Angular Momentum Expansions in Relativistic Field Theory

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As a step toward more general applications of angular momentum expansions in quantum field theory the expansion of the scattering matrix is examined in the case of scattering of spin 0 by spin $\frac{1}{2}$ particles. The matrix is represented in terms of its eigenvalues and eigenvectors, the latter being eigenstates of total angular momentum. Using eigenstates of helicity to simplify the discussion, the eigenvectors may sometimes be obtained from the conservation laws alone. The eigenvalues are computed only to second order in a Yukawa interaction, but the results are more useful than the usual second-order matrix elements. Since angular momentum expansions lead effectively to solutions of operator equations, the expressions derived facilitate the relativistic application of Heitler's unitary approximation (with an exact solution of the equation relating the transition operator T and the Hermitian reaction operator K) or the determinantal method of Schwinger and Baker.

1. INTRODUCTION

SINCE the general utility of angular momentum expansions in scattering problems is so well recognized, it is surprising that such expansions have not been exploited very systematically in the problems of relativistic field theory. The purpose of this paper is to show that it is convenient and practical to calculate relativistic scattering matrices in such a way that the contribution of each angular momentum state is explicitly exhibited. Attention is restricted to two-particle processes of the type $a+b\rightarrow c+d$. The aim is to present the two-particle submatrix of a scattering operator M in the spectral form

$$\langle\beta|M|\alpha\rangle=\sum_\lambda \langle\beta|\lambda\rangle M_\lambda\langle\lambda|\alpha\rangle, \tag{1.1}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are two-particle, plane wave states and λ represents a set of eigenvalues of some complete set of commuting observables including the total angular momentum. The term scattering operator refers to any operator M having the same eigenvectors as the collision operator S. Some examples are the transition operator $T=S-1$, the Hermitian reaction operator K defined by $S=(1-\frac{1}{2}iK)(1+\frac{1}{2}iK)^{-1}$, and the Hermitian phase-shift operator η defined by $S=\exp(2i\eta)$. In seeking this representation (1.1) we have in mind the obvious possibility of replacing infinite-dimensional matrix equations by numerical equations. For instance, the solution to the Heitler equation[1]

$$\langle\beta|T|\alpha\rangle+(i/2)\int d\gamma\,\langle\beta|K|\gamma\rangle\langle\gamma|T|\alpha\rangle=-i\langle\beta|K|\alpha\rangle \tag{1.2}$$

is simply

$$\langle\beta|T|\alpha\rangle=\sum_\lambda \langle\beta|\lambda\rangle T_\lambda\langle\lambda|\alpha\rangle,$$

where T_λ is given by

$$T_\lambda+(i/2)K_\lambda T_\lambda=-iK_\lambda,$$

because of the orthogonality of the states $|\lambda\rangle$.

In two-particle problems in which conserved quantum numbers specify completely the initial and final states the exact eigenvectors may be determined easily. When there is no such complete specification by conserved quantities the eigenvectors depend on dynamical details. Still, one may isolate the angular dependence of matrix elements in orthogonal functions and reduce infinite-dimensional matrix equations to finite-dimensional ones.

Since the dynamical effects are treated only in second-order perturbation theory in this paper, the calculation is to be regarded as principally for orientation. Even so, the results can be used to account for what would be regarded as higher order effects in S-matrix perturbation theory. With the ability to solve operator equations, several different approximation schemes may be based on the coupling constant expansion of operators or other quantities related to S. For instance, Hermitian perturbation approximations to K or K^{-1} yield unitary approximations to S via the solution of (1.2). Previously this method of Heitler has been applied in field theory only with approximate solutions of (1.2) or in isolated cases in which the solution of (1.2) was obvious without the spectral forms. The determinantal method,[2] which is presumably a great improvement on S-matrix perturbation theory, also requires the approximate computation of eigenvalues of K.

In view of particular problems to be discussed in another paper the K matrix for the scattering of a spin zero Boson and a spin one-half Fermion is treated, assuming invariance under T, C, and P separately. The eigenvalues are calculated from Yukawa interactions, with all choices of relative parities and masses of the initial and final particles. The methods used work as well in other scattering problems and are also applicable in fourth-order calculations.

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[1] B. A. Lippmann and J. Schwinger, Phys. Rev. 79, 469 (1950).

[2] M. Baker, Ann. Phys. 4, 271 (1958); J. J. Giambiagi and T. W. B. Kibble, Ann. Phys. 7, 39 (1959); The application of determinantal methods in meson-nucleon scattering was advocated by J. Schwinger in two lectures at Harvard University, February, 1955 (unpublished).

Our procedure is to expand the initial and final plane wave states in angular momentum states before carrying out the integrations of the perturbation method. Instead, one could decompose the known perturbation formulas for the plane wave representation of scattering matrices into separate angular momentum contributions. But this method is not much easier than the one used here, and it is somewhat less explicit and informative. Moreover, in higher orders in the coupling constant the plane wave matrices cannot always be given in closed form. In such cases our method should provide a shorter route to the desired result.

It should be pointed out that higher order terms of scattering matrix eigenvalues may be of considerable interest. The determinantal method[2] and possibly also the expansion of the inverse of K [3] show some promise of yielding sufficiently rapid convergence (for observed strengths of coupling) to allow some crude quantitative estimates of the predictions of meson theory. These techniques, which are systematic and renormalizable, might provide a useful complement to the studies of analytic properties of scattering matrices which have recently been in the center of attention. This is particularly so since one would still like to test the old ideas (e.g., the Yukawa interaction), and this does not seem possible without improvements in approximation methods.

2. SPECTRAL FORM TO SECOND ORDER

The initial state will be $|i\rangle=|qp\sigma\alpha\rangle$, where $q=(q_0,\mathbf{q})$ is the four momentum of a spin zero Boson of mass μ and $p=(p_0,\mathbf{p})$ the four momentum of a spin one-half Fermion of mass m. q will also be used for the magnitude of the three momentum in the center-of-mass system when no confusion can arise. The helicity, σ, is the eigenvalue of $\mathbf{s}\cdot\mathbf{p}/|\mathbf{p}|=\mathbf{s}\cdot\hat{p}$, where $\mathbf{s}$ is the Fermion spin operator, and α is a channel index containing whatever further information is necessary to specify the state. The final state $|f\rangle=|q'p'\sigma'\alpha'\rangle$ will be similar but with masses μ' and m'.

The second-order K matrix is proportional to the second-order S matrix.[4] Since $K_{(2)}=iS_{(2)}$, we have $\langle f|K_{(2)}|i\rangle=i(\eta^C C+\eta^D D)$ where η^C and η^D are numerical factors arising from the charge space and C and D are the "crossed" and "direct" terms which can be written in terms of the interaction picture operators and states as follows[5]:

$$\begin{aligned}iC&=gg'\left\langle f\left|\int d^4(x'x)[\phi^{(-)}(x)\bar{\psi}^{(-)}(x')\right.\right.\\&\qquad\left.\left.\times\Gamma' G_{(0)}(x'-x)\Gamma\psi^{(+)}(x)\phi^{(+)}(x')]\right|i\right\rangle,\\iD&=gg'\left\langle f\left|\int d^4(x'x)[\phi^{(-)}(x')\bar{\psi}^{(-)}(x')\right.\right.\\&\qquad\left.\left.\times\Gamma' G_{(0)}(x'-x)\Gamma\psi^{(+)}(x)\phi^{(+)}(x)]\right|i\right\rangle.\end{aligned}\tag{2.1}$$

Here $\hbar=c=1$ and g and g' are unrationalized coupling constants.

$$G_{(0)}(x'-x)=(2\pi)^{-4}\int\frac{\gamma\cdot k+m_i}{k^2-m_i^2+i\epsilon}e^{-ik\cdot(x'-x)}d^4k,\tag{2.2}$$

where m_i is the mass of the Fermion in the intermediate state and the metric is $g^{00}=-g^{kk}=1$. All possibilities for the vertex matrices Γ and Γ' will be considered.

The term iC, which contains all angular momentum states, will be treated first. Later we can write down easily the simpler term iD containing only $j=\frac{1}{2}$. Using the adiabatic decoupling hypothesis and the commutation relations in the usual way iC becomes

$$\begin{aligned}iC=gg'(2\pi)^{-6}W^{-1}\int d^4(x'x)\, e^{i(q'x+p'x')}u_{\sigma'}(p')\\ \times\Gamma' G_{(0)}\Gamma u_\sigma(p)e^{-i(qx'+px)},\end{aligned}\tag{2.3}$$

where

$$W^{-1}=(m'm/4q_0'p_0'q_0p_0)^{\frac{1}{2}}.$$

After inserting (2.2) we do not follow the usual procedure of reversing completely the order of space and momentum integrations. Instead of eliminating the wave functions immediately we first expand them in angular momentum states. To deal with angular momentum one must work in three dimensions, so the time integrations are carried out first, and then the k_0 integration. This replaces $G_{(0)}(x'-x)$ by the time independent Green function

$$(2\pi)^{-2}\delta(\epsilon_f-\epsilon_i)\times\int\frac{\gamma_0(p_0-q_0')-\boldsymbol{\gamma}\cdot\mathbf{k}+m_i}{(p_0-q_0')^2-k^2-m_i^2+i\epsilon}e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}d^3k,\tag{2.4}$$

where ϵ_f and ϵ_i are the final and initial energies of the system. The following notation will be used for the three-dimensional Green functions:

$$\begin{aligned}\mathcal{G}_{(0)}{}^3(x;\,\mathcal{E},\mu)&=\int\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathcal{E}^2-k^2-\mu^2+i\epsilon}d^3k,\\G_{(0)}{}^3(x;\,\mathcal{E},m)&=(\gamma_0\mathcal{E}+i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\mathcal{G}_{(0)}{}^3(x;\,\mathcal{E},m).\end{aligned}\tag{2.5}$$

By contour integration one proves that

$$\mathcal{G}_{(0)}{}^3(x;\,\mathcal{E},m)=-2\pi^2x^{-1}\exp(i\mathcal{K}-\epsilon/2\mathcal{K})x,\tag{2.6}$$

[3] R. L. Warnock, doctoral thesis, Harvard, 1959 (unpublished). The author has learned that the expansion of K^{-1}, suggested to him by Dr. Harold Weitzner, has been considered in more detail by Kenneth Wilson. The hope for the K^{-1} and determinantal methods is based on encouraging experience with the static model.

[4] J. Pirenne, Phys. Rev. 86, 395 (1952).

[5] Operator and state normalizations and notation not otherwise explained will be the same as in S. Schweber, H. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), Vol. I.

where $\mathcal{K}=(\mathcal{E}^2-m^2)^{\frac{1}{2}}$. When $\mathcal{E}^2<m^2$, $\mathcal{K}$ is to be interpreted as $i|\mathcal{K}|$ and we have a propagator which decreases rapidly with the separation of the spatial points, corresponding to the limited propagation of a virtual particle with energy less than the rest mass of the corresponding real particle. Differentiating,

$$G_{(0)}{}^3(x;\,\mathcal{E},m)=-2\pi^2[\gamma_0\mathcal{E}-i\boldsymbol{\gamma}\cdot\hat{x}(x^{-1}-i\mathcal{K})+m]x^{-1}\\ \times\exp[(i\mathcal{K}-\epsilon/2\mathcal{K})x]. \quad (2.7)$$

In (2.4) we have $\mathcal{E}=p_0-q_0'$ where $(p_0-q_0')^2<m_i^2$ for the cases of interest. $\mathcal{K}$ is imaginary, so all remaining integrals will be well defined if we set ϵ equal to zero at this stage. In fourth-order scattering the case of real $\mathcal{K}$ occurs.

Since $G_{(0)}{}^3$ has a simple dependence on $\mathbf{x}'-\mathbf{x}$ a change of variables is called for. The transformation $\mathbf{x}'-\mathbf{x}=\mathbf{r}$, $\mathbf{x}'=\mathbf{s}$ gives the exponents in (2.3) a convenient form,

$$-i(\mathbf{q}'\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}')+i(\mathbf{q}\cdot\mathbf{x}'+\mathbf{p}\cdot\mathbf{x})\\ =-i(\mathbf{q}'+\mathbf{p}'-\mathbf{q}-\mathbf{p})\cdot\mathbf{s}+i(\mathbf{q}'-\mathbf{p})\cdot\mathbf{r}.$$

Now the $\mathbf{s}$ integration produces the delta function of total three momentum. Transforming to the zero momentum frame and substituting (2.7) we have

$$iC=-gg'\delta^4(\Delta P)[2(2\pi)^3W]^{-1}\\ \times\int d^3r\,\bar{u}_{\sigma'}(p_0',-\mathbf{q}')e^{i\mathbf{q}'\cdot\mathbf{r}}\\ \times\Gamma'[\gamma_0(p_0-q_0')-i\boldsymbol{\gamma}\cdot\hat{r}(r^{-1}+\rho^{-1})+m_i]\\ \times\Gamma u_\sigma(p_0,-\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}r^{-1}e^{-r/\rho}, \quad (2.8)$$

where $\rho^{-1}=-i\mathcal{K}=[m_i^2-(p_0-q_0')^2]^{\frac{1}{2}}$. After expansion of the initial and final wave functions in eigenstates of j the integration of $\mathbf{r}$ over solid angle is trivial. This is true in spite of the angle dependent term $\boldsymbol{\gamma}\cdot\hat{r}$, as we shall see.

To discuss the expansion of the wave functions the following representation of the Dirac matrices is particularly convenient:

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix};\quad \gamma^k=\begin{pmatrix}0&\sigma_k\\-\sigma_k&0\end{pmatrix};\\ \gamma_5=\begin{pmatrix}0&-i\\-i&0\end{pmatrix}. \quad (2.9)$$

The sign of the mass term in the Dirac equation

$$[-i\gamma^\mu(\partial/\partial x_\mu)+m]\psi(x^\mu)=0 \quad (2.10)$$

is appropriate for the propagator (2.2). A maximal set of linearly independent plane wave solutions of (2.10) is

$$w_\mu(p_0,\mathbf{p})e^{-ipx}=\left(\frac{p_0+m}{2m}\right)^{\frac{1}{2}}\\ \times\begin{bmatrix}\xi_{\frac{1}{2}}{}^\mu\\ [\boldsymbol{\sigma}\cdot\mathbf{p}/(p_0+m)]\xi_{\frac{1}{2}}{}^\mu\end{bmatrix}e^{-ipx}, \quad (2.11)$$

where $\mu=\pm\frac{1}{2}$, $p_0=\pm(p^2+m^2)^{\frac{1}{2}}$ and the $\xi_{\frac{1}{2}}{}^\mu$ are the eigenvectors of σ_3. Since the normalization is given by $\bar{w}w=p_0/|p_0|$, the expectation value of the Fermion particle density operator with respect to one-Fermion states is always $(2\pi)^{-3}$. The functions (2.11) are eigenfunctions of spin only if $\mathbf{p}$ is in the z direction. μ is the helicity if $\mathbf{p}$ is in the positive z direction. Since helicity is invariant under rotations, one may obtain an helicity state with arbitrary momentum direction by rotation of a z direction state. Looking first at the Pauli spinors we notice that an eigenvector of the Pauli helicity operator $\frac{1}{2}\boldsymbol{\sigma}\cdot\hat{p}$ with eigenvalue σ is

$$\textstyle\sum_\mu D_{\mu\sigma}{}^{\frac{1}{2}}(-\phi_p,-\theta_p,0)\xi_{\frac{1}{2}}{}^\mu,$$

where θ_p and ϕ_p are the polar and azimuthal angles of $\mathbf{p}$ and the rotation group representation $D_{m'm}{}^j(\alpha,\beta,\gamma)$ is as defined by Edmonds.[6] Now it is easy to see that the Dirac helicity states are as follows:

$$u_\sigma(p_0,\mathbf{p})=\textstyle\sum_\mu D_{\mu\sigma}{}^{\frac{1}{2}}(-\phi_p,-\theta_p,0)w_\mu(p_0,\mathbf{p}). \quad (2.12)$$

The angular dependence of the right-hand side is not entirely in $D_{\mu\sigma}{}^{\frac{1}{2}}$, since the spin operator is not the generator of rotations of Dirac states [i.e., $w_\mu(p_0,\mathbf{p})$ contains orbital angular momentum]. However, the angular dependence may be transferred entirely to $D_{\mu\sigma}{}^{\frac{1}{2}}$ through the following identity:

$$\textstyle\sum_\mu D_{\mu\sigma}{}^{\frac{1}{2}}\boldsymbol{\sigma}\cdot\hat{p}\xi_{\frac{1}{2}}{}^\mu=(-1)^{\sigma-\frac{1}{2}}\sum_\mu D_{\mu\sigma}{}^{\frac{1}{2}}\xi_{\frac{1}{2}}{}^\mu. \quad (2.13)$$

This is easy to prove using the explicit functional form of the group representation. The particular spinor we need is

$$u_\sigma(p_0,-\mathbf{q})=i\left(\frac{p_0+m}{2m}\right)^{\frac{1}{2}}\sum_\mu D_{\mu-\sigma}{}^{\frac{1}{2}}(-\phi_q,-\theta_q,0)\\ \times\begin{bmatrix}\xi_{\frac{1}{2}}{}^\mu\\ [(-1)^{\sigma-\frac{1}{2}}q/(p_0+m)]\xi_{\frac{1}{2}}{}^\mu\end{bmatrix}, \quad (2.14)$$

where we have used (2.13) and the following relation between the D's corresponding to directions $-\mathbf{q}$ and $\mathbf{q}$:

$$D_{\mu\sigma}{}^{\frac{1}{2}}(-\phi-\pi,\theta-\pi,0)=iD_{\mu-\sigma}{}^{\frac{1}{2}}(-\phi,-\theta,0). \quad (2.15)$$

The expansion of $u_\sigma(p_0,-\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}$ in angular momentum states, which is needed in (2.8), may be obtained directly from (2.14) and the expansion of $e^{i\mathbf{q}\cdot\mathbf{r}}$ in spherical harmonies. Alternatively, we may begin with the z direction state

$$u_\sigma(p_0,0,0,-q)e^{iqz}, \quad (2.16)$$

expand it in j states, and then rotate. Following the latter procedure we define two-component, orthogonal functions by

$$\chi_\kappa{}^\sigma(r)=\textstyle\sum_\mu\langle l(\kappa),\tfrac{1}{2},\sigma-\mu,\mu|l(\kappa),\tfrac{1}{2},|\kappa|-\tfrac{1}{2},\sigma\rangle\\ \times Y_{l(\kappa)}{}^{\sigma-\mu}(r)\xi_{\frac{1}{2}}{}^\mu, \quad (2.17)$$

[6] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

where

$$\kappa=\pm1,\pm2,\cdots;\quad \begin{array}{ll} l(\kappa)=\kappa, & \kappa>0 \\ l(\kappa)=|\kappa|-1, & \kappa<0. \end{array}$$

These would be nonrelativistic eigenstates of j, with $j=|\kappa|-\frac{1}{2}$, $j_z=\sigma$. The Clebsch-Gordan coefficients in (2.17) will be as defined by Condon and Shortley. The following properties of the functions (2.17) are to be noted[7]:

$$\boldsymbol{\kappa}\chi_\kappa{}^\sigma=-(\boldsymbol{\sigma}\cdot\mathbf{L}+1)\chi_\kappa{}^\sigma=\kappa\chi_\kappa{}^\sigma, \tag{2.18a}$$

$$\boldsymbol{\sigma}\cdot\hat{r}\chi_\kappa{}^\sigma(r)=-\chi_{-\kappa}{}^\sigma(r). \tag{2.18b}$$

The plane wave (2.16) is expanded in the eigenfunctions u_κ:

$$u_\sigma(p_0,0,0,-q)e^{iqz}=\sum_\kappa u_\kappa =\sum_\kappa\begin{bmatrix}\eta_1 j_{l(\kappa)}(qr)\chi_\kappa{}^{-\sigma}(r)\\ \eta_2 j_{l(-\kappa)}(qr)\chi_{-\kappa}{}^{-\sigma}(r)\end{bmatrix}. \tag{2.19}$$

u_κ is an eigenfunction of the two operators

$$\mathbf{j}^2=(\mathbf{L}+\tfrac{1}{2}\boldsymbol{\sigma})^2,\quad \boldsymbol{\kappa}=-\gamma_0(1+\boldsymbol{\sigma}\cdot\mathbf{L}),$$

but not an eigenfunction of the helicity operator (κ and σ are complementary observables). After expanding the exponential in spherical harmonics one can use the orthogonality of the $\chi_\kappa{}^\sigma$ and the values of the Clebsch-Gordan coefficients appearing in (2.17) to obtain the coefficients

$$\left.\begin{array}{c}\eta_1\\ \eta_2\end{array}\right\}=i\left(\frac{2\pi|\kappa|}{m}\right)^{\frac{1}{2}} s_{-\kappa}{}^{\sigma-\frac{1}{2}}\left\{\begin{array}{l} i^{l(\kappa)}(p_0+m)^{\frac{1}{2}}\\ i^{l(-\kappa)}(p_0-m)^{\frac{1}{2}},\end{array}\right. \tag{2.20}$$

where s_x is the sign of x. Then we have, finally, for the rotated states,

$$u_\sigma(p_0,-\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}} =i\sum_{\kappa,\mu}\left(\frac{2\pi|\kappa|}{m}\right)^{\frac{1}{2}} s_{-\kappa}{}^{\sigma-\frac{1}{2}}D_{\mu-\sigma}{}^{|\kappa|-\frac{1}{2}}(-\phi_q,-\theta_q,0) \times\begin{bmatrix}(p_0+m)^{\frac{1}{2}}i^{l(\kappa)}j_{l(\kappa)}(qr)\chi_\kappa{}^\mu(r)\\ (p_0-m)^{\frac{1}{2}}i^{l(-\kappa)}j_{l(-\kappa)}(qr)\chi_{-\kappa}{}^\mu(r)\end{bmatrix}, \tag{2.21a}$$

$$\bar{u}_\sigma(p_0,-\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}} =-i\sum_{\kappa,\mu}\left(\frac{2\pi|\kappa|}{m}\right)^{\frac{1}{2}} s_{-\kappa}{}^{\sigma-\frac{1}{2}}D_{\mu-\sigma}{}^{|\kappa|-\frac{1}{2}*}(-\phi_q,-\theta_q,0) \times\begin{bmatrix}(p_0+m)^{\frac{1}{2}}i^{l(\kappa)}j_{l(\kappa)}(qr)\chi_\kappa{}^{\mu*}(r)\\ -(p_0-m)^{\frac{1}{2}}i^{l(-\kappa)}j_{l(-\kappa)}(qr)\chi_{-\kappa}{}^{\mu*}(r)\end{bmatrix}. \tag{2.21b}$$

(2.21b) was obtained from (2.21a) by complex conjugation, multiplication by γ_0, and the transformation $\mathbf{r}\to-\mathbf{r}$.

[7] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

After substituting (2.21) in (2.8) and integrating over solid angle one may read off the various terms in the matrix element, making use of (2.18b) and the orthogonality of the $\chi_\kappa{}^\mu$. There will be "transitions" from a term in the initial state involving $\chi_\kappa{}^\mu$ to terms with both $\chi_\kappa{}^\mu$ and $\chi_{-\kappa}{}^\mu$ in the final state, because of (2.18b). Of course, this just corresponds to the fact that l is not conserved relativistically.

Before stating the results of the integration we introduce some notation. Defining $\mathbf{K}$ by $\langle f|K|i\rangle =2\pi\delta^4(\Delta P)\langle f|\mathbf{K}|i\rangle$ the K matrix will be given in the form,

$$\langle f|\mathbf{K}|i\rangle=\sum_{j\mu s_\kappa s_\kappa'}(\mathbf{q}'\sigma'|j\mu s_\kappa')\langle js_\kappa'\alpha'|\mathbf{K}|js_\kappa\alpha\rangle \times(j\mu s_\kappa|\mathbf{q}\sigma). \tag{2.22}$$

The spin-angle functions[8]

$$\begin{aligned}(\mathbf{q}\sigma|j\mu-)&=c_jD_{-\sigma\mu}{}^j(0,\theta_q,\phi_q),\\ (\mathbf{q}\sigma|j\mu+)&=(-1)^{\sigma-\frac{1}{2}}c_jD_{-\sigma\mu}{}^j(0,\theta_q,\phi_q),\end{aligned} \tag{2.23}$$

satisfy the orthonormality condition

$$\sum_{\sigma=-\frac{1}{2}}^{\frac{1}{2}}\int d^3q\,\delta(\epsilon(q)-E)(j'\mu's_\kappa'|\mathbf{q}\sigma)(\mathbf{q}\sigma|j\mu s_\kappa) =\delta^{jj',\mu\mu',s_\kappa s_\kappa'}, \tag{2.24}$$

where $\epsilon(q)=(m^2+q^2)^{\frac{1}{2}}+(\mu^2+q^2)^{\frac{1}{2}}$ and E is any energy accessible to the system. This normalization, which implies the factor

$$c_j=\left(\frac{2j+1}{8\pi q}\cdot\frac{\epsilon}{p_0q_0}\right)^{\frac{1}{2}}, \tag{2.25}$$

is motivated in the following section. The rounded bracket notation has been used for the functions (2.23) since they differ (trivially) from the representatives of the initial and final states (see next section). The different choices of the vertex matrices will be lettered as follows:

$$\begin{array}{ll}\text{(a) }\Gamma=\Gamma'=1; & \text{(b) }\Gamma=\Gamma'=\gamma_5;\\ \text{(c) }\Gamma=\gamma_5,\quad\Gamma'=1; & \text{(d) }\Gamma=1,\quad\Gamma'=\gamma_5.\end{array} \tag{2.26}$$

In cases (a) and (b) κ is conserved, while in (c) and (d) the sign of κ changes. The matrix elements corresponding to these two situations will be written

$$\begin{aligned}\langle j-\alpha'|\mathbf{K}|j-\alpha\rangle&=K_{j\alpha'\alpha}{}^{(-)}=\eta^CC_j{}^{(-)}+\eta^DD_j{}^{(-)},\\ \langle j+\alpha'|\mathbf{K}|j-\alpha\rangle&=K_{j\alpha'\alpha}{}^{(+-)}=\theta^CC_j{}^{(+-)}+\theta^DD_j{}^{(+-)},\end{aligned} \tag{2.27}$$

where η's and θ's are charge space factors, and similar expressions hold when $(-)$ and $(+)$ are interchanged. For the crossed terms we have

[8] In obtaining (2.23) from the result of the integration we have used the identity

$$D_{m'm}{}^j(-\phi,-\theta,0)=D_{mm'}{}^{j*}(0,\theta,\phi).$$

$$C_j^{(-)}=AB_j^{(-)}$$
$$=A(-1)^{j-\frac{1}{2}}\left(\frac{q'}{p_0'+m'}J_{+-}+\frac{q}{p_0+m}J_{-+}-(p_0-q_0'\pm m_i)J_{--}+\frac{(p_0-q_0'\mp m_i)q'q}{(p_0'+m')(p_0+m)}J_{++}\right),\quad(2.28)$$
$$C_j^{(+-)}=\mp iAB_j^{(+-)}$$
$$=\mp iA(-1)^{j-\frac{1}{2}}\left(J_{+-}+\frac{q'q}{(p_0'+m')(p_0+m)}J_{-+}-q'\frac{(p_0-q_0'\mp m_i)}{p_0'+m'}J_{--}+\frac{q(p_0-q_0'\pm m_i)}{p_0+m}J_{++}\right),$$

where

$$A=\frac{gg'}{4\pi}\frac{[q'q(p_0'+m')(p_0+m)]^{\frac{1}{2}}}{2\pi\epsilon},$$

$$J_{\pm\mp}=\int_0^\infty e^{-x/\rho}(x\rho^{-1}+1)j_{j\pm\frac{1}{2}}(q'x)j_{j\mp\frac{1}{2}}(qx)dx=J_{j\pm\frac{1}{2},j\mp\frac{1}{2}},\quad(2.29)$$
$$J_{\pm\pm}=\int_0^\infty e^{-x/\rho}xj_{j\pm\frac{1}{2}}(q'x)j_{j\pm\frac{1}{2}}(qx)dx=J_{j\pm\frac{1}{2},j\pm\frac{1}{2}}.$$

Where there is a choice of sign in (2.28) the upper sign corresponds to case (a) or case (c) of (2.26). After the expressions (2.28) are computed it is easy to find the quantities

$$C_j^{(+)}=AB_j^{(+)},\quad C_j^{(-+)}=AB_j^{(-+)},$$

since $B_j^{(+)}$ and $B_j^{(-+)}$ are obtained from the corresponding quantities $B_j^{(-)}$ and $B_j^{(+-)}$ by the interchange

$$J_{++}\leftrightarrow -J_{--},\quad J_{+-}\leftrightarrow J_{-+}.\quad(2.30)$$

The radial integrals (2.29) are evaluated in the Appendix.

The direct terms are

$$D_{\frac{1}{2}}^{(-)}=A(\epsilon\mp m_i)^{-1},$$
$$D_{\frac{1}{2}}^{(+)}=Aqq'[(p_0'\pm m')(p_0+m)(\epsilon\pm m_i)]^{-1},$$
$$D_{\frac{1}{2}}^{(+-)}=\mp iAq'[(p_0'+m')(\epsilon\pm m_i)]^{-1},\quad(2.31)$$
$$D_{\frac{1}{2}}^{(-+)}=\mp iAq[(p_0+m)(\epsilon\mp m_i)]^{-1}.$$

The terms (2.31) were obtained simply by substituting the states of definite helicity (2.14) in the well-known formula

$$iD=\frac{gg'\delta^4(\Delta P)}{(2\pi)^2W}\bar{u}_{\sigma'}(p_0',-\mathbf{q}')\times\Gamma'\frac{\gamma_0\epsilon+m_i}{\epsilon^2-m_i^2}\Gamma u_\sigma(p_0,-\mathbf{q}).\quad(2.32)$$

The method used for the crossed terms is unnecessarily complicated in this case, since there is no term dependent on angles in the denominator of (2.32).

In (2.28) and (2.31) the superscripts $(-)$ and $(+)$ correspond to nonrelativistic states with $l=j-\frac{1}{2}$ and $l=j+\frac{1}{2}$, respectively.

3. GENERAL COMMENTS ON SCATTERING MATRICES

Suppose first that charge space quantum numbers (together with masses, momenta, and helicities) specify the initial and final states completely. Suppose further that these quantum numbers are conserved and that the theory allows no changes in intrinsic parity[9] [i.e., κ is conserved, as in cases (a) and (b)]. When these conditions are met, we designate the situation as case (i). Taking eigenstates of charge space quantum numbers as initial and final states we may choose $\alpha=\alpha'$. The eigenvectors $|M'\rangle$ of the scattering operator M satisfy the equation

$$\sum_\sigma\int d^3p\,d^3q\,\langle p'q'\sigma'\alpha|M|pq\sigma\alpha\rangle\langle pq\sigma\alpha|M'\rangle$$
$$=\sum_\sigma\int d^3q\,\delta[\epsilon_f-\epsilon(q)]\langle p'q'\sigma'\alpha|\mathbf{M}|[p_0(q),-\mathbf{q}]q\sigma\alpha\rangle\times\langle[p_0(q),-\mathbf{q}]q\sigma\alpha|M'\rangle$$
$$=M'\langle p'q'\sigma'\alpha|M'\rangle,\quad(3.1)$$

where $\langle\beta|M|\alpha\rangle=\delta^4(\Delta P)\langle\beta|\mathbf{M}|\alpha\rangle$. The abbreviated notation

$$|\mathbf{q}\sigma\alpha\rangle=|[p_0(q),-\mathbf{q}]q\sigma\alpha\rangle$$

will be used from now on. The representatives of the $|\mathbf{q}\sigma\alpha\rangle$ with which we deal are[10]

$$\langle j\mu s_\kappa\bar{\epsilon}\bar{\alpha}|\mathbf{q}\sigma\alpha\rangle.\quad(3.2)$$

$\bar{\epsilon}$ is an energy eigenvalue, and $\bar{\alpha}$ a charge space index of the same variety as α. (3.2) is proportional to $\delta(\bar{\alpha},\alpha)\delta(\bar{\epsilon}-\epsilon(q))$, at least with appropriate definition of the α and $\bar{\alpha}$ variables. After factoring the δ's out of (3.2) it is difficult to maintain the usual bra-ket notation without special interpretations. Therefore, we define a rounded bracket symbol

$$\langle j\mu s_\kappa\bar{\epsilon}\bar{\alpha}|\mathbf{q}\sigma\alpha\rangle=\delta(\bar{\alpha},\alpha)\delta(\bar{\epsilon}-\epsilon(q))(j\mu s_\kappa|\mathbf{q}\sigma).\quad(3.3)$$

(2.22) is obtained by expressing $\langle f|\mathbf{K}|i\rangle$ in terms of

[9] Ordinarily, if charge space quantum numbers specify the states κ will be conserved automatically.

[10] Both bra and ket depend on three variables with infinite domains and two with finite domains.

(3.3). We write $\langle js_\kappa'\alpha'|\mathbf{K}|js_\kappa\alpha\rangle$ for

$$\langle j\mu s_\kappa'\epsilon\alpha'|\mathbf{K}|j\mu s_\kappa\epsilon\alpha\rangle$$

since the quantity is independent of μ (due to rotational invariance) and its dependence on energy is obvious.

Still considering the simplest case, covered by (3.1), it is clear that the normalization (2.24) is the natural one. That is,

$$\sum_\sigma \int d^3q\,\langle \mathbf{q}'\sigma'\alpha|\mathbf{M}|\mathbf{q}\sigma\alpha\rangle(\mathbf{q}\sigma|j\mu s_\kappa)$$
$$=\langle js_\kappa\alpha|\mathbf{M}|js_\kappa\alpha\rangle(\mathbf{q}'\sigma'|j\mu s_\kappa)=M'(\mathbf{q}'\sigma'|j\mu s_\kappa). \quad (3.4)$$

A more complicated situation is that in which κ is conserved but the other conserved quantum numbers do not specify the states completely [case (ii)]. An example of this kind may be present in the case of the interaction of negative K mesons and nucleons, provided that the parities of the K meson with respect to the relevant baryons are such as to allow no changes of κ. For instance, a $T=0$ state formed from K and nucleon may go either into a similar state or into a $T=0$ state composed of π and Σ. In case (ii) the eigenvectors depend on dynamics. Introducing some nonconserved index α which will distinguish charge states, one would have to find the eigenvectors of the finite-dimensional charge space matrix $\langle js_\kappa\alpha'|\mathbf{M}|js_\kappa\alpha\rangle$. In the example just mentioned α could be an index to distinguish the states $|\bar{K}N\rangle$, $|\pi\Lambda\rangle$, $|\pi\Sigma\rangle$, as well as values of total isotopic spin and charge.

In case (iii) we have changes of κ as well as an incomplete specification of charge states by conserved quantum numbers. This would be the case in the K-nucleon reaction if scalar K-nucleon-hyperon interactions were present in the company of pseudoscalar π-baryon interactions. Of course, in a system in which κ can change there is also the possibility of it remaining constant, since elastic scattering is always possible. Considering all channels on both sides of a reaction one would have the problem of finding the eigenvectors of a matrix with dimensionality twice the number of values taken on by α (i.e., $\langle js_\kappa'\alpha'|\mathbf{M}|js_\kappa\alpha\rangle$). A matrix of the same dimensionality would result if we used eigenstates of helicity as intermediate states.

In practical applications there is usually no reason to find the spectral forms of the finite-dimensional matrices in cases (ii) and (iii). Infinite-dimensional matrix equations may be reduced to finite-dimensional ones which are solved more easily by matrix inversion than by computing spectral forms.

It should be recognized that any spin 0-spin $\frac{1}{2}$ scattering matrix $\langle f|\mathbf{M}|i\rangle$ may be written in the form (2.22) as long as the conservation laws we have assumed are present. Only the asymptotic wave functions and the conservation laws are necessary to obtain this result.[11]

[11] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959). This paper, which uses helicity states to discuss the general form of the scattering matrix, appeared after the present work was finished. Jacob and Wick do not discuss dynamics, and in our problem their point of view is slightly different in that they do not refer explicitly to κ conservation.

4. SOLUTION OF HEITLER EQUATION

In applying the preceding results to the Heitler equation we define $\langle f|\mathbf{T}|i\rangle$ by $\langle f|T|i\rangle=-2\pi i\delta^4(\Delta P)\times\langle f|\mathbf{T}|i\rangle$. Suppressing helicity and channel indices we have

$$\langle f|\mathbf{T}|i\rangle+\pi i\int d^3\bar{q}\,\langle f|\mathbf{K}|\bar{q}\rangle\langle\bar{q}|\mathbf{T}|i\rangle=\langle f|\mathbf{K}|i\rangle, \quad (4.1)$$

below the energy threshold for particle production. With substitution of (2.22) and a similar expression for $\langle f|\mathbf{T}|i\rangle$ the application of (2.24) gives

$$\langle js_\kappa'\alpha'|\mathbf{T}|js_\kappa\alpha\rangle+\pi i\sum\langle js_\kappa'\alpha'|\mathbf{K}|j\bar{s}_\kappa\bar{\alpha}\rangle$$
$$\times\langle j\bar{s}_\kappa\bar{\alpha}|\mathbf{T}|js_\kappa\alpha\rangle=\langle js_\kappa'\alpha'|\mathbf{K}|js_\kappa\alpha\rangle, \quad (4.2)$$

where the summation is over $\bar{s}_\kappa$ and $\bar{\alpha}$. In case (i) of Sec. 3 this reduces to a numerical equation. Using the notation (2.27), the real phase shifts are defined as usual by

$$K_{j\alpha\alpha}{}^{(\mp)}=-\pi^{-1}\tan\delta^{js_\kappa\alpha}. \quad (4.3a)$$

The solution to (4.2) is then

$$T_j{}^{(\mp)}=-\pi^{-1}\sin\delta^{js_\kappa\alpha}\exp(i\delta^{js_\kappa\alpha}). \quad (4.3b)$$

In case (ii) the solution is

$$T^{(\mp)}=(1+\pi iK^{(\mp)})^{-1}K^{(\mp)}, \quad (4.4)$$

where the subscript j has been dropped and matrix multiplication with respect to channel indices is understood. With case (iii) Eq. (4.2) yields

$$\begin{aligned}T^{(-)}+\pi i(K^{(-)}T^{(-)}+K^{(-+)}T^{(+-)})&=K^{(-)},\\ T^{(+-)}+\pi i(K^{(+-)}T^{(-)}+K^{(+)}T^{(+-)})&=K^{(+-)},\end{aligned} \quad (4.5)$$

and two similar equations obtained by the substitution $(+)\leftrightarrow(-)$. Solving the set of four simultaneously we have

$$T^{(-)}=[1+\pi i(K^{(-)}-\pi iK^{(-+)}(1+\pi iK^{(+)})^{-1}K^{(+-)})]^{-1}$$
$$\times[K^{(-)}-\pi iK^{(-+)}(1+\pi iK^{(+)})^{-1}K^{(+-)}],$$
$$T^{(+-)}=[1+\pi i(K^{(+)}-\pi iK^{(+-)}(1+\pi iK^{(-)})^{-1}K^{(-+)})]^{-1}$$
$$\times K^{(+-)}[1-\pi i(1+\pi iK^{(-)})^{-1}K^{(-)}], \quad (4.6)$$

and the additional equations from $(+)\leftrightarrow(-)$. The computation of inverses in (4.6) is not as laborious in practical cases as one might think, since the matrices involved contain lots of holes (i.e., either $K_{\alpha'\alpha}{}^{(\pm)}$ or $K_{\alpha'\alpha}{}^{(\pm\mp)}$ is zero for a particular choice of α,α').

5. IMPLICATIONS OF TIME REVERSAL INVARIANCE

According to (2.28) and (2.31) our K matrix has the symmetries

$$\langle js_\kappa\alpha'|\mathbf{K}|js_\kappa\alpha\rangle=\langle js_\kappa\alpha|\mathbf{K}|js_\kappa\alpha'\rangle, \quad (5.1a)$$

$$\langle js_\kappa'\alpha'|\mathbf{K}|js_\kappa\alpha\rangle=-\langle js_\kappa\alpha|\mathbf{K}|js_\kappa'\alpha'\rangle, \quad (5.1b)$$

where (5.1b) holds when $s_\kappa \neq s_\kappa'$. Of course, these equations are the result of time reversal invariance, and are not special to the second order approximation. In nonrelativistic theory the customary choice of phase factors of states leads to a real, symmetric K matrix. We have an antisymmetric term in K whenever changes of intrinsic parity can occur. Perhaps it is worthwhile to show that this is correct with our choice of states, and to indicate the trivial modification of states necessary to obtain a symmetric K.

The theory is invariant with respect to Wigner time inversion if there is an antiunitary operator $\mathbf{T}$ such that

$$\mathbf{T}H(t,\mathbf{r})\mathbf{T}^{-1}=H(-t,\mathbf{r}), \tag{5.2}$$

where $H(t,\mathbf{r})$ is the interaction Hamiltonian density in the interaction picture.[12] (5.2) implies $\mathbf{T}^{-1}S\mathbf{T}=S^\dagger$, $\mathbf{T}^{-1}K\mathbf{T}=K^\dagger=K$. From the antiunitary property $(\mathbf{T}\psi_1,\mathbf{T}\psi_2)=(\psi_1,\psi_2)^*$ it follows that

$$(\psi_\beta,K\psi_\alpha)=(\psi_\alpha',K\psi_\beta'), \tag{5.3}$$

where $\psi'=\mathbf{T}\psi$. We wish to show that (5.1) is equivalent to (5.3). T may be constructed explicitly as follows:

$$\begin{aligned}\mathbf{T}\phi(t,\mathbf{r})\mathbf{T}^{-1}&=\epsilon\phi(-t,\mathbf{r}),\\ \mathbf{T}\phi^\dagger(t,\mathbf{r})\mathbf{T}^{-1}&=\epsilon^*\phi^\dagger(-t,\mathbf{r}),\\ \mathbf{T}\psi(t,\mathbf{r})\mathbf{T}^{-1}&=\eta T\psi(-t,\mathbf{r}),\\ \mathbf{T}\psi^\dagger(t,\mathbf{r})\mathbf{T}^{-1}&=\eta^*\psi^\dagger(-t,\mathbf{r})T^\dagger.\end{aligned} \tag{5.4}$$

The field operators are in the interaction picture. ϵ, η, and the matrix T must be chosen so that (5.2) is satisfied, and so that Eqs. (5.4) do not contradict the field equations or commutation relations. The latter requirement is met by taking T proportional to $\gamma^1\gamma^3$ [with our representation (2.9)] and $|\epsilon|^2=|\eta|^2=1$.[12] The choice $T=i\gamma^1\gamma^3$ is convenient. To derive a necessary condition for (5.2) consider the scattering process $\alpha+a\rightarrow\beta+b$, where Greek letters correspond to Bosons. The interaction will contain, among other terms,

$$g\bar{\psi}_c\Gamma\psi_a\phi_\alpha+g'\bar{\psi}_b\Gamma'\psi_c\phi_\beta^\dagger. \tag{5.5}$$

Taking account of $\mathbf{T}c=c^*\mathbf{T}$, (5.2), (5.4), (2.9), and (5.5) imply

$$\epsilon_\alpha\eta_a=\epsilon_\beta\eta_b,\quad \Gamma=\Gamma', \tag{5.6a}$$

$$\epsilon_\alpha\eta_a=-\epsilon_\beta\eta_b,\quad \Gamma\neq\Gamma'. \tag{5.6b}$$

Of course, one may show that there exists a set of ϵ's and η's sufficient for (5.2) and consistent with (5.6). Now write the helicity state $|\mathbf{q},\sigma\rangle$ in terms of the field operators:

$$|\mathbf{q},\sigma\rangle=(2\pi)^{-3}\int d^3x\,\bar{\psi}(x)u_\sigma(p_0,-\mathbf{q})e^{i(-\mathbf{q}\cdot\mathbf{x}-p_0x_0)}(p_0/m)^{\frac{1}{2}}$$
$$\times\int d^3y\,\phi^\dagger(y)e^{i(\mathbf{q}\cdot\mathbf{y}-q_0y_0)}(2q_0)^{\frac{1}{2}}|0\rangle. \tag{5.7}$$

[12] We follow the point of view of G. Lüders, Ann. Phys. 2, 1 (1957).

Using $\mathbf{T}c=c^*\mathbf{T}$, $\mathbf{T}|0\rangle=|0\rangle$, (5.7), (5.4), and (2.14), we find that

$$\mathbf{T}|\mathbf{q},\sigma\rangle=\epsilon\eta(-1)^{\sigma+\frac{1}{2}}|-\mathbf{q},\sigma\rangle. \tag{5.8}$$

Combined with (5.3), (5.6), (2.22), and (2.15) this yields exactly Eqs. (5.1).

The antisymmetry (5.1b) arises from the phase relation (5.6b). This may be compensated to give a symmetric K matrix simply by multiplying certain states by i, because of the antilinear nature of $\mathbf{T}$. Construct a maximal set of states $|\mathbf{q},\sigma,A\rangle$ which are connected to each other only by transitions which involve no changes of κ. Multiply these states by i, and leave all other states $|\mathbf{q},\sigma,B\rangle$ unaltered. Then for the A states we have

$$\mathbf{T}i|\mathbf{q},\sigma,A\rangle=-\epsilon\eta(-)^{\sigma+\frac{1}{2}}i|\mathbf{q},\sigma,A\rangle, \tag{5.9}$$

and (5.1b) becomes

$$\langle js_\kappa'B|\mathbf{K}(i)|js_\kappa A\rangle=\langle js_\kappa A|(-i)\mathbf{K}|js_\kappa'B\rangle, \tag{5.10}$$

because (5.6b) is compensated by (5.9). (5.1a) is unaltered.

6. CROSS SECTIONS AND POLARIZATION

Expressions for total and differential cross sections are derived without trouble by using the properties of the group representations. For the total cross section for scattering from channel α to channel α' we find

$$\sigma_{\alpha'\alpha}=2\pi(\pi\hbar/q)^2\sum_j(2j+1)\times(|T_{j\alpha'\alpha}{}^{(-)}|^2+|T_{j\alpha'\alpha}{}^{(+)}|^2), \tag{6.1}$$

or a similar expression with $T^{(+-)}$ and $T^{(-+)}$ replacing $T^{(-)}$ and $T^{(+)}$ if s_κ changes. For the differential cross section without spin sums the result is

$$(d\sigma/d\Omega)_{\alpha'\alpha\sigma'\sigma}=(\pi\hbar/q)^2\Big|\sum_j\langle j\sigma'\alpha'|\mathbf{T}|j\sigma\alpha\rangle\times(2j+1)D_{-\sigma'-\sigma}{}^j(0\theta\phi)\Big|^2, \tag{6.2}$$

where θ and ϕ are the angles of the final Boson in the zero momentum frame with respect to the direction of the initial Boson. In terms of the s_κ representation the matrix in (6.2) is

$$\langle j\sigma'\alpha'|\mathbf{T}|j\sigma\alpha\rangle=\tfrac{1}{2}(T^{(-)}+(-1)^{\sigma'+\sigma-1}T^{(+)}+(-1)^{\sigma'-\frac{1}{2}}T^{(+-)}+(-1)^{\sigma-\frac{1}{2}}T^{(-+)})_{j\alpha'\alpha}. \tag{6.3}$$

(6.2) may be evaluated in various ways.[11] When no polarizations are observed and only $j=\frac{1}{2}$ and $j=\frac{3}{2}$ terms are included one obtains the familiar expression (for $s_\kappa=s_\kappa'$)

$$\begin{aligned}(d\sigma/d\Omega)_{\alpha'\alpha}=(\pi\hbar/q)^2\{&|T_1{}^{(-)}-T_3{}^{(+)}|^2+|T_1{}^{(+)}-T_3{}^{(-)}|^2\\ &+2\,\mathrm{Re}[T_1{}^{(-)}T_1{}^{(+)*}+2(T_1{}^{(-)}T_3{}^{(-)*}+T_1{}^{(+)}T_3{}^{(+)*})\\ &-5T_3{}^{(-)}T_3{}^{(+)*}]\cos\theta+[3(|T_3{}^{(-)}|^2+|T_3{}^{(+)}|^2)\\ &+6\,\mathrm{Re}(T_1{}^{(-)}T_3{}^{(+)*}+T_1{}^{(+)}T_3{}^{(-)*})]\cos^2\theta\\ &+18\,\mathrm{Re}(T_3{}^{(-)}T_3{}^{(+)*})\cos^3\theta\},\end{aligned} \tag{6.4}$$

where 1 and 3 indicate $\frac{1}{2}$ and $\frac{3}{2}$. The less familiar ex-

pression for $s_\kappa \neq s_\kappa'$ is obtained from (6.4) by the substitution $T^{(-)} \to T^{(+-)}$, $T^{(+)} \to T^{(-+)}$.

Since the statistical density operator for an unpolarized initial beam is $\frac{1}{2}\sum_\sigma |\mathbf{q}\sigma\rangle d^3q\langle\mathbf{q}\sigma|$, the polarization of the corresponding final state Fermions is

$$P(\theta)=\sum_{\sigma''\sigma'\sigma}\langle\mathbf{q}\sigma|\mathbf{T}^\dagger|\mathbf{q}'\sigma''\rangle \times\langle\mathbf{q}'\sigma''|\mathbf{s}\cdot(\mathbf{q}\times\mathbf{q}')/\tfrac{1}{2}|\mathbf{q}\times\mathbf{q}'||\mathbf{q}'\sigma'\rangle \times\langle\mathbf{q}'\sigma'|\mathbf{T}|\mathbf{q}\sigma\rangle\Big[\sum_{\sigma'\sigma}\langle\mathbf{q}\sigma|\mathbf{T}^\dagger|\mathbf{q}'\sigma'\rangle \times\langle\mathbf{q}'\sigma'|\rho_F|\mathbf{q}'\sigma'\rangle\langle\mathbf{q}'\sigma'|\mathbf{T}|\mathbf{q}\sigma\rangle\Big]^{-1}, \quad (6.5)$$

with channel indices suppressed.

$\mathbf{s}$ and ρ_F are the Fermion spin density and particle density operators:

$$s=\tfrac{1}{2}\psi^\dagger(x)\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}\psi(x);\quad \rho_F=\psi^\dagger(x)\psi(x).$$

Using (5.7) one finds[13]

$$\langle\mathbf{q}'\sigma''|\mathbf{s}\cdot(\mathbf{q}\times\mathbf{q}')/\tfrac{1}{2}|\mathbf{q}\times\mathbf{q}'||\mathbf{q}'\sigma'\rangle =(2\pi)^{-3}\frac{m}{p_0}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad (6.6a)$$

$$\langle\mathbf{q}'\sigma'|\rho_F|\mathbf{q}'\sigma'\rangle=(2\pi)^{-3}. \quad (6.6b)$$

In (6.6a) the first and second rows correspond to $-\sigma''$ equal to $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and similarly for columns. Including only $j=\frac{1}{2}$ and $j=\frac{3}{2}$ states the polarization for $s_\kappa=s_\kappa'$ is

$$P(\theta)=2(m/p_0)\mathfrak{D}_{\alpha'\alpha}^{-1}\sin\theta\,\mathrm{Im}[T_1^{(-)*}T_1^{(+)} +T_3^{(-)*}T_1^{(-)}+T_1^{(+)*}T_3^{(+)}+T_3^{(+)*}T_3^{(-)} +3(T_3^{(-)*}T_1^{(+)}+T_1^{(-)*}T_3^{(+)})\cos\theta +9T_3^{(-)*}T_3^{(+)}\cos^2\theta]. \quad (6.7)$$

$\mathfrak{D}$ represents the contents of the bracket in (6.4). The corresponding expression for $s_\kappa\neq s_\kappa'$ is found by the substitution $T^{(-)}\to T^{(-+)}$, $T^{(+)}\to T^{(+-)}$ in the numerator, and the different substitution $T^{(-)}\to T^{(+-)}$, $T^{(+)}\to T^{(-+)}$ in the denominator $\mathfrak{D}$.

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APPENDIX

The evaluation of the radial integrals (2.29) is reduced to the integration of rational functions by substitution of the following integral representation of the spherical Bessel function:

$$j_l(x)=x^l(2^{l+1}l!)^{-1}\int_{-1}^{1}e^{ixt}(1-t^2)^l dt.$$

After integration over x it is convenient to translate the integration variables so that denominators are always of the form z^n, z being the (complex) variable of integration. The values of all integrals occurring in $j=\frac{1}{2}$ and $j=\frac{3}{2}$ amplitudes are given below.

$$\begin{aligned}J_{00}&=(q^2)^{-1}[\alpha^{-1}\gamma],\\ J_{10}&=(2q\beta)^{-1}[1-\gamma+2\beta^2\alpha^{-1}\gamma],\\ J_{11}&=-(2q^2\beta)^{-1}[1-\gamma],\\ J_{21}&=(8q\beta^2)^{-1}[(3\alpha-4\beta^2)(1-\gamma)+4\beta^2\alpha^{-1}\gamma],\\ J_{22}&=-(8q^2\beta^2)^{-1}[3\alpha(1-\gamma)+4\beta^2\alpha^{-1}\gamma].\end{aligned} \quad (A.1)$$

α, β, and γ are the following dimensionless quantities:

$$\begin{aligned}\alpha&=(\rho q)^{-2}[1+\rho^2(q^2+q'^2)],\\ \beta&=q'/q,\\ \gamma&=(\alpha/4\beta)\ln[(\alpha+2\beta)/(\alpha-2\beta)].\end{aligned} \quad (A.2)$$

Some of the formulas (A.1) are not convenient at ordinary energies because γ is then very close to one. The logarithm must be expanded to give

$$\gamma=1+\tfrac{1}{3}(2\beta/\alpha)^2+\tfrac{1}{5}(2\beta/\alpha)^4+\cdots. \quad (A.3)$$

The values of the integrals (2.29) for $q=0$, $q'\neq0$ are sometimes needed. We have

$$J_{00}(q',0)=(\rho^{-2}+q'^2)^{-2},\quad J_{10}(q',0)=q'(\rho^{-2}+q'^2)^{-2}, \quad (A.4)$$

and all other integrals vanish.

Estimates of (2.29) obtained by substituting the asymptotes $x^l/(2l+1)!!$ of the Bessel functions are accurate only at rather low energies.

[13] The energy dependence peculiar to relativistic spin matrices [see Eq. (6.6a)] makes it necessary to define polarization in the rest system of the particle if one does not derive this energy dependence from an explicit wave equation as we have done. Thus the polarization expressions of Jacob and Wick (reference 11) are not directly comparable to ours since they are not strictly valid in the center-of-mass system (although they are expressed in terms of center-of-mass scattering angles). For the proper relativistic transformation to the laboratory system see Chou Kuang-Kao and M. I. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 1230 (1958) [translation: Soviet Phys.—JETP **7**, 851 (1958)].