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Formal Paradox in Quantum Electrodynamics*

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The formal paradox concerning the vanishing of the photon self-mass, obtained by a formal manipulation, is examined in Källén's formulation of electrodynamics. It is suggested that this difficulty can be removed, and the formal manipulation retained, by a regularization of the Heisenberg operators. An alternate method of obtaining the spectral function of the photon commutator is described, and a possible consequence of regularization, in connection with the proof of the renormalization constants' divergence, is briefly discussed.

INTRODUCTION

PERHAPS the most striking paradox obtainable by the application of certain commutation rules in quantum electrodynamics concerns the vanishing of the photon's self-mass. This is an old problem with much history, encountered first in perturbation theory and later in the exact (Heisenberg representation) formulation of electrodynamics. The purpose of this note is to remark that this difficulty again appears in Källén's formulation,[1] and is connected in a deceptive way with the magnitude of the renormalization constants. The only precise mathematical conclusion to be drawn from this discussion is the oft repeated statement that one of the formal manipulations employed is improper; however, one may imagine an alternate way out of this difficulty (regularization of the Heisenberg operators) with the effect of preserving the validity of such formal operations.

In the usual formulation of electrodynamics [corresponding to Eqs. (1) and (2) below] an apparent inconsistency arises upon calculating the photon self-mass: a straightforward and eminently reasonable inference drawn from the canonical commutation rules requires that this quantity be zero, but its calculation yields a nonzero, and infinite, result. In lowest order perturbation theory one applies the Pauli-Villars method of regularization[2] to eliminate this divergence, thereby bringing the computed result into agreement with the corresponding commutation relations. In higher orders, one may imagine the application of the quite formal regularization procedure of Gupta[3] as having the same net effect. It will be assumed in this discussion that there exists a consistent method of regularization (allowed to contain unobservable ficticious regulating masses for which an indefinite metric and/or improper statistics may be employed, as in the Gupta method) which may be applied to the interacting fields in the Heisenberg representation, and which will provide agreement between the computed predictions of the theory and corresponding predictions inferred from the canonical commutation rules. Since the formal operations to be performed shall yield mathematically inconsistent results, this assumption appears necessary in order to give meaning to the manipulations employed.

In Källén's formulation of electrodynamics it is not surprising to find the identical "inconsistency"; and again, if one desires to retain the same formal manipulations, this difficulty may be resolved by the assumption of a regularization procedure. In this formulation, however, in addition to the requirement of "consistency," the necessity of such regularization may be related to value of the renormalization constants by the following statement: unless the mechanical mass of the photon is regulated to zero, one of the renormalization constants is necessarily infinite. For the present discussion of Källén's equations the assumption of regularization is equivalent to the single requirement that the photon mass be regulated to zero; in the usual formulation there is an additional complication.[4] It may also be noted that the method of Källén's argument concerning the divergence of the renormalization constants[5] may lead to an inconclusive result when such regularization is considered.

STATEMENT OF THE PARADOX

The equations for the unrenormalized field operators in the usual formulation are

$$\Box A_\mu = J_\mu = -i(e/2)[\bar{\psi},\gamma_\mu\psi], \tag{1}$$

$$(\gamma_\mu\partial_\mu+m)\psi=\delta m\psi+ieA\psi, \tag{2}$$

where $A\equiv\gamma_\mu A_\mu$. In Källén's formulation, one writes the equations for the renormalized operators and charge in the form

$$\Box A_\mu = N^2 J_\mu + L(\Box A_\mu - \partial_\mu\partial_\sigma A_\sigma) \equiv -j_\mu, \tag{3}$$

$$(\gamma_\mu\partial_\mu+m)\psi=K\psi+ieA\psi. \tag{4}$$

From the conditions

$$[A_\mu(x),A_\nu(0)]_{x_0=0}=0, \tag{5}$$

$$[\psi(x),A_\nu(0)]_{x_0=0}=0, \tag{6}$$

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[1] G. Källén, Helv. Phys. Acta **24**, 417 (1952).
[2] W. Pauli and F. Villars, Revs. Modern Phys. **21**, 434 (1949). Equivalently, such regularization causes the equal-times commutator of two fermion currents, constructed from free-field operators, to vanish.
[3] S. N. Gupta, Proc. Phys. Soc. (London) **A66**, 129 (1953).
[4] The discussion leading to Eq. (13) indicates that the photon spectral function will contain a term proportional to $\delta'(\kappa^2)$; and it is difficult to understand how this can arise from a sum-over-states derivation.
[5] G. Källén, Kgl. Danske Vidensk. Selskab, Mat-fys. Medd. **27**, No. 12 (1953).

assumed valid in both formulations, and the respective equations for ψ, it follows that

$$[\partial_0\psi(x),A_\nu(0)]_{x_0=0}=0. \tag{7}$$

From Eqs. (6) and (7) one may then infer the relation[6]

$$[\partial_0 J_\mu(x),A_\nu(0)]_{x_0=0}=0. \tag{8}$$

The photon commutator

$$D_{\mu\nu}(x)=-i\langle 0|[A_\mu(x),A_\nu(0)]|0\rangle$$

has the spectral representation

$$D_{\mu\nu}(x)=\int_0^\infty d\kappa^2\,\{\rho_1(\kappa^2)\delta_{\mu\nu}+\rho_2(\kappa^2)\partial_\mu\partial_\nu\}D(x;\kappa^2).$$

In the usual formulation one may use Eq. (1) to obtain

$$\Box D_{\mu\nu}(x)=-i\langle 0|[J_\mu(x),A_\nu(0)]|0\rangle,$$

or

$$\langle 0|[\partial_0 J_\mu(x),A_\nu(0)]|0\rangle =-\frac{i}{(2\pi)^3}\int_0^\infty d\kappa^2\int dk\,\epsilon(\kappa)e^{ik\cdot x}\delta(k^2+\kappa^2) \times\kappa^2 k_0\{\delta_{\mu\nu}\rho_1-k_\mu k_\nu\rho_2\}. \tag{9}$$

At $x_0=0$ the left side of Eq. (9) must vanish which implies

$$\int_0^\infty d\kappa^2\,\kappa^2\rho_1(\kappa^2)=0. \tag{10}$$

In meson theory, this application of Eq. (8) yields the mechanical mass of the meson,[7]

$$\mu_0{}^2=\int_0^\infty d\kappa^2\,\kappa^2\rho(\kappa^2)\bigg/\int_0^\infty d\kappa^2\,\rho(\kappa^2);$$

in electrodynamics one obtains a statement of the vanishing of the photon's mechanical mass.[8] The "inconsistency" then arises because Eq. (10) is incompatible with the derived form of $\rho_1:\rho_1(\kappa^2)=a\delta(\kappa^2)+\sigma(\kappa^2);\sigma(\kappa^2)>0$.

In Källén's formulation one may use Eq. (3) to obtain

$$\Box D_{\mu\nu}(x)=i\langle 0|[j_\mu(x),A_\nu(0)]|0\rangle,$$

or

$$(1-L)\Box D_{\mu\nu}(x)+L\partial_\mu\partial_\sigma D_{\sigma\nu}(x) =-iN^2\langle 0|[J_\mu(x),A_\nu(0)]|0\rangle.$$

The expression derived by Källén for $D_{\mu\nu}(x)$ is

$$D_{\mu\nu}(x)=\frac{i}{(2\pi)^3}\int_0^\infty d\kappa^2\int dk\,\epsilon(k)e^{ik\cdot x}\delta(k^2+\kappa^2)Q_{\mu\nu},$$

$$Q_{\mu\nu}=\delta_{\mu\nu}\left[\delta(\kappa^2)+\frac{\pi(\kappa^2)}{\kappa^2}\right]+k_\mu k_\nu\left[\frac{\pi(\kappa^2)}{\kappa^2\kappa^2}-2M\delta(\kappa^2)\right],$$

from which one can calculate

$$N^2\langle 0|[\partial_0 J_\mu(x),A_\nu(0)]|0\rangle =\frac{i}{(2\pi)^3}\int_0^\infty d\kappa^2\int dk\,\epsilon(k)e^{ik\cdot x}\delta(k^2+\kappa^2)k_0R_{\mu\nu},$$
$$R_{\mu\nu}=(1-L)\delta_{\mu\nu}\pi(\kappa^2)+k_\mu k_\nu\left[(1-L)\frac{\pi(\kappa^2)}{\kappa^2}-L\delta(\kappa^2)\right]. \tag{11}$$

At $x_0=0$ the left side of Eq. (11) must vanish, which, for $N\neq 0$, implies

$$(1-L)\int_0^\infty d\kappa^2\,\pi(\kappa^2)=0. \tag{12}$$

Unless $L=1$, the mechanical mass of the photon must again be set equal to zero. However, this choice of L is, at best, ambiguous, since the defining equation for A_μ, Eq. (3), is then meaningless; further, the charge renormalization, $Z_3{}^{-1}=(1-L)^{-1}$, is then infinite, implying that the integral of Eq. (12) will diverge. As was the case with the integral of Eq. (10), the only precise statement to be made is that these integrals are necessarily infinite; assuming the consistency of the remainder of the theory, the application of Eq. (8) is mathematically improper. If the condition of Eq. (8) is to be retained, a regularization procedure must be assumed. The requirement that the integral of Eq. (12) must vanish was found by Goto and Imamura[9] as necessary in order to obtain agreement with a formal application of the canonical commutation rules.

DISCUSSION

As is evident from the above considerations, the source of this difficulty lies in the form of the spectral representation for the commutator of the fermion current with the photon field. Because $J_\mu(x)$ satisfies the relation $\partial_\mu J_\mu\equiv 0$, one can write

$$\langle 0|[J_\mu(x),A_\nu(0)]|0\rangle =\int_0^\infty d\kappa^2\,\Omega(\kappa^2)[\delta_{\mu\nu}\kappa^2-\partial_\mu\partial_\nu]D(x;\kappa^2),$$

where the spectral function $\Omega(\kappa^2)$ is composed of two essentially distinct parts: one part proportional to $\delta(\kappa^2)$, and the other proportional to that spectral function appearing in the commutator of two current operators. The requirement

$$\langle 0|[J_\mu(x),A_\nu(0)]|0\rangle_{x_0=0}=0 \tag{13}$$

will be satisfied[10] if $\int_0^\infty d\kappa^2\,\Omega(\kappa^2)=0$, but the additional requirement of Eq. (8) implies that $\int_0^\infty d\kappa^2\,\kappa^2\Omega(\kappa^2)=0$,

[6] This "inferred" commutation rule and that of Eq. (13) are the mathematically dangerous suppositions.

[7] H. Lehmann, Nuovo cimento **11**, 342 (1954).

[8] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[9] T. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) **14**, 396 (1955). The same point has recently been noted by J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

[10] In Källén's formulation this condition reproduces the defining equation for L in terms of $\pi(\kappa^2)$.

which cannot also be satisfied without the assumption of regularization.

It may be remarked that the necessity of such regularization can be demonstrated in an independent manner, by noting that there exists an alternate procedure for the calculation of the photon commutator from that used by Källén. This method proceeds from the observation that the commutator obeys a second order partial differential equation, with specified "initial values" given by the equal-times commutation relations. One need not assume relations between matrix elements of the photon operators and their asymptotic values, but instead one may write the equations

$$\Box D_{\mu\nu}(x)=E_{\mu\nu}(x)\equiv i\langle 0|[j_\mu(x),A_\nu(0)]|0\rangle,$$
$$\Box E_{\mu\nu}(x)=\pi_{\mu\nu}(x)\equiv -i\langle 0|[j_\mu(x),j_\nu(0)]|0\rangle,$$

which have the unique solutions

$$D_{\mu\nu}(x)=\int d^3y\, D(\mathbf{x}-\mathbf{y},x_0;0)G_{\mu\nu}(\mathbf{y})+\int d^4y\,\mathfrak{D}(x,y)E_{\mu\nu}(y),$$

$$E_{\mu\nu}(x)=\int d^3y\, D(\mathbf{x}-\mathbf{y},x_0;0)H_{\mu\nu}(\mathbf{y})+\int d^3y\,\partial_0 D(\mathbf{x}-\mathbf{y},x_0;0)I_{\mu\nu}(\mathbf{y})+\int d^4y\,\mathfrak{D}(x,y)\pi_{\mu\nu}(y),$$

where $\mathfrak{D}(x,y)$ is that Green's function satisfying[11]

$$\Box_x\mathfrak{D}=\delta(x-y),\quad \mathfrak{D}|_{x_0=0}=\partial_{x0}\mathfrak{D}|_{x_0=0}=0.$$

The quantities $G_{\mu\nu}(\mathbf{x})$ and $I_{\mu\nu}(\mathbf{x})$ represent appropriate initial conditions for $D_{\mu\nu}$ and $E_{\mu\nu}$, inferred from the canonical commutation relations; $H_{\mu\nu}(\mathbf{x})$ is obtained from the requirement: $\partial_\mu E_{\mu\nu}(x)\equiv 0$, and is proportional to the integral of Eq. (12). Upon dropping all such photon mass terms, straightforward integration reproduces Källén's expressions for $E_{\mu\nu}$ and $D_{\mu\nu}$.

Finally, one may question whether a regularization procedure used in conjunction with Källén's discussion of the magnitude of the renormalization constants does not of itself imply an inconclusive result. This is suggested by the following argument. Suppose that Källén's conjecture concerning the nature of the positive definite function $\pi(\kappa^2)$ is correct; that is, there exists a lower bound to this quantity, say $\pi'(\kappa^2)$, which possesses the property[12]

$$\lim_{\kappa^2\to\infty}\pi'(\kappa^2)\to\xi\lim_{\kappa^2\to\infty}\pi^{(0)}(\kappa^2),$$

where ξ is a dimensionless constant and $\pi^{(0)}(\kappa^2)$ represents the lowest order (Born) approximation:

$$\pi^{(0)}(\kappa^2)=\eta\theta(\kappa^2-4m^2)(1+2m^2/\kappa^2)(1-4m^2/\kappa^2)^{\frac{1}{2}},$$

with η a constant independent of m. It would then follow that the integral $\int_0^\infty(d\kappa^2/\kappa^2)\pi'(\kappa^2)$ diverges at its upper limit, which, following Källén, is equivalent to the statement that at least one of the renormalization constants is infinite. However, suppose that, in analogy with the perturbation theory procedure, the effect of regularization is to replace $\pi'(\kappa^2)$ by $\pi_{\text{reg}}'(\kappa^2)=\sum_i C_i'\pi_i'(\kappa^2)$, where the subscript i denotes the contribution of the ith fermion field, and the C_i' represent a set of coefficients so chosen as to ensure the vanishing of the integral

$$\lim_{\Lambda^2\to\infty}\int_0^{\Lambda^2}d\kappa^2\,\pi_{\text{reg}}'(\kappa^2). \tag{14}$$

From Källén's conjecture one would expect

$$\lim_{\kappa^2\to\infty}\pi_i'(\kappa^2)\to\xi_i\lim_{\kappa^2\to\infty}\pi_i^{(0)}(\kappa^2)=\eta\xi_i,$$

and in order for the integral of Eq. (14) to vanish at its upper limit one would then, among other conditions, require $\sum_i C_i'\xi_i=0$. (Actually, the C_i' should be chosen such that the complete photon mass integral vanishes; but if the number of regulating fields allowed has only a lower bound, as in perturbation theory, this condition should always be realizable. Further, one might expect that the ξ_i would turn out to be independent of i, whereupon this condition would reduce to one of those familiar from perturbation theory and anticipated as necessary for the vanishing of the complete self-mass integral.) But the asymptotic limit of $\pi_{\text{reg}}'(\kappa^2)$ is then zero, and the renormalization integral constructed from this function is "indeterminate" [of form $\int^\infty(d\kappa^2/\kappa^2)\times 0$] at its upper limit; hence no statement concerning the convergence of this integral can be inferred from this type of argument.

Note added in proof.—It should perhaps be emphasized that the regularized renormalization constant $(1-L_{\text{reg}})^{-1}$ be considered as distinct from the unregularized renormalization constant $(1-L)^{-1}$.

In fact, one might even expect a regularization procedure to yield arbitrary values for the renormalization constants. This is suggested by considering the effect of regularization on that integral constructed from the

[11] H. M. Fried, Phys. Rev. **115**, 220 (1959). The necessary integrals of this Green's function are

$$\int d^4y\,\mathfrak{D}(x,y)D(y;\kappa^2)=\frac{1}{\kappa^2}[D(x;\kappa^2)-D(x;0)],$$

and

$$\int d^4y\,\mathfrak{D}(x,y)\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial y_\nu}D(y;\kappa^2)=\frac{\partial_\mu\partial_\nu}{\kappa^2}[D(x;\kappa^2)-D(x;0)]-\delta_{\mu4}\delta_{\nu4}D(x;0).$$

[12] This point has been questioned by S. G. Gasiorowicz, D. R. Yennie, and H. Suura, Phys. Rev. Letters **2**, 513 (1959).

lowest order approximation $\pi^{(0)}$. The quantity

$$\lim_{\Lambda^2\to\infty}\int_0^{\Lambda^2}\frac{d\kappa^2}{\kappa^2}\pi^{(0)}(\kappa^2)\to\eta\left[-\frac{5}{3}+2\ln\left(\frac{\Lambda}{m}\right)\right]$$

is logarithmically divergent and remains so when regulated by means of the standard conditions[2] $\sum_i C_i=\sum_i C_i m_i^2=0$, where $C_0=1$ and m_0 denotes the electron's mass. It should be noted, however, that these relations represent the minimum number of such conditions required to ensure the vanishing of the photon mass integral. There is nothing to prevent the adoption of the further condition $\sum_i C_i \ln(m_i/m_0)=P$, where P is zero or any selected finite number. The use of such an "extended" regularization procedure requires an extra regulating field and changes the values of the coefficients C_i, for $i\geqslant 1$ (they are now logarithmically divergent with the regulating masses), but in no way alters the results of the lowest order vacuum polarization calculation. Regularizing in this manner, the value of this renormalization integral is proportional to the arbitrary number P.

This discussion should, of course, not be regarded as exact in any way, or even correct; but rather as merely a kind of plausibility argument. Certainly, any precise statements concerning the effect of regularization on the magnitude of the renormalization constants must await the explicit demonstration of a consistent regularization procedure for the coupled Heisenberg fields.

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Moment of Inertia of Superfluid Many-Fermion Systems*

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The effects of possible superfluidity on the cranking moment of a large many-fermion system moving under periodic boundary conditions are investigated within the framework of the theory of superconductivity recently formulated by Bogolyubov. The Hamiltonian is initially subjected to Bogolyubov's general unitary transformation. The collective excitations of the fermions are then considered in the usual pair approximation; the appropriate cranking terms are linear in the boson pair operators. On performing a unitary transformation which transforms away these linear terms, one obtains an expression for the moment of inertia of the system which includes both the effects of possible superfluidity and collective excitation. This expression, by virtue of its being stationary with respect to arbitrary variations in the amplitude associated with the latter unitary transformation, is then utilized as a variational principle for the moment of inertia. For the normal state, the result previously obtained by the author, that the moment of inertia has the rigid value, is rederived in more compact form. For the superfluid state, one finds that collective excitations effect a marked increase in the superfluid moment at intermediate coupling strengths although the resulting moment is still quite small compared to the rigid value. In contrast to the normal state case, where particle-hole pairs play a major role, this increase is almost entirely due to excitations consisting of particle-pairs or hole-pairs. The precise magnitude of the apparent resonance in the moment produced by the d-wave part of the cranking interaction is dependent to some extent on the features of the particle-particle potential which leads to the superfluid state. Variational expressions for the moment are exhibited for both Yukawa and delta-function shell potentials. These results are identical in charged and neutral Fermi systems. A calculation of the cranking moment at finite temperatures is presented in an Appendix along with an interpretation of it in terms of Bardeen's two-fluid model of superconductivity.

I. INTRODUCTION

IN a previous work by the author[1] (hereafter referred to as I), some of the consequences of particle-particle interaction on the cranking moment of a large many-fermion system moving under periodic boundary conditions were investigated. In particular, it was shown that the shift in the rigid moment of inertia due to collective excitations consisting in mass-renormalized particle-hole pairs could be obtained exactly without recourse to the usual perturbation theory.[2] Further, this shift was found to vanish exactly, although, initially, it had seemed likely that pair excitations would furnish the major contribution to such an interaction shift. One noted that stability requirements[3,4] in I re-

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[1] R. M. Rockmore, Phys. Rev. **116**, 469 (1959).

[2] The effect of interparticle forces in the lowest order of perturbation theory has recently been investigated by R. Amado and K. Brueckner, Phys. Rev. **115**, 778 (1959).

[3] K. Sawada and R. Rockmore, Phys. Rev. **116**, 1618 (1959); A. E. Glassgold, W. Heckrotte, and K. M. Watson, Ann. Phys. **6**, 1 (1959).

[4] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau, Inc., New York, 1959).