)i^{l'}\left(\frac{4\pi}{2l'+1}\right)^{\frac{1}{2}}\sum_{l'=J-1}^{J+1}U_{ll'}{}^{Js}F_{l'}{}^{Jm_sm_s}(\hat{\mathbf{k}},\hat{\mathbf{r}})$$
$$\times C(Jm_s;l'0;sm_s).$$

In the triplet state, the attraction appears first in the s state, for which Eq. (2.3) reduces to (we give the relevant sum over m_s values only)

$$\sum_{m_s}(\mathbf{k}'1m_s|K|\mathbf{k}1m_s)|_{l=0}$$
$$=3\int d\mathbf{r}\,\gamma_0(k'r)\sum_{l'=0}^{2}V_{0l'}{}^{11}(r)U_{0l'}{}^{11}(r). \quad (2.4)$$

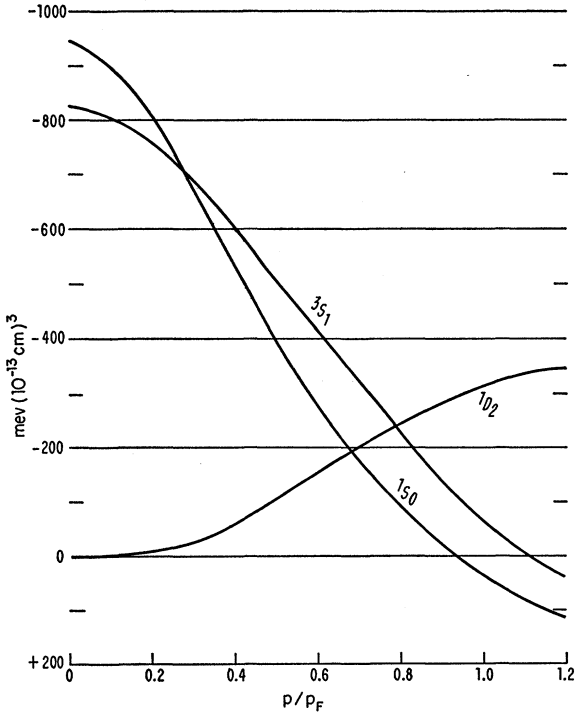

FIG. 1. Diagonal elements of the K matrix as a function of momentum for nuclear matter.

[6] The details of the angular momentum reduction of the K-matrix equation are given in reference 4.

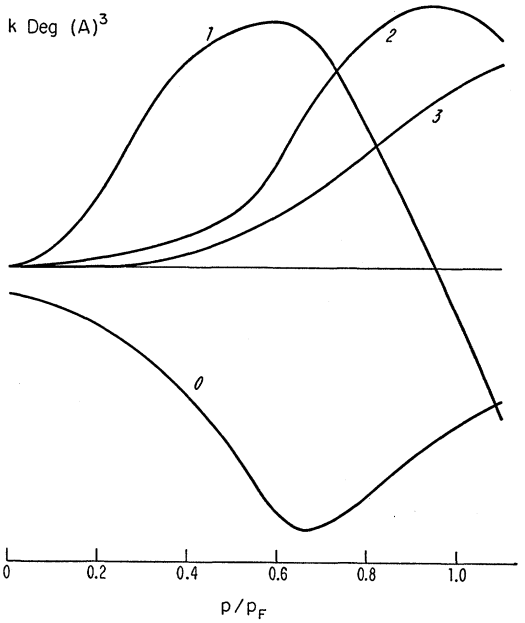

FIG. 2. Diagonal elements of the K matrix as a function of momentum for liquid He³.

This exhibits explicitly the coupling to the $l'=2$ state which is very important in determining the magnitude of the attraction.

The diagonal elements with $k'=k$ are given as a function of k/k_F for nuclear matter in Fig. 1 and for liquid He³ in Fig. 2. These show that in He³ the effective interaction at the Fermi surface is repulsive in the state of angular momentum $l=0$, weak and repulsive in the $l=1$ state, and strongly attractive in the $l=2$ state. In nuclear matter the $^1S_0$ state is repulsive,[7] the $^3S_1$ attractive, and $^1D_2$ attractive. We now turn to a consideration of the effects of these interactions on level structure.

III. LEVEL STRUCTURE

The effect on level structure is expected to be appreciable only near the Fermi surface, where the relative momentum

$$\mathbf{k}=\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2) \quad (3.1)$$

for the BCS type of pairing with $k_1=k_2$, is equal to k_F in magnitude. The interaction in the lth angular momentum state may then be approximated by

$$H_l'=-(2l+1)V_l\sum_{\mathbf{kk}'}C_{\mathbf{k}}{}^*C_{-\mathbf{k}}{}^*C_{-\mathbf{k}'}C_{\mathbf{k}'}P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'), \quad (3.2)$$

with

$$V_l=-(k_F|K_l|k_F). \quad (3.3)$$

[7] A slightly different conclusion has been reached by V. J. Emery (private communication) who finds that the $^1S_0$ K matrix is still very weakly attractive at the Fermi surface. This discrepancy may be due to slight differences in the methods used to solve the K matrix operations.

For an attractive interaction, V_l as defined in Eq. (3.3) is positive. (For l even, the spins of $\mathbf{k}$ and $-\mathbf{k}$ must be antiparallel; for l odd, they must be parallel.)

We now follow BCS and make the replacement

$$C_{-\mathbf{k}}C_{\mathbf{k}}=b_{\mathbf{k}}. \tag{3.4}$$

The Hamiltonian can then be written in BCS form as

$$H=\sum_{k>k_F} 2\epsilon_k b_{\mathbf{k}}^* b_{\mathbf{k}}-\sum_{k<k_F} 2|\epsilon_k| b_{\mathbf{k}} b_{\mathbf{k}}^* - (2l+1)V_l \sum_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}}^* b_{\mathbf{k}'} P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'). \tag{3.5}$$

In this equation the ϵ_k are the single particle energies measured relative to the Fermi surface, i.e.,

$$\epsilon_k=(k^2-k_F^2)/2M^*. \tag{3.6}$$

The value of M^* is that determined from the evaluation of the level structure in the normal state.

As a trial function, we again use the form of BCS and take

$$\bar{\psi}=\Pi_k \frac{1+A_{\mathbf{k}}b_{\mathbf{k}}^*}{(1+|A_{\mathbf{k}}|^2)^{\frac{1}{2}}}\bar{\psi}_0, \tag{3.7}$$

with $\bar{\psi}_0$ the vacuum state. The expectation value of the Hamiltonian then is

$$E=2\sum_{k>k_F}\epsilon_k\frac{|A_{\mathbf{k}}|^2}{1+|A_{\mathbf{k}}|^2}-2\sum_{k<k_F}|\epsilon_k|\frac{1}{1+|A_{\mathbf{k}}|^2} - (2l+1)V_l\sum_{\mathbf{k}\mathbf{k}'}\frac{A_{\mathbf{k}}^*}{1+|A_{\mathbf{k}}|^2}\frac{A_{\mathbf{k}'}}{1+|A_{\mathbf{k}'}|^2}P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'). \tag{3.8}$$

Equation (3.9) shows that the interaction term vanishes if $A_{\mathbf{k}}$ is a scalar. Consequently we take for $A_{\mathbf{k}}$ the form[8]

$$A_{\mathbf{k}}=\alpha_k Y_l^k(\hat{\mathbf{k}},\hat{\mathbf{k}}_0), \tag{3.9}$$

with $Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)$ a normalized spherical harmonic and $\hat{\mathbf{k}}_0$ an arbitrary unit vector. A more general form would involve a sum over azimuthal quantum numbers, but we have not treated this case.

We now carry out the angular integral in Eq. (3.8), using the expansion theorem for $P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')$. The result is

$$E=2\sum_{k>k_F}\epsilon_k\frac{\alpha_k^2|Y_l^k(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2}{1+\alpha_k^2|Y_l^k(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2} -2\sum_{k<k_F}|\epsilon_k|\frac{1}{1+\alpha_k^2|Y_l^k(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2}-V_l\Delta^2, \tag{3.10}$$

with

$$\Delta=\sum_{\mathbf{k}}\frac{\alpha_k|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2}{1+\alpha_k^2|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2}. \tag{3.11}$$

[8] A slightly more accurate ansatz can be shown to be $A(1-A^2)^{-1}\propto Y_l^m$, but the results do not differ physically (or within the accuracy of other approximations in the theory). This more exact computation will be published in the future, together with more complete results on the thermal and other behavior.

The optimum choice of α_k is now determined by minimizing the energy. We follow the procedure of BCS with the result

$$E=N(0)(\Delta E)^2\int\frac{d\Omega_k}{4\pi} \times\left\{1-\left[1+\frac{\epsilon_0^2}{(\Delta E)^2}|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2\right]^{\frac{1}{2}}\right\}. \tag{3.12}$$

To obtain this result we have defined $N(0)$ as the level density at the Fermi surface and replaced the momentum integrals by

$$\int d\mathbf{k}=N(0)\int_{-\Delta E}^{\Delta E}d\epsilon_k. \tag{3.13}$$

We discuss the choice of the cutoff energy ΔE in the next section.

The characteristic energy ϵ_0 in Eq. (3.12) is determined by the equation

$$\frac{1}{V_l}=\frac{1}{2}\sum_{\mathbf{k}}\frac{|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2}{[\epsilon_k^2+\epsilon_0^2|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2]^{\frac{1}{2}}}. \tag{3.14}$$

The single particle energy is given by the methods of BCS to be

$$E_k=[\epsilon_k^2+\epsilon_0^2|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2]^{\frac{1}{2}}. \tag{3.15}$$

Thus we see that there is no true energy gap, the excitation energy vanishing for orientations of $\mathbf{k}$ such that $|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|$ is zero. In particular, for the best (i.e., lowest energy) cases the gap vanishes at a point and the density of states rises only quadratically with energy at first. In the weak coupling approximation, with $\epsilon_0\ll\Delta E$, we find

$$E=-\tfrac{1}{2}N(0)\epsilon_0^2 \tag{3.16}$$

and

$$\epsilon_0=2\Delta E \exp\left[-\left(\frac{1}{N(0)V_l}+\int\frac{d\Omega_k}{4\pi}|Y_l^m|^2\ln|Y_l^m|\right)\right]. \tag{3.17}$$

The temperature variation of ϵ_0 is also of interest. This is determined as in BCS by the equation

$$\frac{1}{V_l}=\frac{1}{2}\sum_{k}\frac{\tanh(\frac{1}{2}\beta E_k)}{E_k}|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^2, \tag{3.18}$$

with E_k given by Eq. (3.15). At the transition point where ϵ_0 vanishes, Eq. (3.18) gives simply

$$\frac{1}{N(0)V_l}=\int_0^{\Delta E}\frac{\tanh(\frac{1}{2}\beta_c\epsilon)}{\epsilon}d\epsilon, \tag{3.19}$$

or

$$kT_c=1.14\Delta E\exp\{-[1/N(0)V_l]\}. \tag{3.20}$$

We note that the transition temperature is independent of the azimuthal quantum number m chosen in the trial function of Eq. (3.9).

The change in specific heat at the transition is also easily determined. The result is, following BCS,

$$\left(\frac{C_s-C_n}{C_n}\right)_{T_c}=1.52[4\pi\int d\Omega_k|Y_l^m(\hat{\mathbf{k}},\hat{\mathbf{k}}_0)|^4]^{-1}. \quad (3.21)$$

The angular integral in this result comes from the angular dependence of the single particle energy.

The above results are given for the case of general m. For the l values of interest ($l=0,1,2$), we find for the energy

$$\begin{aligned}
&E/N(0)(\Delta E)^2\\
&=-2\exp[-2/N(0)V_0], \quad l=0, \quad m=0\\
&=-\frac{2}{3}\exp\left[-\frac{2}{N(0)V_1}+\frac{2}{3}\right], \quad l=1, \quad m=0\\
&=-\frac{1}{3}\exp\left[-\frac{2}{N(0)V_1}+\frac{5}{3}\right], \quad l=1, \quad m=1\\
&=-\frac{2}{5}\exp\left[-\frac{2}{N(0)V_2}-\frac{4}{9}\sqrt{3}\ln\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{32}{15}\right],\\
&\qquad\qquad l=2, \quad m=0\\
&=-\frac{1}{15}\exp\left[-\frac{2}{N(0)V_2}+\frac{47}{15}\right], \quad l=2, \quad m=1,2.
\end{aligned} \quad (3.22)$$

Evaluation of the numerical constants shows that for $l=1$, the $m=1$ case has a lower energy than $m=0$ by about 36% and for $l=2$, the degenerate $m=1$ and $m=2$ cases are lower in energy than the $m=0$ case by about 24%. These energy differences are relatively small so that considerable mixing of modes of excitation will probably occur at finite temperatures, particularly near the transition point where all m modes for a given l are degenerate. This situation complicates the analysis of the thermodynamic behavior; we have not carried out the study of the case of general mixing of m modes.

The jump in the specific heat at the transition point for the l values given in Eq. (3.22) follows from Eq. (3.21). The result is

$$\begin{aligned}
\left(\frac{C_s-C_n}{C_s}\right)_{T_c}&=1.52, \quad l=0, \quad m=0\\
&=0.84, \quad l=1, \quad m=0\\
&=1.27, \quad l=1, \quad m=1\\
&=0.71, \quad l=2, \quad m=0\\
&=1.06, \quad l=2, \quad m=1\\
&=1.06, \quad l=2, \quad m=2.
\end{aligned} \quad (3.23)$$

Thus the change is strongly m dependent and a specific prediction is not possible in the approximation we have used.

IV. APPLICATION

We now apply these results to liquid He^3 and nuclear matter. In the former case, the calculations of Brueckner and Gammel give at a volume per particle corresponding to a radius of 2.55 A for the $l=0$, $l=1$, and $l=2$ state

$$\begin{aligned}
V_0&=-(k_F|K_0|k_F)=-233k \text{ deg (A)}^3,\\
&=-(k_F|K_1|k_F)=-\ 60k \text{ deg (A)}^3,\\
&=-(k_F|K_2|k_F)=\ \ 78k \text{ deg (A)}^3.
\end{aligned} \quad (4.1)$$

Thus a change in level structure is according to these results not to be expected in the $l=0$ or $l=1$ state. It should be emphasized, however, that the calculations of the total energy by Brueckner and Gammel gave a binding energy per particle of 0.96 degrees instead of the observed 2.53 degrees. To correct for this deficiency, an increase in potential energy of about 21% is required, which could change the sign of the interaction in the $l=1$ state.

Taking the results of Eq. (4.1), we expect pairing only in the state with $l=2$, which is a singlet state. The level density is

$$[N(0)]_{\text{singlet}}=k_FM^*/2\pi^2\hbar^2, \quad (4.2)$$

with the effective mass at the Fermi surface

$$M^*=1.81M. \quad (4.3)$$

Thus we find

$$V_2N(0)=0.337, \quad (4.4)$$

and

$$kT_c=0.059\Delta E. \quad (4.5)$$

If we take as a rough estimate for ΔE the kinetic energy at the Fermi surface, which is $2.58k$ degrees (for an effective mass of $1.81M$), then

$$T_c=0.154 \text{ degree}. \quad (4.6)$$

This answer is, of course, very sensitive to the values of V_2 and M^*. A decrease in M^* from $1.81M$ to M reduces T_c by about a factor of four.

For nuclear matter, the repulsive $^1S_0$ interaction prevents $l=0$ pairing for identical particles. These instead pair in the $l=2$ state, where

$$V_2=-(k_F|K_2|k_F)=0.66 \text{ Mev } (10^{-13} \text{ cm})^3. \quad (4.7)$$

The level density for identical particles in the singlet state is as given in Eq. (4.2) with for the nuclear case

$$M^*=0.73M. \quad (4.8)$$

This gives

$$V_2N(0)=0.079, \quad (4.9)$$

which for any m value gives an energy shift of the order of $10^{-6}\Delta E$ which is negligible.

For the $^3S_1$ state the interaction is attractive, giving

$$V_0=-(k_F|K_0|k_F)=65 \text{ Mev } (10^{-13} \text{ cm})^3. \quad (4.10)$$

Taking the level density from Eq. (4.2) and inserting a factor of three for the higher density of triplet states, and the effective mass from Eq. (4.8), we obtain the result

$$[N(0)V_0]_{\text{triplet}}=0.155. \quad (4.11)$$

The resulting energy gap is

$$\epsilon_0=0.00325\Delta E. \quad (4.12)$$

Taking as a rough estimate for ΔE the kinetic energy at the Fermi surface, which is 57.6 Mev for an effective mass of $0.73M$, we find

$$\epsilon_0=0.19 \text{ Mev}. \quad (4.13)$$

This result agrees roughly with the energy gap which has been suggested from the analysis of nuclear level structure.

V. DISCUSSION

We have shown that in the presence of an effective interaction which is repulsive in the $l=0$ state, attraction in a higher state of angular momentum will not lead to an energy gap but instead only to a changed level density. Consequently, the phenomena of superfluidity (or superconductivity) are expected to be considerably modified if present in such fermion liquids. A rather interesting type of correlation does occur, however, since the increase in binding energy of the system is possible only through the introduction of a directional correlation in the liquid as in Eq. (3.9). The Fermi surface is no longer spherical but distorted with the perturbation depending on the first angular momentum state in which attraction occurs. Although we have not attempted to analyze this phenomenon in detail, it is likely that this correlation will persist over a range given by a formula similar to that derived by BCS, i.e.,

$$R\sim\hbar v_F/\epsilon_0,$$

which gives for He^3 a correlation range of roughly 400 A. Consequently the liquid will break up into small cells of roughly this dimension in which a correlation axis will exist. These effects should be apparent in the specific heat and other properties at temperatures close to T_c.

In nuclear matter we have seen that the superconducting type of pair correlation is likely to exist for identical nucleons only in the $l=2$ state, but that neutron-proton pairs probably show weak correlation in the $l=0$ state. These conclusions may not be relevant to the case of finite nuclei since the change from nuclear matter in level structure and surface density will alter the effective interaction. The conclusion from nuclear matter that the $^3S_1$ pairing is more pronounced than the $^1S_0$ pairing is, however, probably also valid in finite nuclei as long as the neutron and proton orbits are not too different; although for such nuclei the cooperative effects of superconducting type may not exist.

VI. ACKNOWLEDGMENT

We are indebted to Dr. J. L. Gammel for supplying numerical values of the K-matrix elements.